GENERALIZED LIMITS IN GENERAL ANALYSIS*

SECOND PAPER

BY

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In a previous paper of the same title† I have developed the fundamental principles of a general theory which includes as particular instances the theories of Cesàro and Hölder summability of divergent series and divergent integrals. I further made use of these fundamental principles to prove a general theorem which includes as special cases several important theorems in the above mentioned special theories.

In the present paper the general theory referred to above is extended to the case of multiple limits and the theorem mentioned is likewise generalized. The theorem thus obtained includes as special cases the extension to multiple series of the Knopp-Schnee-Ford theorem‡ on the equivalence of Cesàro and Hölder summability for divergent series, the extension to multiple integrals of the analogous theorem of Landau‡ for the case of divergent integrals, and the extension to partial derivatives of a corresponding theorem with regard to the equivalence of certain generalized derivatives. Once the principles of the theory are set forth, the proof of this general theorem is fully as simple as the proofs of any of the special theorems would be. Thus we have exhibited the greater power of the methods of General Analysis as compared with the methods of classical analysis.

The basis of our general theory may be indicated as follows:

$$(\mathfrak{A};\ \mathfrak{P}_1;\ \mathfrak{P}_2;\ \ldots;\ \mathfrak{P}_m;\ \mathfrak{S}_1;\ \mathfrak{S}_2;\ \ldots;\ \mathfrak{S}_m;\ \mathfrak{G}^{\text{on } \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m, \text{ to } \mathfrak{A}};$$

$$\mathfrak{H}_i^{\text{on } \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m \text{ to } \mathfrak{A}} \qquad (i = 1, 2, \ldots, m);$$

$$\mathfrak{F}_i^{\text{on } \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m \text{ to } \mathfrak{A}} \qquad (i = 1, 2, \ldots, m);$$

$$\mathfrak{H}^{\text{on } \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m \text{ to } \mathfrak{A}};\ \mathfrak{F}^{\text{on } \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m \text{ to } \mathfrak{A}};\ \varphi_0^{(m)};$$

$$J_i^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H}_i;\ \text{on } \mathfrak{H}_i \text{ to } \mathfrak{F}_i} \qquad (i = 1, 2, \ldots, m);$$

$$J^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H};\ \text{on } \mathfrak{H} \text{ to } \mathfrak{F}}),$$

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† These Transactions, vol. 24 (1922), pp. 79–88.

‡ For references to the literature dealing with the special theorems referred to, see Paper I.

where $\mathfrak{A} \equiv [a]$ denotes the class of all real numbers a, $\mathfrak{P}_i \equiv [p_i]$ denotes a class of elements p_i $(i = 1, 2, \ldots, m)$, and $\mathfrak{S}_i \equiv [\sigma^{(i)}]$ denotes a class of sets $\sigma^{(i)}$ of elements p_i of the range $\mathfrak{P}_i$ $(i = 1, 2, \ldots, m)$; $\mathfrak{G} \equiv [\gamma]$, $\mathfrak{H}_i \equiv [\eta^{(i)}]$ $(i = 1, 2, \ldots, m)$, $\mathfrak{F}_i \equiv [\varphi^{(i)}]$ $(i = 1, 2, \ldots, m)$, $\mathfrak{H} \equiv [\eta]$, and $\mathfrak{F} \equiv [\varphi]$ are $(2m + 3)$ classes of functions, γ, $\eta^{(1)}, \ldots, \eta^{(m)}$, $\varphi^{(1)}, \ldots, \varphi^{(m)}$, η, and φ respectively on $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m$ to $\mathfrak{A}$ (we consider only single-valued functions); $\varphi_0^{(m)}$ is a special function of the class $\mathfrak{F}$; J_i is a functional operation turning a function of the class $\mathfrak{G}$ into a function of the class $\mathfrak{H}_i$ or a function of the class $\mathfrak{H}_i$ into a function of the class $\mathfrak{F}_i$, denoted by $J_i\gamma$ or $J_i\eta^{(i)}$ respectively; and J is a functional operation turning a function of the class $\mathfrak{G}$ into a function of the class $\mathfrak{H}$ or a function of the class $\mathfrak{H}$ into a function of the class $\mathfrak{F}$, denoted by $J\gamma$ or $J\eta$ respectively.

