THE INTERSECTION NUMBERS*

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1. In his first memoir on analysis situs† Poincaré defined a number $N(\Gamma_k, \Gamma_{n-k})$ which had previously been considered, at least in special cases, by Kronecker. With certain conventions as to sign this number represents the excess of the number of positive over the number of negative intersections of a k-dimensional circuit Γ_k with an $(n-k)$-dimensional circuit Γ_{n-k} when both are immersed in an n-dimensional oriented manifold. The purpose of the present paper is to show how to calculate this number when the manifold is defined combinatorially as a collection of cells and the circuits are composed of sets of these cells; and to show how the matrices which represent the intersectional relations between the k-circuits and the $(n-k)$-circuits depend on the matrices of orientation of the manifold. We also define certain modulo 2 intersection numbers and discuss the matrices connected with them.

The terminology and notations of the *Cambridge Colloquium Lectures on Analysis Situs* (New York, 1922) will be used without further explanation, and the references not otherwise indicated will be to that book.

2. Let a manifold M_n be given as the set of all points of a complex C_n. Let C'_n be a complex dual to C_n constructed as explained on page 88 by means of a complex $\overline{C}_n$ which is a regular subdivision both of C_n and of C'_n. Every k-cell a_j^k of C_n has a single point P_j^k (cf. p. 85) in common with a single $(n-k)$-cell of C'_n which is called b_j^{n-k}. Our first problem will be to assign a positive or negative sign to the intersection of a_j^k with b_j^{n-k}.

In order to do this, we suppose M_n to be oriented as explained in Chapter IV and that all cells, circuits, etc., are oriented. Moreover, in the regular complex $\overline{C}_n$, in which each i-cell is uniquely determined by its $i+1$ vertices, the orientation of the i-cell will be denoted by the order in which its vertices are written, and the following two conventions will be followed: (1) if $A_0 A_1 \cdots A_k$ denotes a given oriented k-cell $(k = 1, 2, \ldots, n)$ any even permutation of $A_0 A_1 \cdots A_k$ denotes the same oriented k-cell and any odd permutation denotes its negative; (2) the oriented $(k-1)$-cell $A_1 A_2 \cdots A_k$ is positively related to the oriented k-cell $A_0 A_1 \cdots A_k$.

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* Presented to the Society under a different title, April 24, 1920.

† Journal de l'École Polytechnique, ser. 2, vol. 1 (1895).

A simple argument by mathematical induction could, but will not here, be given to prove that these notations and conventions are consistent with themselves and with the definition of oriented cells.

3. The k-cell a_j^k of C_n is made up of a number of k-cells of $\bar{C}_n$ having P_j^k as their common vertex. Using the notation of page 86, let one of these be denoted by

$$P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k,$$

the points P being chosen, as is always possible, so that the orientation of this k-cell agrees with that of a_j^k. In like manner, b_j^{n-k} is made up of a number of $(n-k)$-cells of $\bar{C}_n$ having P_j^k as their common vertex, and we let any one of these be denoted by

$$P_j^k P_l^{k+1} \cdots P_s^n,$$

the points P being chosen this time so that the sense of the k-cell which they represent agrees with that of b_j^{n-k}. According as the oriented n-cell

$$P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$$

is positively or negatively oriented, we say that the intersection of a_j^k with b_j^{n-k} is positive or negative. In the first case we write

$$N(a_j^k, b_j^{n-k}) = 1$$

and in the second case

$$N(a_j^k, b_j^{n-k}) = -1.$$

From the definition of the points P it follows directly that this definition is independent of the particular cells of $\bar{C}_n$ which it employs. It also follows that the function N is such that

$$\begin{aligned} N(a_j^k, b_j^{n-k}) &= -N(-a_j^k, b_j^{n-k}) \\ &= -N(a_j^k, -b_j^{n-k}). \end{aligned} \tag{3.1}$$

