

A NEW SET OF POSTULATES FOR BETWEENNESS, WITH PROOF OF COMPLETE INDEPENDENCE*

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INTRODUCTION

The paper on betweenness published by E. V. Huntington and J. R. Kline in 1917 started with a basic list of twelve postulates:

A, B, C, D, 1, 2, 3, 4, 5, 6, 7, 8,

from which eleven sets of independent postulates were selected, as follows:

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| (1) A, B, C, D, 1, 2; | (5) A, B, C, D, 1, 8; | (9) A, B, C, D, 3, 4, 6; |
| (2) A, B, C, D, 1, 5; | (6) A, B, C, D, 2, 4; | (10) A, B, C, D, 3, 4, 7; |
| (3) A, B, C, D, 1, 6; | (7) A, B, C, D, 2, 5; | (11) A, B, C, D, 3, 4, 8. |
| (4) A, B, C, D, 1, 7; | (8) A, B, C, D, 3, 5; | |

Eight of these sets contain six postulates each, and three contain seven postulates each.†

In the present paper a new postulate, called postulate 9, is added to the basic list. This new postulate leads to a twelfth set of independent postulates:

(12) A, B, C, D, 9,

in which the number of postulates is reduced to five. Moreover, the new postulate 9 itself is easier to remember and more convenient to handle than any of the other postulates 1—8.

The addition of this new postulate makes desirable an extension of the discussion of the earlier paper so as to include all thirteen of the basic postulates; and this extension has been made in the present paper.

Finally, the postulates of the new set (12) are shown to be completely independent in the sense of E. H. Moore. (In regard to the other sets, a

* Presented to the Society, December 27, 1923.

† E. V. Huntington and J. R. Kline, *Sets of independent postulates for betweenness*, these Transactions, vol. 18 (1917), pp. 301–325.

recent paper by Mr. W. E. Van de Walle* has shown that sets (1)—(10) are completely independent, while set (11) is not.)

It is hoped that the material now available on the simple relation of "betweenness," including as it does, 12 sets of postulates with the "complete existential theory" of each set, and no less than 200 demonstrated theorems (116 on deducibility and 84 on non-deducibility), may prove of special interest to students of logic, since it provides the most elaborate known example of an abstract deductive theory.

THE BASIC LIST OF THIRTEEN POSTULATES

The universe of discourse consists of all systems K, R , where K is a class of elements, A, B, C, \dots , and $R(ABC)$ is a triadic relation; among these systems (K, R) we designate as "betweenness" systems those that satisfy the following thirteen conditions, or postulates.

POSTULATE A. $ABC \supset CBA$.

(That is, if ABC is true, then CBA is true.)

POSTULATE B. $A \neq B. B \neq C. C \neq A \supset : BAC \sim CAB \sim ABC \sim CBA \sim ACB \sim BCA$.

(That is, if A, B, C are distinct, then *at least one* of the six possible permutations will form a true triad.)

POSTULATE C. $A \neq X. X \neq Y. Y \neq A \supset : AXY. AYX. = .0$.

(That is, if A, X, Y are distinct, then we cannot have AXY and AYX both true at the same time.)

POSTULATE D. $ABC \supset : A \neq B. B \neq C. C \neq A$.

(That is, if ABC is true, then the elements A, B , and C are distinct.)

POSTULATES 1-8. If A, B, X, Y , are distinct, then:

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| 1. $XAB. ABY. \supset . XAY.$ | |
| 2. $XAB. AYB. \supset . XAY.$ | |
| 3. $XAB. AYB \supset . XYB.$ | |
| 4. $AXB. AYB. \supset . AXY \sim AYX.$ | |
| 5. $AXB. AYB. \supset . AXY \sim YXB.$ | |
| 6. $XAB. YAB. \supset . XYA \sim YXB.$ | |
| 7. $XAB. YAB. \supset . XYA \sim YXA.$ | |
| 8. $XAB. YAB. \supset . XYA \sim YXB.$ | |

* W. E. Van de Walle, *On the complete independence of the postulates for betweenness*, in the present number of these Transactions, pp. 249-256.

POSTULATE 9. If A, B, C, X are distinct, then $ABC \cdot X \supset \cdot ABX \sim XBC$.

The new postulate 9 may be read as follows: If ABC is true, and if X is any fourth element distinct from A and B and C , then X must lie either on the right of the middle element (giving ABX), or else on the left of the middle element (giving XBC).

In regard to certain peculiarities of postulates 5 and 8, see under Theorem 5k, below.