In order to show the relationship of our general theorem to the special cases of it to which we have referred, we will indicate here what the general basis reduces to in the particular instances III and IV.

$$\mathfrak{P}_i^{\mathrm{III}} \equiv [\text{all } n_i = 1, 2, 3, \ldots] \qquad (i = 1, 2, \ldots, m);$$

$$\mathfrak{S}_i \equiv [\sigma_{n_i} \equiv (1, 2, \ldots, n_i) | n_i] \qquad (i = 1, 2, \ldots, m);$$

$$\mathfrak{G} \equiv \mathfrak{H}_i \equiv \mathfrak{F}_i \equiv \mathfrak{H} \equiv \mathfrak{F} \equiv [\text{all } \gamma,\ \eta^{(i)},\ \varphi^{(i)},\ \eta,\ \varphi^{\text{on } \mathfrak{S}_1, \ldots, \mathfrak{S}_m \text{ to } \mathfrak{A}}] \qquad (i = 1, 2, \ldots, m);$$

$$\varphi_0^{(m)}(\sigma_{n_1}, \sigma_{n_2}, \ldots, \sigma_{n_m}) = n_1 n_2 \cdots n_m \qquad (n_i;\ i = 1, 2, \ldots, m);$$

$$(J_i\,\theta)(\sigma_{n_1}, \ldots, \sigma_{n_m}) = \sum_{k_i=1}^{k_i=n_i} \theta(\sigma_{n_1}, \ldots, \sigma_{k_i}, \ldots, \sigma_{n_m}) \qquad (n_i;\ i = 1, 2, \ldots, m;\ \theta = \gamma, \eta^{(i)});$$

$$(J\theta)(\sigma_{n_1}, \ldots, \sigma_{n_m}) = \sum_{k_1=1}^{k_1=n_1} \cdots \sum_{k_m=1}^{k_m=n_m} \theta(\sigma_{k_1}, \ldots, \sigma_{k_m}); \qquad (n_i;\ i = 1, 2, \ldots, m;\ \theta = \gamma, \eta).$$

$$\mathfrak{P}_i^{\mathrm{IV}} \equiv [\text{all } a_i > 0] \qquad (i = 1, 2, \ldots, m);$$

$$\mathfrak{S}_i \equiv [\sigma_a^{(i)} \equiv (\text{all } x_i \text{ such that } 0 < x_i \leq a_i) \quad (a_i > 0;\ i = 1, 2, \ldots, m)];$$

$\mathfrak{G} \equiv$ [all functions that are finite in any finite region $(0 < x_i \leq a_i;$ $i = 1, 2, \ldots, m)$ and are integrable (Lebesgue) with respect to each of the variables x_i ($i = 1, \ldots, m$) on every finite interval ($0 < x_i \leqq a_i$; $i = 1, \ldots, m$)];

$$\mathfrak{H}_i \equiv \text{all } \eta_i^{(i)} = \int_0^{x_i} \gamma(x_1, x_2, \ldots, x_i, \ldots, x_m)\, dx_i \quad (x_i > 0; i = 1, \ldots, m);$$

$$\mathfrak{F}_i \equiv \text{all } \varphi^{(i)} = \int_0^{x_i} \eta^{(i)}(x_1, x_2, \ldots, x_i, \ldots, x_m)\, dx_i \quad (x_i > 0; i = 1, \ldots, m);$$

$$\mathfrak{H} \equiv \text{all } \eta = \int_0^{x_1} \cdots \int_0^{x_m} \gamma(x_1, \ldots, x_m) \qquad (x_i > 0;\ i = 1, \ldots, m);$$

$$\mathfrak{F} \equiv \text{all } \varphi = \int_0^{x_1} \cdots \int_0^{x_m} \eta(x_1, \ldots, x_m) \qquad (x_i > 0;\ i = 1, \ldots, m);$$