Since the relation between C_n and C'_n is reciprocal, the definition given here determines the meaning of $N(b_j^{n-k}, a_j^k)$, and a simple count of transpositions in the notation gives the formula

$$N(b_j^{n-k}, a_j^k) = (-1)^{k(n-k)} N(a_j^k, b_j^{n-k}). \tag{3.2}$$

4. The cells of C_n and $\bar{C}_n$ are so oriented (cf. p. 123) that

$$E'_k = \bar{E}_{n-k+1},$$

which means that a_j^k is positively or negatively related to a_i^{k-1} according as b_i^{n-k+1} is positively or negatively related to b_j^{n-k}. Now the points P may be so chosen that $P_a^0 P_b^1 \cdots P_i^{k-1}$ represents an oriented cell on a_i^{k-1} and $P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$ represents an oriented cell on b_i^{n-k+1}. By the definition in § 2 above, the oriented cell $P_a^0 P_b^1 \cdots P_i^{k-1}$ is positively or negatively related to $P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k$, and therefore to a_i^k, according as $(-1)^k$ is positive or negative. On the other hand, $P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$ is positively related to $P_j^k P_l^{k+1} \cdots P_s^n$, and therefore to b_i^{n-k}. Hence if b_i^{n-k+1} is positively related to b_i^{n-k}, a_i^{k-1} is positively related to a_i^k and $(-1)^k N(a_i^{k-1}, b_i^{n-k+1})$ is positive or negative according as

$$P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k P_l^{k+1} \cdots P_s^n$$

is positively or negatively oriented. A similar result holds if b_i^{n-k+1} is negatively related to b_j^{n-k}. Hence

$$N(a_j^k, b_j^{n-k}) = (-1)^k N(a_i^{k-1}, b_i^{n-k+1}).$$

By repeated application of this formula we obtain

$$N(a_j^k, b_j^{n-k}) = (-1)^{k(k+1)/2} N(a_a^0, b_a^n).$$

But all the n-cells b_i^n are similarly oriented. Hence the value of $N(a_a^0, b_a^n)$ is the same for all zero cells a_a^0, and consequently the value of $N(a_j^k, b_j^{n-k})$ is independent of j. Hence if the notation is so chosen that b_i^n is positively oriented,*

$$\begin{aligned} N(a_i^0, b_i^n) &= 1, \\ N(a_i^1, b_i^{n-1}) &= -1, \\ N(a_i^2, b_i^{n-2}) &= 1, \\ N(a_i^3, b_i^{n-3}) &= -1, \\ &\ \vdots \end{aligned}$$

and all these equations are independent of i.

* Cf. Poincaré, Proceedings of the London Mathematical Society, vol. 32 (1900) p. 280.

5. An oriented complex Γ_k composed of the oriented k-cells $a_1^k\, a_2^k \cdots a_{\alpha_k}^k$ counted x^1 times, x^2 times, $\cdots$, x^{α_k} times, respectively, is represented by the notation

$$\Gamma_k = (x^1, x^2, \ldots, x^{\alpha_k}). \tag{5.1}$$

Let Γ'_{n-k} be an arbitrary oriented complex of C'_k, so that

$$\Gamma'_{n-k} = (y^1, y^2, \ldots, y^{\alpha_k}). \tag{5.2}$$

By the number of intersections of Γ_k with Γ'_{n-k}, having regard to sign, we shall mean the number $N(\Gamma_k, \Gamma'_{n-k})$ defined by means of the equation

$$\begin{aligned} N(\Gamma_k, \Gamma'_{n-k}) &= \sum_{j=1}^{\alpha_k} x^j\, y^j\, N(a_j^k\, b_j^{n-k}) \\ &= (-1)^{k(k+1)/2} \sum_{j=1}^{\alpha_k} x^j\, y^j. \end{aligned} \tag{5.3}$$

If we recall that there are no intersections of cells of Γ_k of dimensionality less than k with cells of Γ'_{n-k} and that no cell a_i^k intersects a cell b_j^{n-k} unless $i = j$, it is clear that this definition is in accordance with geometric intuition.