THEOREMS ON DEDUCIBILITY

Besides the 71 theorems on deducibility which were stated and proved in the earlier paper, there are found to be 45 new theorems involving the new postulate 9. The proofs of these new theorems are given below, and the complete list of 116 theorems is set forth in Table I'.

The following proofs are supplementary to those given in the earlier paper. In each proof, the number of times that any postulate is used is indicated by an exponent.

THEOREM 1e. *Proof of 1 from A, C, 9.*

To prove: $XAB \cdot ABY \cdot \supset \cdot XAY$. By A, $ABY \cdot \supset \cdot YBA$. By 9, $XAB \cdot Y \cdot \supset \cdot XAY \sim YAB$. But YAB conflicts with YBA , by C. Hence XAY .

THEOREM 2j. *Proof of 2 from A, C, 9.*

To prove: $XAB \cdot AYB \cdot \supset \cdot XAY$. By A, $AYB \cdot \supset \cdot BYA$. By 9, $XAB \cdot Y \cdot \supset \cdot XAY \sim YAB$. But if YAB , then by A, BAY , which conflicts with BYA , by C. Hence XAY .

THEOREM 3f. *Proof of 3 from A, C², 9².*

To prove: $XAB \cdot AYB \cdot \supset \cdot XYB$. Suppose XYB is false. First, by 9, $AYB \cdot X \cdot \supset \cdot AYX \sim XYB$; hence AYX , whence by A, XYA . Second, by 9, $XAB \cdot Y \cdot \supset \cdot XAY \sim YAB$; but YAB conflicts with AYB , by A and C; hence XAY . But thirdly, XYA and XAY conflict with each other, by C. Therefore XYB must be true.

THEOREM 3g. *Proof of 3 from A, 1², 9².*

To prove: $XAB \cdot AYB \cdot \supset \cdot XYB$. Suppose XYB is false. By 9, $AYB \cdot X \cdot \supset \cdot AYX \sim XYB$; hence AYX , whence, by A, XYA . By 9, $XAB \cdot Y \cdot \supset \cdot XAY \sim YAB$.

Case 1. If YAB , then by 1, $XYA \cdot YAB \cdot \supset \cdot XYB$.

Case 2. If XAY , then by A and 1, $BYA \cdot YAX \cdot \supset \cdot BYX$, whence, by A, XYB .

THEOREM 3h. *Proof of 3 from A, B, 2⁴, 9.*

To prove: $XAB \cdot AYB \cdot \supset \cdot XYB$. Suppose XYB is false. Then, by B and A, $YXB \sim XBY$.

Case 1. If YXB , then by 2, $YXB \cdot XAB \cdot \supset \cdot YXA$; hence, by 2 and A, $BYA \cdot YXA \cdot \supset \cdot BYX$, whence, by A, XYB .

Case 2. If XBY , then by 2 and A, $YBX \cdot BAX \cdot \supset \cdot YBA$.

Now by 9, $AYB \cdot X \cdot \supset \cdot AYX \sim XYB$; but XYB is false; hence AYX ; whence, by A, XYA . Then by 2, $XYA \cdot YBA \cdot \supset \cdot XYB$.

THEOREM 3i. *Proof of 3 from A, 2³, 6, 9.*

To prove: $XAB \cdot AYB \cdot \supset \cdot XYB$. Suppose XYB is false. By 9, $AYB \cdot X \cdot \supset \cdot AYX \sim XYB$; hence AYX . Then by A and 2, $BAX \cdot AYX \cdot \supset \cdot BAY$, whence, by A, YAB . Then by 6, $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$; but XYB is false; hence YXB . Then by 2, $YXB \cdot XAB \cdot \supset \cdot YXA$. Hence, by A and 2, $BYA \cdot YXA \cdot \supset \cdot BYX$, whence, by A, XYB .

THEOREM 3j. *Proof of 3 from A, 2⁴, 7, 9.*

To prove: $XAB \cdot AYB \cdot \supset \cdot XYB$. Suppose XYB is false. By 9, $AYB \cdot X \cdot \supset \cdot AYX \sim XYB$; hence AYX , whence, by A, XYA . Then by A and 7, $BYA \cdot XYA \cdot \supset \cdot BXY \sim XBY$.

Case 1. If BXY , then by A and 2, $YXB \cdot XAB \cdot \supset \cdot YXA$. Then by A and 2, $BYA \cdot YXA \cdot \supset \cdot BYX$, whence by A, XYB .

Case 2. If XBY , then by A and 2, $YBX \cdot BAX \cdot \supset \cdot YBA$. Then by 2, $XYA \cdot YBA \cdot \supset \cdot XYB$.