$$(\varphi_0^{(m)})(\sigma_{a_1}, \ldots, \sigma_{a_m}) = a_1\, a_2 \cdots a_m \qquad (a_i;\ i = 1, \ldots, m);$$

$$(J_i\theta)\,(\sigma_{a_1}, \ldots, \sigma_{a_m}) = \int_0^{a_i} \theta\, dx_i \qquad (a_i;\ i = 1, \ldots, m;\ \theta = \gamma,\ \eta^{(i)});$$

$$(J\theta)\,(\sigma_{a_1}, \ldots, \sigma_{a_m}) = \int_0^{a_1} \cdots \int_0^{a_m} \theta \qquad (a_i; i = 1, \ldots, m;\ \theta = \gamma,\ \eta).$$

With regard to each of the classes $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_m$ we make definitions analogous to those made for the class $\mathfrak{S}$ in Paper I, and we further postulate analogous properties. These properties will be referred to by the same letters as in the previous paper with a subscript or index attached to indicate the particular class to which reference is made. When any two functions θ, α on $\mathfrak{S}_1, \ldots, \mathfrak{S}_m$ to $\mathfrak{A}$ are regarded as functions of a single set $\sigma^{(i)}$, the other sets being held fixed, we define the notation $(D_i\,\theta)\,(\sigma^{(1)}, \ldots, \sigma^{(m)}) = \alpha\,(\sigma^{(1)}, \ldots, \sigma^{(m)})$

in a manner entirely analogous to that in which the notation $(D\theta)(\sigma) = \alpha(\sigma)$ was defined in Paper I.

When functions of the class $\mathfrak{G}$ are regarded as functions of a single set $\sigma^{(i)}$, the other sets being held fixed, we postulate for them the properties of class $\mathfrak{G}$ of Paper I, these properties to be referred to by the same letter with suitable index or subscript. Analogous properties for the classes $\mathfrak{H}$ and $\mathfrak{F}$, designated in similar fashion, are also postulated. Furthermore for the classes $\mathfrak{H}_i$ and $\mathfrak{F}_i$, regarded as functions of the set $\sigma^{(i)}$ alone, we postulate the properties of classes $\mathfrak{H}$ and $\mathfrak{F}$ of Paper I and indicate them in like manner. When the functions of classes $\mathfrak{G}$ and $\mathfrak{H}_i$ that are involved in the operation J_i are regarded as functions of the set $\sigma^{(i)}$ alone, we require J_i to have all the properties required of J in Paper I, which properties we shall designate by the same symbols with subscript or index i. We further postulate that any of the operations $J_1, J_2, \ldots, J_m$ is interchangeable with any other of the set, which property we designate as (I). We also postulate as to the relationship between J and $J_1, J_2, \ldots, J_m$ that

$$(N) \qquad (J\eta)(\sigma^{(1)}, \ldots, \sigma^{(m)}) \equiv (J_1(J_2 \cdots (J_m\eta) \cdots))(\sigma^{(1)}, \ldots, \sigma^{(m)}).$$

With regard to the special function $\varphi_0^{(m)}(\sigma^{(1)}, \ldots, \sigma^{(m)})$ we postulate that

$$(X) \qquad \varphi_0^{(m)}(\sigma^{(1)}, \ldots, \sigma^{(m)}) = \varphi_0(\sigma^{(1)})\,\varphi_0(\sigma^{(2)}) \cdots \varphi_0(\sigma^{(m)}),$$

where $\varphi_0(\sigma^{(i)})$ as function of $\sigma^{(i)}$, for $i = 1, 2, \ldots, m$, is the same function as $\varphi_0(\sigma)$ of Paper I as function of σ. We also postulate for the class $\mathfrak{F}$ that $(J_i\varphi)(\sigma^{(1)}, \ldots, \sigma^{(m)})$, for $i = 1, 2, \ldots, m$, is of the class $\mathfrak{F}$, which property we designate as (K).