6. The last equation has as obvious corollaries the equations

$$N(\Gamma_k + \Delta_k, \Gamma'_{n-k}) = N(\Gamma_k, \Gamma'_{n-k}) + N(\Delta_k, \Gamma'_{n-k}), \tag{6.1}$$

$$N(\Gamma_k, \Gamma'_{n-k} + \Delta'_{n-k}) = N(\Gamma_k, \Gamma'_{n-k}) + N(\Gamma_k, \Delta'_{n-k}), \tag{6.2}$$

from which it follows that if $\Gamma_k^i\,(i = 1, 2, \ldots, \alpha_k)$ is any set of k-dimensional complexes on which all k-dimensional complexes of C_n are linearly dependent and $\Gamma_{n-k}^i\,(i = 1, 2, \ldots, \alpha_k)$ a set of $(n-k)$-dimensional complexes on which all $(n-k)$-dimensional complexes of C'_n are linearly dependent, then if

$$\Gamma_k = \sum_{i=1}^{\alpha_k} x_i\, \Gamma_{n-k}^i \tag{6.3}$$

and

$$\Gamma_{n-k} = \sum_{i=1}^{\alpha_k} y_j\, \Gamma_{n-k}^j \tag{6.4}$$

where the x's and y's are integers, then

$$N(\Gamma_k, \Gamma_{n-k}) = \sum_{i=1}^{\alpha_k} \sum_{j=1}^{\alpha_k} x_i y_j N(\Gamma_k^i, \Gamma_{n-k}^j). \tag{6.5}$$

Hence the intersection numbers of all k-dimensional complexes with all $(n-k)$-dimensional complexes depend on the matrix of numbers $N(\Gamma_k^i, \Gamma_{n-k}^j)$. By choosing the complexes Γ_k^i and Γ_{n-k}^j in the normal manner described in the Colloquium Lectures this matrix may be given a very simple form, which we shall determine in the next three sections.

7. As proved on page 116 of the Colloquium Lectures, a set of k-dimensional complexes upon which all the complexes formed from cells of C_n are linearly dependent may be so chosen as to consist of (1) a set of P_{k-1} non-bounding circuits which we shall denote by $\Gamma_k^i (i = 1, \cdots, P_k - 1)$, or in Poincaré's notation,

$$\Gamma_k^i \equiv 0; \tag{7.1}$$

(2) a set of τ_k circuits $\varDelta_k^i$ $(i = 1, \ldots, \tau_k)$ which satisfy the homologies

$$t_i^k \varDelta_k^i \sim 0 \tag{7.2}$$

in which t_i^k represents a k-dimensional coefficient of torsion; (3) a set of $r_{k+1} - \tau_k$ bounding circuits Θ_k^i

$$\Theta_k^i \sim 0; \tag{7.3}$$

and (4) and (5) two sets of complexes Φ_k^i and Ψ_k^i which are not circuits but satisfy the following congruences:

$$\Phi_k^i \equiv \Theta_{k-1}^i, \qquad 0 < i < r_k - \tau_{k-1}, \tag{7.4}$$

$$\Psi_k^i \equiv t_i^{k-1} \varDelta_{k-1}^i, \qquad 0 < i < \tau_{k-1}, \tag{7.5}$$

in which Θ_{k-1}^i and $\varDelta_{k-1}^i$ are defined by replacing k by $k-1$ in (7.3) and (7.2).

These relations are derived from the matrix equation

$$E_k \cdot D_k = C_{k-1} \cdot E_k^* \tag{7.6}$$

which arises in reducing (cf. p. 108) the orientation matrix E_k to normal form. The matrix E_k^* is one in which all elements are zero except the first r_k elements of the main diagonal. The first $r_k - \tau_{k-1}$ of the non-zero elements are 1 and the remaining τ_{k-1} are the coefficients of torsion of dimensionality $k-1$.