THEOREM 3k. *Proof of 3 from A, 2², 8, 9.*

To prove: $XAB \cdot AYB \cdot \supset \cdot XYB$. Suppose XYB is false. By 9, $AYB \cdot X \cdot \supset \cdot AYX \sim XYB$; hence AYX , whence by A, XYA . By 2, $XAB \cdot AYB \cdot \supset \cdot XAY$, whence by A, YAX . Then by A and 8, $YAX \cdot BAX \cdot \supset \cdot YBA \sim BYX$, whence by A, $YBA \sim XYB$; but XYB is false; hence YBA . Then by 2, $XYA \cdot YBA \cdot \supset \cdot XYB$.

THEOREM 4k. *Proof of 4 from A, C, 9².*

To prove: $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$. Suppose both AXY and AYX are false.

By 9, $\neg XB \cdot Y \cdot \supset \cdot AXY \sim YXB$; hence YXB .

By 9, $AYB \cdot X \cdot \supset \cdot AYX \sim XYB$; hence XYB .

But YXB and XYB conflict with each other, by A and C. Hence $AXY \sim AYX$.

THEOREM 4l. *Proof of 4 from A, 1², 7², 9.*

To prove: $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$. Suppose both AXY and AYX are false. By 9, $AXB \cdot Y \cdot \supset \cdot AXY \sim YXB$; hence YXB .

Then by 7, $AXB \cdot YXB \cdot \supset \cdot AYZ \sim YAX$; hence YAX .

By 1, $YAX \cdot AXB \cdot \supset \cdot YAB$.

By 1 and A, $XAY \cdot AYB \cdot \supset \cdot XAB$.

Then by 7, $YAB \cdot XAB \cdot \supset \cdot YXA \sim XYA$. Hence by A, $AXY \sim AYZ$.

THEOREM 4m. *Proof of 4 from A, 3², 7², 9².*

To prove: $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYZ$. Suppose both AXY and AYZ are false.

By 9, $AXB \cdot Y \cdot \supset \cdot AXY \sim YXB$. Hence YXB , whence, by A, BXY .

By 9, $AYB \cdot X \cdot \supset \cdot AYZ \sim XYB$. Hence XYB , whence, by A, BYX .

Then by 7, $AXB \cdot YXB \cdot \supset \cdot AYZ \sim YAX$; hence YAX , whence, by A, XAY .

By 3, $BYX \cdot YAX \cdot \supset \cdot BAX$; and by 3, $BXY \cdot XAY \cdot \supset \cdot BAY$.

Then by 7 and A, $YAB \cdot XAB \cdot \supset \cdot YXA \sim XYA$, whence, by A, $AXY \sim AYZ$.

THEOREM 4n. *Proof of 4 from A, 7, 8², 9².*

To prove: $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYZ$. Suppose both AXY and AYZ are false.

By 9, $AXB \cdot Y \cdot \supset \cdot AXY \sim YXB$; hence YXB . By 9, $AYB \cdot X \cdot \supset \cdot AYZ \sim XYB$; hence XYB .

Then by 8, $AXB \cdot YXB \cdot \supset \cdot AYZ \sim YAB$; hence YAB .

Also, by 8, $AYB \cdot XYB \cdot \supset \cdot AXY \sim XAB$; hence XAB .

Then by 7, $YAB \cdot XAB \cdot \supset \cdot YXA \sim XYA$, whence, by A, $AXY \sim AYZ$.

THEOREM 4o. *Proof of 4 from C², 7³, 8², 9².*

To prove: $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYZ$. Suppose both AXY and AYZ are false.

By 9, $AXB \cdot Y \cdot \supset \cdot AXY \sim YXB$; hence YXB .

By 9, $AYB \cdot X \cdot \supset \cdot AYZ \sim XYB$; hence XYB .

Then by 7, $AXB \cdot YXB \cdot \supset \cdot AYZ \sim YAX$; hence YAX . And by 7, $AYB \cdot XYB \cdot \supset \cdot AXY \sim XAY$; hence XAY .

Also by 8, $AXB \cdot YXB \cdot \supset \cdot AYZ \sim YAB$; hence YAB and by 8, $AYB \cdot XYB \cdot \supset \cdot AXY \sim XAB$; hence XAB .

Then by 7, $YAB \cdot XAB \cdot \supset \cdot YXA \sim XYA$.

But YXA conflicts with YAX , by C, and XYA conflicts with XAY , by C. Therefore $AXY \sim AYZ$.

THEOREM 4p. *Proof of 4 from 2², 9².*

To prove: $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYZ$. Suppose both AXY and AYZ are false.