For the sake of brevity we shall agree to represent, in all cases where no loss of clearness is involved, the set of elements $p^{(1)}, \ldots, p^{(m)}$ by the single symbol p, the set of classes $\mathfrak{P}^{(1)}, \ldots, \mathfrak{P}^{(m)}$ by $\mathfrak{P}$, the set of classes $\mathfrak{S}^{(1)}, \ldots, \mathfrak{S}^{(m)}$ by $\mathfrak{S}$, and the set of sets $\sigma^{(1)}, \ldots, \sigma^{(m)}$ by σ. Analogous to the definition of the notation $\lim_\sigma \theta(\sigma) = a$ in Paper I, we define the corresponding notation in the case that σ represents the set of sets $\sigma^{(1)}, \ldots, \sigma^{(m)}$ to mean that corresponding to every positive e there exist sets $\sigma_e^{(1)}, \ldots, \sigma_e^{(m)}$ such that for sets $\sigma^{(i)} > \sigma_e^{(i)}$ $(i = 1, 2, \ldots, m)$ we have $|\theta(\sigma) - a| < e$.

We then postulate as to the class $\mathfrak{H}$ the property (B) defined by

(B) If $\lim_\sigma \eta(\sigma)$ exists and is equal to a, then $|\eta(\sigma)| < a_1(\sigma)$.

If we also for the sake of brevity agree to represent a group of properties such as $R_1, R_2, \ldots, R_m$ by the single letter R, we may indicate the foundation of our theory as follows:

$$\sum \equiv \Big(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{S}^{UAR};\ \mathfrak{G}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\,.\,LP};\ \mathfrak{H}_i^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\,.\,L_i P_i S_G^{(i)}} \qquad (i = 1, \ldots, m);$$

$$\mathfrak{F}_i^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\,.\,L_i P_i S_H^{(i)} C_i \Delta_i} \qquad (i = 1, \ldots, m); \qquad \mathfrak{H}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\,.\,LPS_G B};$$

$$\mathfrak{F}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\,.\,LPS_H C \Delta K}; \qquad \varphi_0^{(m)\mathfrak{F}\,.\,X};$$

$$J_i^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H}_i\,.\,\text{on } \mathfrak{H}_i \text{ to } \mathfrak{F}_i\,.\,M_1^{(i)} M_2^{(i)} I_D^{(i)} I_J^{(i)} I} \qquad (i = 1, \ldots, m);$$

$$J^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H}\,.\,\text{on } \mathfrak{H} \text{ to } \mathfrak{F}\,.\,N}\Big).$$

We now set

$$\varphi_{0n}^{(m)}(\sigma) = \varphi_{0n}(\sigma^{(1)})\,\varphi_{0n}(\sigma^{(2)}) \cdots \varphi_{0n}(\sigma^{(m)}), \tag{1}$$

where $\varphi_{0n}(\sigma^{(i)})$, as function of $\sigma^{(i)}$, is the same function as $\varphi_{0n}(\sigma)$, as function of σ, defined by equation (3) of Paper I. We are then ready to define the two generalized limits with which we shall be concerned. Given any function $\eta(\sigma)$, we set

$$(C_n \eta)(\sigma) = \frac{(n!)^m}{\varphi_{0n}^{(m)}(\sigma)}(J^n \eta)(\sigma) \qquad (n), \tag{2}$$

$$(M \eta)(\sigma) = \frac{1}{\varphi_0^{(m)}(\sigma)}(J\eta)(\sigma), \tag{3}$$

$$(H_n \eta)(\sigma) = (M^n \eta)(\sigma) \qquad (n). \tag{4}$$

If for a fixed n $\lim_\sigma (C_n \eta)(\sigma)$ exists, we define this limit as the generalized limit of type (Cn) for $\eta(\sigma)$. If $\lim_\sigma (H_n \eta)(\sigma)$ exists, we define this limit as the generalized limit of type (Hn) for $\eta(\sigma)$.