The first $r_k - \tau_{k-1}$ columns of D_k represent the complexes Φ_k^i, the next τ_{k-1} columns represent the complexes Ψ_k^i, the next $P_k - 1$ columns represent the circuits Γ_k^i, the next $r_{k+1} - \tau_k$ columns represent the circuits Θ_k^i, the next τ_k columns represent the circuits Δ_k^i. Thus, for example, if the jth column of D_k $(0 < j < r_k - \tau_{k-1})$ is $(x_{1j}, x_{2j}, \ldots, x_{a_k j})$ we have

$$\Phi_k^j = (x_{1j}, x_{2j}, \ldots, x_{a_k j}). \tag{7.7}$$

The columns of the matrix C_{k-1} are the same as the columns of D_{k-1} in a different order, and each complex represented by a column of D_k is bounded by the circuit represented by the corresponding column of the matrix $C_{k-1} \cdot E_k^*$. It is from this fact that the congruences (7.4) and (7.5) are derived. The fact that Γ_k^i, Δ_k^i, Θ_k^i are circuits is a consequence of the fact that all elements of E_k^* subsequent to the r_kth column are zero.

The homologies (7.2) and (7.3) arise by similar reasoning from the matrix equation

$$E_{k+1} \cdot D_{k+1} = C_k \cdot E_{k+1}^* \tag{7.8}$$

in which it is to be remembered that the columns of C_k are the same as those of D_k in a different order.

8. The $(n-k)$-dimensional complexes required in the formulas of § 6 may be determined by the same process as described in § 7, from the matrices of the dual complex C_n'. The matrices of the dual complex are related to those of C_n by the equation (cf. p. 123).

$$\bar{E}_{n-k} = E_{k+1}' \tag{8.1}$$

in which $\bar{E}_{n-k}$ is the matrix of the relations between $(n-k-1)$-cells and $(n-k)$-cells of C_n' and E_{k+1}' is the matrix obtained by interchanging rows and columns of E_{k+1}. The equation (7.8) gives the following:

$$C_k^{-1} \cdot E_{k+1} \cdot D_{k+1} = E_{k+1}^*,$$

$$D_{k+1}' \cdot E_{k+1}' \cdot C_k^{-1'} = E_{k+1}^*,$$

$$\bar{E}_{n-k} \cdot C_k^{-1'} = \bar{E}_{n-k}^* \cdot D_{k+1}^{-1'}.$$

The columns of $C_k^{-1'}$ determine a linearly independent set of complexes analogous to those determined by the columns of D_k. They are described by the following homologies and congruences, written in the order of the columns of $C_k^{-1'}$:

$$\bar{\Phi}^j_{n-k} \equiv \bar{\Theta}^j_{n-k-1}, \qquad 0<j \leqq r_{k+1}-\tau_k; \tag{8.2}$$

$$\bar{\Psi}^j_{n-k} \equiv t_j^{n-k-1}\bar{\Delta}^j_{n-k-1}, \qquad 0<j \leqq \tau_k; \tag{8.3}$$

$$\bar{\Gamma}^j_{n-k} \equiv 0, \qquad 0<j \leqq P_k-1; \tag{8.4}$$

$$\bar{\Theta}^j_{n-k} \sim 0, \qquad 0<j \leqq r_k-\tau_{k-1}; \tag{8.5}$$

$$t_j^{n-k}\Delta^j_{n-k} \sim 0, \qquad 0<j \leqq \tau_{k-1}. \tag{8.6}$$

9. Since the columns of D_k are the same as those of C_k in a different order, and the columns of $C_k^{-1'}$ are the same as the rows of C_k^{-1}, the matrix equation

$$C_k^{-1}\cdot C_k = 1 \tag{9.1}$$

implies the relations

$$\sum_{i=1}^{\alpha_k} x_{ij}\, x'_{ip} = \begin{cases} 1 \text{ if } j=p \\ 0 \text{ if } j \neq p \end{cases} \tag{9.2}$$

between the columns $(x_{1j}, x_{2j}, \ldots, x_{\alpha_k j})$ of D_k and the columns $(x'_{1p}, x'_{2p}, \ldots, x'_{\alpha_k p})$ of C_k^{-1}. But by (5.3) this implies that the intersection numbers of Γ^j_k, Δ^j_k, etc., with $\bar{\Gamma}^j_{n-k}$, $\bar{\Delta}^j_{n-k}$, etc., are zero except in the following α_k cases, written in the order of the columns of C_k^{-1}:

$$N(\Theta^j_k,\ \bar{\Phi}^j_{n-k}) = (-1)^{k(k+1)/2}, \qquad 0<j \leqq r_{k+1}-\tau_k; \tag{9.3}$$