By 9, $AXB \cdot Y \cdot \supset \cdot AXY \sim YXB$; hence YXB .

By 9, $AYB \cdot X \cdot \supset \cdot AYZ \sim XYB$; hence XYB .

Then by 2, $AYB \cdot YXB \supset AYX$, and by 2, $AXB \cdot XYB \supset AXY$.
Therefore $AXY \sim AYX$.

THEOREM 5k. *Proof of 5 from 9.*

To prove: $AXB \cdot AYB \supset AXY \sim YXB$.

By 9, $AXB \cdot Y \supset AXY \sim YXB$; which was to be proved. It will be observed that only the first part of the hypothesis is used in the proof. Postulate 9 is "stronger" than postulate 5. By interchanging X and Y , postulate 5 may also be written in the form

$$AYB \cdot AXB \supset AYX \sim XYB;$$

which may be proved as follows: By 9,

$$AYB \cdot X \supset AYX \sim XYB;$$

which was to be proved. Here again, only the first part of the hypothesis is used in the proof.

Furthermore, since " AXB and AYB " is logically equivalent to " AYB and AXB ", postulate 5 may be written in either of the following forms:

$$\begin{aligned} &AXB \cdot AYB \supset AYX \sim XYB. \\ &AYB \cdot AXB \supset AXY \sim YXB. \end{aligned}$$

It is interesting to notice, however, that no one of these four forms is a significant statement, unless one part of the hypothesis is recognized specifically as the "first part" and the other as the "second part" — a distinction which, strictly speaking, introduces a foreign element into the statement of the proposition.

In order to avoid the necessity of making this arbitrary distinction between the "first" and the "second" term of a pair connected by a simple "and", we may restate postulate 5 in the following less objectionable form:

$$5'. \quad AXB \cdot AYB : \supset : (AXY \sim YXB) \cdot (AYX \sim XYB).$$

This should not be regarded as merely a combination of two of the separate statements mentioned above, since, without employing the distinction between "first" and "second", we cannot tell which part of the hypothesis is supposed to be paired with which part of the conclusion. It is only when the statement (5') is taken as a whole that it can be translated into significant words, without using the distinction between the "first" and "second" parts of the simple conjunction which forms the hypothesis.

Thus, 5' may be read as follows: "The two triads in the hypothesis contain the same initial element, A , and the same terminal element, B , but different middle elements, X and Y (which we may call the "odd elements"). The conclusion also consists of two parts. One part says that at least one of the following triads is true:

(A) (one odd) (the other odd) or (the other odd) (the one odd) (B); the other part says that at least one of the following is true:

(A) (the other odd) (the one odd) or (the one odd) (the other odd) (B)."

Now neither of these parts alone gives us any definite information unless we are able to recognize the "one" as X and the "other" as Y (or vice versa); but the two parts together give an unequivocal conclusion whether the "one" = X and the "other" = Y , or the "one" = Y and the "other" = X .

Precisely the same remarks apply to postulate 8, which may be re-stated more strictly as follows:

$$8'. \quad XAB . YAB : \supset : (XYA \sim YXB) . (YXA \sim XYB).$$

Fortunately, these logical refinements do not affect the essential reasoning, provided the precaution already stated in the footnote on page 318 of the earlier paper is observed.

THEOREM 6k. *Proof of 6 from A, B, C², 9.*

To prove: $XAY . YAB . \supset . XYB \sim YXB$. By B and A, $XYB \sim YXB \sim XBY$. Suppose XBY . Then by 9, $XBY . A . \supset . XBA \sim ABY$. But XBA conflicts with XAB , by C; and $ABY . \supset . YBA$, by A, which conflicts with YAB , by C. Hence $XYB \sim YXB$.

THEOREM 6l. *Proof of 6 from A, C, 7, 9.*

To prove: $XAB . YAB . \supset . XYB \sim YXB$.

By 7, $XAB . YAB . \supset . XYA \sim YXA$.

Case 1. If XYA , then by 9, $XYA . B . \supset . XYB \sim BYA$. But BYA conflicts with YAB , by C and A. Hence, in Case 1, XYB .

Case 2. If YXA , then by 9, $YXA . B . \supset . YXB \sim BXA$. But BXA conflicts with XAB , by C and A. Hence, in Case 2, YXB .

THEOREM 6m. *Proof of 6 from A, 2⁴, 7³, 9².*

To prove: $XAB . YAB . \supset . XYB \sim YXB$. Suppose both XYB and YXB (and hence, by A, also BYX and BXY) are false.