Before proceeding to the proof of the equivalence theorem we introduce the following notations:

$$(5)\qquad (M_i\gamma)(\sigma) = [1/\varphi_0(\sigma^{(i)})](J_i\gamma)(\sigma) \qquad (i = 1, \ldots, m),$$

$$(6)\quad (C_n^{i,i+1,\ldots,i+k}\eta)(\sigma) = \frac{(n!)^{k+1}}{\varphi_{0n}(\sigma^{(i)})\cdots\varphi_{0n}(\sigma^{(i+k)})}(J_i\cdots(J_{i+k}\eta)\cdots)(\sigma) \quad (n),$$

$$(7)\quad \begin{aligned} &\gamma_n^{(i)}(\sigma) = \varphi_0(\sigma^{(i)})\,\varphi_0(\sigma_1^{(i)})\cdots\varphi_0(\sigma_{n-2}^{(i)})\,\gamma(\sigma) \qquad (n>2;\ i=1,\ldots,m),\\ &\gamma_2^{(i)}(\sigma) = \varphi_0(\sigma^{(i)})\,\gamma(\sigma), \qquad \gamma_1^{(i)}(\sigma) = \gamma(\sigma) \qquad (i = 1, \ldots, m),\end{aligned}$$

where $\sigma_1^{(i)}$, $\sigma_2^{(i)}$, ... are defined with regard to $\sigma^{(i)}$ in the same manner as σ_1, σ_2, ... with regard to σ in Paper I;

$$(8)\quad (S_n^{(i)}\gamma)(\sigma) = \left(\left(\frac{n-1}{n}M_i + \frac{1}{n}E\right)\gamma\right)(\sigma) \qquad (n),$$

$$(9)\quad (S_n^{(1,2,\ldots,i)}\gamma)(\sigma) = (S_n^{(1)}(S_n^{(2)}\cdots(S_n^{(i)}\gamma)\cdots))(\sigma) \quad (n;\ i = 1, \ldots, m),$$

$$(10)\quad (S_n\gamma)(\sigma) = (S_n^{(1)}(S_n^{(2)}\cdots(S_n^{(m)}\gamma)\cdots))(\sigma) \qquad (n),$$

$$(11)\quad (T_n^{(i)}\gamma)(\sigma) = n\gamma(\sigma) - \frac{n(n-1)}{\varphi_{0n}(\sigma^{(i)})}(J_i\gamma_n^{(i)})(\sigma) \qquad (n;\ i = 1, \ldots, m).$$

We are now ready for the proof of our theorem; we begin by proving some lemmas.

LEMMA 1. *If we define S_n as in* (10), *we have the identity*

$$(12)\qquad (S_n(C_n\eta))(\sigma) = (M(C_{n-1}\eta))(\sigma) \qquad (n).$$

We have from Lemma 1 of Paper I and the interchangeability of the various operations involved

$$(S_n(C_n\eta))(\sigma) = \left(S_n^{(1,2,\dots,m-1)}(C_n^{(1,2,\dots,m-1)}(S_n^{(m)}(C_n^{(m)}\eta)))\right)(\sigma)$$

$$= \left(S_n^{(1,2,\dots,m-1)}(C_n^{(1,2,\dots,m-1)}(M_m(C_{n-1}^{(m)}\eta)))\right)(\sigma)$$

$$= \left(S_n^{(1,2,\dots,m-2)}\left(C_n^{(1,2,\dots,m-2)}\left(S_n^{(m-1)}(C_n^{(m-1)}(M_m(C_{n-1}^{(m)}\eta)))\right)\right)\right)(\sigma)$$

$$= \left(S_n^{(1,2,\dots,m-2)}\left(C_n^{(1,2,\dots,m-2)}(M_{m-1}(M_m(C_{n-1}^{(m-1,m)}\eta)))\right)\right)(\sigma)$$

$$= \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$= \left(M_1(M_2\cdots(M_n(C_{n-1}\eta))\cdots)\right)(\sigma) = (M(C_{n-1}\eta))(\sigma).$$

Our lemma is therefore established.