$$N(\Delta^j_k,\ \bar{\Psi}^j_{n-k}) = (-1)^{k(k+1)/2}, \qquad 0<j \leqq \tau_k; \tag{9.4}$$

$$N(\Gamma^j_k,\ \bar{\Gamma}^j_{n-k}) = (-1)^{k(k+1)/2}, \qquad 0<j \leqq P_k-1; \tag{9.5}$$

$$N(\Phi^j_k,\ \Theta^j_{n-k}) = (-1)^{k(k+1)/2}, \qquad 0<j \leqq r_k-\tau_{k-1}; \tag{9.6}$$

$$N(\Psi^j_k,\ \Delta^j_{n-k}) = (-1)^{k(k+1)/2}, \qquad 0<j \leqq \tau_{k-1}. \tag{9.7}$$

Thus, each k-circuit Γ_k^i intersects the corresponding $(n-k)$-circuit once and intersects no other of the fundamental $(n-k)$-dimensional complexes. None of the other k-circuits (Λ_k^i or Θ_k^i) intersects any $(n-k)$-circuits, but each Θ_k^i intersects a complex Φ_{n-k}^i which is bounded by Θ_{n-k-1}^i; and each Λ_k^i intersects a complex Ψ_{n-k}^i which is bounded by Λ_{n-k-1}^i counted t_i^k times. Thus we may say that each Θ_k^i *links* one and only one Θ_{n-k-1}^i once and each Λ_k^i links one Λ_{n-k-1}^i in a manner which may be described as a fractional number of times, $\pm 1/\tau_i^k$. A further study of these linkages would carry us beyond the bounds of the present paper.

10. The matrix spoken of at the end of § 6 is now seen to consist entirely of zeros except for α_k elements whose value, 1 in every case, is given by equations (9.3), ..., (9.7). If we limit attention to circuits the only non-zero terms which remain are those given by the intersections of $\Gamma_k^1, \ldots, \Gamma_k^{P_1-1}$ with the corresponding non-bounding $(n-k)$-circuits. The matrix is therefore one which consists entirely of zeros except for the first $P_1 - 1$ terms of the main diagonal which are all 1's. For any k-circuit Γ_k of C_n we have

$$(10.1) \qquad \Gamma_k = \sum_{i=1}^{P_k-1} x_i \Gamma_k^i + \sum_{i=1}^{\tau_k} y_i \Lambda_k^i + \sum_{i=1}^{r_{k-1}-\tau_k} z_i \Theta_k^i$$

and for any $(n-k)$-circuit of C_n' we have

$$(10.2) \qquad \overline{\Gamma}_{n-k} = \sum_{i=1}^{P_k-1} x_i' \Gamma_{n-k}^i + \sum_{i=1}^{\tau_k} y_i' \Lambda_k^i + \sum_{i=1}^{r_{k+1}-\tau_k} z_i' \Theta_k^i.$$

When these expressions are substituted in (6.5) there results

$$(10.3) \qquad N(\Gamma_k, \overline{\Gamma}_{n-k}) = (-1)^{k(k+1)/2} \sum_{i=1}^{P_k-1} x_i x_i'.$$

Thus we have the theorem that *if*

$$(10.4) \qquad \Gamma_k \sim \sum_{i=1}^{P_k-1} x_i \Gamma_k^i + \sum_{i=1}^{\tau_k} y_i \Lambda_k^i$$

and

$$(10.5) \qquad \Gamma_{n-k} \sim \sum_{i=1}^{P_k-1} x_i' \overline{\Gamma}_{n-k}^i + \sum_{i=1}^{\tau_k} y_i' \overline{\Lambda}_{n-k}^i,$$

then the intersection number of Γ_k with Γ_{n-k} is given by (10.3).

This theorem has the corollary that

$$N(\Gamma_k, \Gamma_{n-k}) = 0$$

if and only if at least one of the homologies

$$p\Gamma_k \sim 0 \quad or \quad q\Gamma_{n-k} \sim 0$$

is satisfied for some integer value of p or q. In other words, the statement $p\Gamma_k \sim 0$ is equivalent to the equation

$$N(\Gamma_k, \Gamma_{n-k}) = 0$$

for the one circuit Γ_k and all circuits Γ_{n-k}.