By 7, $XAB . YAB . \supset . XYA \sim YXA$.

Case 1. If XYA , then by 9, $XYA . B . \supset . XYB \sim BYA$. But XYB is false. Hence BYA , and by A, AYB . Then by 7, $BYA . XYA . \supset . BXY \sim XBY$.

But BXY is false. Hence XYB . Then by 2 and A, $YBX . BAX . \supset . YBA$. Hence by 2, $XYA . YBA . \supset . XYB$.

Case 2. If YXA , then by 9, $YXA . B . \supset . YXB . BXA$. But YXB is false. Hence BXA . Then by 7 and A, $BXA . YXA . \supset . BYX \sim YBX$. But BYX is false. Hence YBX . Then by 2 and A, $XYB . BAY . \supset . XBA$. Hence by 2, $YXA . XBA . \supset . YXB$.

Therefore $XYB \sim YXB$.

THEOREM 6n. *Proof of 6 from A, 2, 8³, 9.*

To prove: $XAB . YAB . \supset . XYB \sim YXB$. Suppose both XYB and YXB are false. Then, by A, both BYX and BXY are false.

By 8, $XAB . YAB . \supset . XYA \sim YXB$; hence XYA .

By 8, $YAB . XAB . \supset . YXA \sim XYB$; hence YXA .

By 9, $XYA . B . \supset . XYB \sim BYA$; hence BYA .

By 8, $BYA . XYA . \supset . BXY \sim XBA$; hence XBA .

By 2, $YXA . XBA . \supset . YXB$ contrary to supposition.

THEOREM 6o. *Proof of 6 from A, 4, 8², 9².*

To prove: $XAB . YAB . \supset . XYB \sim YXB$. Suppose both XYB and YXB are false.

By 8, $XAB . YAB . \supset . XYA \sim YXB$; hence XYA , and by A, AYX .

By 8, $YAB . XAB . \supset . YXA \sim XYB$; hence YXA , and by A, AXY .

By 9, $XYA . B . \supset . XYB \sim BYA$; hence BYA .

By 9, $YXA . B . \supset . YXB \sim BXA$; hence BXA .

Then by 4, $BXA . BYA . \supset . BXY \sim BYX$. Hence, by A, $XYB \sim BXY$.

THEOREM 6p. *Proof of 6 from A, 7, 8⁴, 9².*

To prove: $XAB . YAB . \supset . XYB \sim YXB$. Suppose both XYB and YXB are false. Then by A, BYX and BXY are false.

By 8, $XAB . YAB . \supset . XYA \sim YXB$; hence XYA .

By 8, $YAB . XAB . \supset . YXA \sim XYB$; hence YXA .

By 9, $XYA . B . \supset . XYB \sim BYA$; hence BYA .

By 9, $YXA . B . \supset . YXB \sim BXA$; hence BXA .

By 8, $BYA . XYA . \supset . BXY \sim XBA$; hence XBA .

By 8, $BXA . YXA . \supset . BYX \sim YBA$; hence YBA .

Then by 7, $XBA . YBA . \supset . XYB \sim YXB$.

THEOREM 6q. *Proof of 6 from C³, 8⁴, 9².*

To prove: $XAB . YAB . \supset . XYB \sim YXB$. Suppose both XYB and YXB are false.

By 8, $XAB . YAB . \supset . XYA \sim YXB$; hence XYA .

By 8, $YAB \cdot XAB \cdot \supset \cdot YXA \sim XYB$; hence YXA .

By 9, $XYA \cdot B \cdot \supset \cdot XYB \sim BYA$; hence BYA .

By 9, $YXA \cdot B \cdot \supset \cdot YXB \sim BXA$; hence BXA .

By 8, $BYA \cdot XYA \cdot \supset \cdot BXY \sim XBA$. But XBA conflicts with XAB by C. Hence BXY .

By 8, $BXA \cdot YXA \cdot \supset \cdot BYX \sim YBA$. But YBA conflicts with YAB by C. Hence BYX .

Now BXY and BYX conflict with each other, by C.

Therefore $XYB \sim YXB$.

THEOREM 7k. *Proof of 7 from A, B, C⁴, 9³.*

To prove: $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$.

By B, $XYB \sim YXB \sim XBY$.

Case 1. If XYB , then by 9, $XYB \cdot A \cdot \supset \cdot XYA \sim AYB$. But AYB conflicts with BAY , by C and A. Hence in Case 1, XYA .

Case 2. If YXB , then by 9, $YXB \cdot A \cdot \supset \cdot YXA \sim AXB$. But AXB conflicts with XAB , by C and A. Hence in Case 2, YXA .