We define $\varphi_n^{(i)}$ in a manner analogous to the definition of $\gamma_n^{(i)}$ in (7). We also set

$$\varphi_{0n\cdot i}(\sigma) = \varphi_{0n}(\sigma^{(i)}).$$

We then prove

LEMMA 2. *If* $\lim_\sigma \varphi(\sigma)$ *exists and is equal to* a *and* $|\varphi(\sigma)| < a_1$ *for every* σ, *then* $\lim_\sigma [\varphi_{0n}(\sigma^{(i)})]^{-1}(J_i\,\varphi_n^{(i)})(\sigma)$ *will exist and be equal to* a/n *and we shall have*

$$|[\varphi_{0n}(\sigma^{(i)})]^{-1}(J_i\,\varphi_n^{(i)})(\sigma)| < a_1^{(i)} \qquad (\sigma,\ n;\ i = 1,\ \dots,\ m).$$

Given a positive e, we choose σ_e' so that $a-(e/4) < \varphi(\sigma) < a+(e/4)$ for $\sigma \geqq \sigma_e'$.* We have

$$[\varphi_{0n}(\sigma^{(i)})]^{-1}(J_i\,\varphi_n^{(i)})(\sigma)$$

$$(13) \qquad = [\varphi_{0n}(\sigma^{(i)})]^{-1}(J_i\,\varphi_n^{(i)})(\sigma^{(1)},\ \dots,\ \sigma_e^{(i)'},\ \dots,\ \sigma^{(m)})]$$

$$+[\varphi_{0n}(\sigma^{(i)})]^{-1}[(J_i\,\varphi_n^{(i)})(\sigma)-(J_i\,\varphi_n^{(i)})(\sigma^{(1)},\ \dots,\ \sigma_e^{(i)'},\ \dots,\ \sigma^{(m)})].$$

* It should be remembered throughout that σ_e' is an abbreviation for $(\sigma_e^{(1)'}, \sigma_e^{(2)'}, \dots, \sigma_e^{(m)'})$, and that $\sigma \geqq \sigma_e'$ is an abbreviation for the set of relationships $\sigma_e^{(1)} \geqq \sigma_e^{(1)'}, \dots, \sigma_e^{(m)} \geqq \sigma_e^{(m)'}$.

Analogous to (18) of Paper I we have the relationship

$$(14)\qquad (J_i\, \varphi_n^{(i)})(\sigma) = \left(J_i \left[\frac{1}{n}(D_i\, \varphi_{0n\cdot i})\,\varphi\right]\right)(\sigma).$$

Making use of (14), and postulates $M_2^{(i)}$ and $I_J^{(i)}$, we see that the second term on the right hand side of (13) lies between

$$\frac{1}{n}\left(a-\frac{e}{4}\right)\left[1-\frac{\varphi_{0n}(\sigma_e^{(i)'})}{\varphi_{0n}(\sigma^{(i)})}\right] \text{ and } \frac{1}{n}\left(a+\frac{e}{4}\right)\left[1-\frac{\varphi_{0n}(\sigma_e^{(i)'})}{\varphi_{0n}(\sigma^{(i)})}\right].$$

From (IV) of Paper I it follows that for a proper choice of $\sigma_e'' > \sigma_e'$ the above expression differs from a/n by a quantity that is less in absolute value than $\frac{1}{2}e$ for all $\sigma > \sigma_e''$.

The first term on the right side of (13) is seen from (14), $M_1^{(i)}$, and $I_J^{(i)}$ to be less in absolute value than

$$\frac{a_1}{n}\left|\frac{\varphi_{0n}(\sigma_e^{(i)})}{\varphi_{0n}(\sigma^{(i)})}\right|.$$

From IV of Paper I it follows that we can choose $\sigma_e''' > \sigma_e'$ so as to make this expression less in absolute value than $\frac{1}{2}e$ for $\sigma > \sigma_e'''$. If now we choose for σ_e the greater of σ_e'' and σ_e''', it follows from (13) that for $\sigma > \sigma_e$,

$$\left|[\varphi_{0n}(\sigma^{(i)})]^{-1}(J_i\,\varphi_n^{(i)})(\sigma) - (a/n)\right| < e,$$

and the first part of our conclusion is established.