From this it follows that if Γ_k' is any k-circuit composed of cells of C_n and such that

$$\Gamma_k \sim \Gamma_k'$$

then

$$N(\Gamma_k, \Gamma_{n-k}) = N(\Gamma_k', \Gamma_{n-k}).$$

11. Incidentally it may be remarked that (10.4) and (10.5) give rise to the following "homologies with division allowed":

$$\Gamma_k \sim \sum_{i=1}^{P_k-1} x_i \Gamma_k^i, \qquad \overline{\Gamma}_{n-k} \sim \sum_{i=1}^{P_k-1} y_i \overline{\Gamma}_{n-k}^i.$$

Whenever these homologies are satisfied the equation (10.3) is satisfied. As remarked by Poincaré, it is because the intersection numbers are more closely related to the homologies with division allowed than to the ordinary homologies that his attempt to prove the Euler theorem and the theorem about the duality of the Betti numbers by means of the intersection numbers was unsuccessful.

12. The fundamental sets of circuits which appear in the formulas of § 10 are chosen in a very special manner. A perfectly arbitrary fundamental set of k-circuits is however related to this special set by homologies

$$\bar{\Gamma}_k^i \sim \sum_{j=1}^{P_k-1} \alpha_j^i \Gamma_k^j + \sum_{j=P_k}^{P_k-1+\tau_k} \alpha_j^i \Delta_k^{j-P_k+1}$$

in which the $(P_k - 1 + \tau_k)$-rowed determinant $|\alpha_j^i| = \pm 1$. A general fundamental set of $(n-k)$-circuits $\bar{\Gamma}_{n-k}^i$ is related to the special set by an analogous set of homologies. Hence the matrix of the intersection numbers

$$N(\bar{\Gamma}_k^i, \bar{\Gamma}_{n-k}^j)$$

is one of $P_k - 1 + \tau_k$ rows and $P_k - 1 + \tau_{k-1}$ columns, of rank $P_k - 1$ and having all its invariant factors unity.

13. For some purposes it is desirable to introduce intersection numbers which do not distinguish between positive and negative intersections. The theory of these numbers is much simpler than that which we have been developing because all the determinations of algebraic sign in §§ 2, 3, 4, 5 can be omitted. We simply replace the definitions of § 3 by the agreement that

$$M(a_i^k, b_j^{n-k}) = 1 \text{ or } 0$$

according as a_i^k and b_j^{n-k} have a common point or not. Then the definition in § 5 is replaced by

$$M(\Gamma_k, \Gamma'_{n-k}) = \sum_{j=1}^{\alpha_k} x^j y^j,$$

the sum being taken modulo 2.

The determination of the intersection numbers of fundamental sets of k-circuits and $(n-k)$-circuits in §§ 7, 8, 9 is replaced by an analogous theory based on the matrices A_{k-1} and B_k which arise in the reduction of the incidence matrix H_k to normal form (cf. p. 79 and following pages). The result obtained is that there exist a set of k-circuits $\Gamma_k^1, \Gamma_k^2, \ldots, \Gamma_k^{R_k-1}$ and a set of $(n-k)$-circuits $\Gamma_{n-k}^1, \Gamma_{n-k}^2, \ldots, \Gamma_{n-k}^{R_k-1}$ such that

$$M(\Gamma_k^i, \Gamma_{n-k}^j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases};$$

and if

$$\Gamma_k = \sum_{i=1}^{R_k - 1} x_i \Gamma_k^i,$$

$$\Gamma_{n-k} = \sum_{i=1}^{R_k - 1} y_i \Gamma_{n-k}^i,$$

then

$$M(\Gamma_k, \Gamma_{n-k}) = \sum_{i=1}^{R_k - 1} x_i y_i \pmod 2.$$

It should be observed that these formulas cannot be obtained by reducing the formulas of § 10, modulo 2, because the formulas of the present section take account of non-orientable circuits which do not enter into the theory of oriented intersections.

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