Case 3. Suppose XBY . Then by 9, $XBY \cdot A \cdot \supset \cdot XBA \sim ABY$. But XBA conflicts with XAB , by C; and ABY conflicts with YAB , by C and A. Hence Case 3 is impossible.

Therefore $XYA \sim YXA$.

THEOREM 7l. *Proof of 7 from A, C², 6, 9².*

To prove: $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$.

By 6, $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$.

Case 1. If XYB , then by 9, $XYB \cdot A \cdot \supset \cdot XYA \sim AYB$. But AYB conflicts with YAB , by C and A. Hence, in Case 1, XYA .

Case 2. If YXB , then by 9, $YXB \cdot A \cdot \supset \cdot YXA \sim AXB$. But AXB conflicts with XAB , by C and A. Hence, in Case 2, YXA .

Therefore $XYA \sim YXA$.

THEOREM 7m. *Proof of 7 from A, 4, 8², 9².*

To prove: $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$. Suppose both XYA and YXA are false.

By 8, $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$; hence YXB .

By 8, $YAB \cdot XAB \cdot \supset \cdot YXA \sim XYB$; hence XYB .

Then by 9, $YXB \cdot A \cdot \supset \cdot YXA \sim AXB$; hence AXB .

And by 9, $XYB \cdot A \cdot \supset \cdot XYA \sim AYB$; hence AYB .

Then by 4, $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$. Hence by A, $XYA \sim YXA$.

THEOREM 7n. *Proof of 7 from A, 1², 4³, 6³, 9².*

To prove: $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$. Suppose both XYA and YXA are false. Then by A, AYX and AXY are false.

By 6, $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$.

Case 1. If XYB is true and YXB false, then by 4, $XAB \cdot XYB \cdot \supset \cdot XAY \sim XYA$; hence XAY . Then by 6 and A, $XAY \cdot BAY \cdot \supset \cdot XBY \sim BXY$, whence, by A, $YBX \sim YXB$. But YXB is false, hence YBX . Now by 9, $XYB \cdot A \cdot \supset \cdot XYA \sim AYB$; hence AYB . Then by 1, $AYB \cdot YBX \cdot \supset \cdot AYX$, which is false.

Case 2. If YXB is true and XYB false, then by 4, $YXB \cdot YAB \cdot \supset \cdot YXA \sim YAX$; hence YAX . Then by 6 and A, $YAX \cdot BAX \cdot \supset \cdot YBX \sim BYX$, whence, by A, $XBY \sim XYB$. But XYB is false; hence XBY . By 9, $YXB \cdot A \cdot \supset \cdot YXA \sim AXB$; hence AXB . Then by 1, $AXB \cdot XBY \cdot \supset \cdot AXY$, which is false.

Case 3. If XYB and YXB are both true, then by 9, $XYB \cdot A \cdot \supset \cdot XYA \sim AYB$; hence AYB . And by 9, $YXB \cdot A \cdot \supset \cdot YXA \sim AXB$; hence AXB . Then by 4, $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$. Therefore by A, $XYA \sim YXA$.

THEOREM 8n. *Proof of 8 from A, B, C³, 9².*

To prove: $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$. By B, $XBY \sim XYB \sim YXB$.

Case 1. Suppose XBY ; then by 9, $XBY \cdot A \cdot \supset \cdot XBA \sim ABY$. But XBA conflicts with XAB , by C; and ABY conflicts with YAB , by A and C.

Case 2. If XYB , then by 9, $XYB \cdot A \cdot \supset \cdot XYA \sim AYB$. But AYB conflicts with BAY , by A and C.

Therefore $XYA \sim YXB$.

THEOREM 8o. *Proof of 8 from A, B, 1², 6², 9.*

To prove: $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$. Suppose both XYA and YXB are false. By 6, $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$; hence XYB . Then by 9, $XYB \cdot A \cdot \supset \cdot XYA \sim AYB$; hence AYB .

By B and A, $YXA \sim XAY \sim XYA$; hence YXA or XAY . But if YXA , then by 1, $YXA \cdot XAB \cdot \supset \cdot YXB$, which is false; hence XAY . Then by 6 and A, $XAY \cdot BAY \cdot \supset \cdot XBY \sim BXY$. But if BXY , then by A, YXB , which is false; hence XBY , whence, by A, YBX . Then by 1, $AYB \cdot YBX \cdot \supset \cdot AYX$, whence, by A, XYA , which is false.

Therefore $XYA \sim YXB$.