Making use of (14) and $M_1^{(i)}$, we have

$$\frac{-a_1}{n} < [\varphi_{0n}(\sigma^{(i)})]^{-1}(J_i\,\varphi_n^{(i)})(\sigma) < \frac{a_1}{n},$$

which establishes the second part of our conclusion.

LEMMA 3. *If* $\lim_\sigma \varphi(\sigma)$ *exists and is equal to a, and* $|\varphi(\sigma)| < a_1$ *for every* σ, *then* $\lim_\sigma (S_n \varphi)(\sigma)$ *will exist and be equal to a and we shall have* $|(S_n \varphi)(\sigma)| < a_2$ *for every* σ.

By virtue of definition (10) the operation S_n is equivalent to a succession of operations $S_n^{(i)}$ for $i = 1, 2, \ldots, m$. It follows from Lemma 2, for the case $n = 1$, that if a function $\varphi(\sigma)$ remains finite for all σ and approaches a limit as to σ, the same is true for the function resulting from the operation $S_n^{(i)}$ applied to $\varphi(\sigma)$. Hence by a succession of m applications of Lemma 2, we obtain the conclusion of the present lemma.

We now set

$$\varphi_i'(\sigma) = (S_n^{(i)} \varphi)(\sigma) \qquad (i = 1, 2, \ldots, m). \tag{15}$$

We then prove

LEMMA 4. *If for any* i $\lim_\sigma \varphi_i'(\sigma)$ *exists and is equal to a, and* $|\varphi_i'(\sigma)| < a_1$ *for every* σ, *then* $\lim_\sigma \varphi(\sigma)$ *will exist and be equal to a, and we shall have* $|\varphi(\sigma)| < a_2$ *for every* σ.

By a procedure analogous to that used in the proof of Lemma 3 of Paper I we may transform equation (15) into the form

$$\varphi(\sigma) = (T_n^{(i)} \varphi_i')(\sigma) \quad (n \geqq 2;\ i = 1, \ldots, m), \tag{16}$$

where $T_n^{(i)}$ is defined by equation (11). Our lemma then follows from Lemma 2 for $n \geqq 2$. For $n = 1$ it is an obvious consequence of (15) and (8).

LEMMA 5. *If* $\lim_\sigma (S_n \varphi)(\sigma)$ *exists and is equal to a and* $|(S_n \varphi)(\sigma)| < a_1$ *for every* σ, *then* $\lim_\sigma \varphi(\sigma)$ *will exist and be equal to a and we shall have* $|\varphi(\sigma)| < a_2$ *for every* σ.

Making use of the definition of S_n given in equation (10), we see that this lemma may be established by successive applications of Lemma 4.

Noting that S_n and M are interchangeable operations, we have from successive applications of (12), in a manner analogous to the corresponding reductions in Paper I by means of equation (14) of that paper,

$$(H_n \eta)(\sigma) = \Big(S_1\Big(S_2 \cdots \Big(S_3 (S_{n-1} (S_n (C_n \eta))) \cdots \Big)\Big)\Big)(\sigma) \qquad (n). \tag{17}$$

We are now ready to prove our theorem:

THEOREM. *If* $\lim_\sigma (C_n\,\eta)(\sigma)$ *exists and is equal to* a, *then* $\lim_\sigma (H_n\,\eta)(\sigma)$ *will exist and be equal to* a, *and conversely.*

From (17), (B), and successive applications of Lemma 3, we obtain the result:

If there exists $\lim_\sigma (C_n\,\eta)(\sigma) = a$, *then there exists* $\lim_\sigma (H_n\,\eta)(\sigma) = a\,(n)$.

From (17), (B), and successive applications of Lemma 5, we obtain the result:

If there exists $\lim_\sigma (H_n\,\eta)(\sigma) = a$, *then there exists* $\lim_\sigma (C_n\,\eta)(\sigma) = a\,(n)$.

Our theorem is therefore established.

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