THEOREM 8p. *Proof of 8 from A, C, 6, 9.*

To prove: $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$. Suppose both XYA and YXB are false. By 6, $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$; hence XYB . Then by 9, $XYB \cdot A \cdot \supset \cdot XYA \sim AYB$. But AYB conflicts with YAB by C and A; and XYA is false.

Therefore $XYA \sim YXB$.

THEOREM 8q. *Proof of 8 from A, C, 7, 9.*

To prove: $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$. We may vary the method of proof, as follows: By 7, $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$. If XYA , the

theorem is established. Suppose YXA ; then by 9, $YXA.B \supset YXB \sim BXA$. If YXB , the theorem is established. Suppose BXA . By A, $XAB \supset BAX$, which conflicts with BXA , by C. Therefore the theorem must be true.

THEOREM 8r. *Proof of 8 from A, 1, 4, 6², 9.*

To prove: $XAB.YAB \supset XYA \sim YXB$. Suppose both XYA and YXB are false. By 6, $XAB.YAB \supset XYB \sim YXB$; hence XYB . Then by 4, $XYB.XAB \supset XYA \sim XAY$; hence XAY ; and by 9, $XYB.A \supset XYA \sim AYB$; hence AYB .

By A and 6, $BAY.XAY \supset BXY \sim XBY$, whence, by A, $YXB \sim YBX$; hence YBX . Then by 1, $AYB.YBX \supset AYX$, whence, by A, XYA , which is false.

Therefore $XYA \sim YXB$.

THEOREM 8s. *Proof of 8 from A, 2³, 7², 9.*

To prove: $XAB.YAB \supset XYA \sim YXB$. Suppose both XYA and YXB are false. By 7, $XAB.YAB \supset XYA \sim YXA$; hence YXA . Then by 9, $YXA.B \supset YXB \sim BXA$; hence BXA . Then by 7, $YXA.BXA \supset YBX \sim BYX$.

Case 1. If YBX , then by A and 2, $XBY.BAY \supset XBA$. Then by 2, $YXA.XBA \supset YXB$.

Case 2. If BYX , then by A and 2, $XYB.YAB \supset XYA$.

Therefore $XYA \sim YXB$.

THEOREM 9a. *Proof of 9 from A, B, 1, 2.*

To prove: $ABC.X \supset ABX \sim XBC$. By B and A we have $XBC \sim BCX \sim BXC$. If BCX , then $ABC.BCX \supset ABX$, by 1. If BXC , then $ABC.BXC \supset ABX$, by 2. Hence $ABX \sim XBC$.

THEOREM 9b. *Proof of 9 from A, B³, C², 1⁴, 5.*

To prove: $ABC.X \supset ABX \sim XBC$. Suppose both ABX and XBC are false. Then, by B and A, $AXB \sim BAX$ and by B and A, $BXC \sim BCX$.

Case 1. If BCX , then by 1, $ABC.BCX \supset ABX$.

Case 2. If BAX , then by 1 and A, $CBA.BAX \supset CBX$, whence, by A, XBC .

Case 3. If BXC and AXB , use B again: $ACX \sim CAX \sim CXA$.

Suppose ACX . Then by 1 and A, $ACX.CXB \supset ACB$, contrary to ABC , by C.

Suppose CAX . Then by 1, $CAX.AXB \supset CAB$, contrary to ABC , by C and A.

Suppose CXA . Then by 5 and A, $CBA.CXA \supset CBX \sim XBA$. Hence, by A, $ABX \sim XBC$.

THEOREM 9c. *Proof of 9 from A, B², C⁴, 1², 6².*

To prove: $ABC.X \supset .ABX \sim XBC$.

Suppose both ABX and XBC are false. Then, by B and A, $AXB \sim BAX$ and, by B and A, $BXC \sim BCX$.

Case 1. If BAX and BXC , then by A and 1, $XAB.ABC \supset .XAC$, whence by A and 6, $BAX.CAX \supset .BCX \sim CBX$; but both BCX and CBX conflict with BXC , by C and A.

Case 2. If AXB and BXC , then, by A and 6, $AXB.CXB \supset .ACB \sim CAB$, both of which conflict with ABC , by C and A.

Case 3. If BCX , then, by 1, $ABC.BCX \supset .ABX$, which is false. Therefore $ABX \sim XBC$.

THEOREM 9d. *Proof of 9 from A, B², C², 1⁴, 7.*

To prove: $ABC.X \supset .ABX \sim XBC$. Suppose both ABX and XBC are false. Then, by B and A, $BAX \sim AXB$, and, by B and A, $BCX \sim CXB$.

Case 1. If BAX , then by 1 and A, $CBA.BAX \supset .CBX$, whence XBC , by A.

Case 2. If BCX , then by 1, $ABC.BCX \supset .ABX$.

Case 3. If CXB and AXB , then by 7, $CXB.AXB \supset .CAX \sim ACX$. But if CAX , then by 1, $CAX.AXB \supset .CAB$, and if ACX , then by 1, $ACX.CXB \supset .ACB$, both of which conflict with ABC , by C.

Therefore $ABX \sim XBC$.

THEOREM 9e. *Proof of 9 from A, B², C³, 1², 8².*

To prove: $ABC.X \supset .ABX \sim XBC$. Suppose both ABX and XBC were false. Then, by B and A, $AXB \sim BAX$, and, by B and A, $CXB \sim BCX$.

Case 1. If BCX , then, by 1, $ABC.BCX \supset .ABX$.

Case 2. If BAX , then, by 1 and A, $CBA.BAX \supset .CBX$, whence, by A, XBC .

Case 3. Suppose AXB and CXB , then, by 8, $AXB.CXB \supset .ACX \sim CAB$ and, by 8, $CXB.AXB \supset .CAX \sim ACB$. But ACX and CAX conflict with each other, by A and C; CAB conflicts with ABC , by A and C; and ACB conflicts with ABC , by C. Hence Case 3 is impossible.

Therefore $ABX \sim XBC$.

THEOREM 9f. *Proof of 9 from A, B², C², 2², 4.*

To prove: $ABC.X \supset .ABX \sim XBC$. Suppose both ABX and XBC were false. Then, by B and A, $BXA \sim BAX$, and, by B and A, $BXC \sim BCX$.

Case 1. If BXC , then, by 2, $ABC.BXC \supset .ABX$.

Case 2. If BXA , then, by 2 and A, $CBA.BXA \supset .CBX$, whence, by A, XBC .

Case 3. If BAX and BCX , then, by 4, $BAX \cdot BCX \cdot \supset \cdot BAC \sim BCA$, both of which conflict with ABC , by C.

Hence $ABX \sim XBC$.

THEOREM 9g. *Proof of 9 from A, B², C³, 2², 5².*

To prove: $ABC \cdot X \cdot \supset \cdot ABX \sim XBC$. Suppose both ABX and XBC were false. Then, by B and A, $BAX \sim BXA$ and $BCX \sim BXC$.

Case 1. If BXC , then, by 2, $ABC \cdot BXC \cdot \supset \cdot ABX$.

Case 2. If BXA , then, by 2 and A, $CBA \cdot BXA \cdot \supset \cdot CBX$, whence, by A, XBC .

Case 3. If BAX , and BCX , then, by 5, $BAX \cdot BCX \cdot \supset \cdot BAC \sim CAX$, and, by 5, $BCX \cdot BAX \cdot \supset \cdot BCA \sim ACX$. But BAC conflicts with ABC , by C and A; BCA conflicts with ABC , by C and A; and CAX and ACX conflict with each other, by C and A. Hence $ABX \sim XBC$.

THEOREM 9h. *Proof of 9 from A, B, 3², 5.*

To prove: $ABC \cdot X \cdot \supset \cdot ABX \sim XBC$. By A, CBA . By B and A, $XAC \sim XCA \sim AXC$.

If XAC , then, by 3, $XAC \cdot ABC \cdot \supset \cdot XBC$.

If XCA , then, by 3, $XCA \cdot CBA \cdot \supset \cdot XBA$, whence ABX by A.

If AXC , then, by 5, $ABC \cdot AXC \cdot \supset \cdot ABX \sim XBC$.

Hence, in any case, $ABX \sim XBC$.

THEOREM 9i. *Proof of 9 from A, B², C², 3², 4, 6.*

To prove: $ABX \cdot X \cdot \supset \cdot ABX \sim XBC$. Suppose both ABX and XBC were false. Then by B and A, $BAX \sim AXB$ and $BCX \sim CXB$.

Case 1. Suppose BAX and BCX . Then, by 4, $BAX \cdot BCX \cdot \supset \cdot BAC \sim BCA$.

Case 2. Suppose BAX and CXB . Then, by 3 and A, $CXB \cdot XAB \cdot \supset \cdot CAB$.

Case 3. Suppose AXB and BCX . Then, by 3 and A, $AXB \cdot XCB \cdot \supset \cdot ACB$.

Case 4. Suppose AXB and CXB . Then, by 6, $AXB \cdot CXB \cdot \supset \cdot ACB \sim CAB$.

Hence, in any case, by A, $CAB \sim ACB$, both of which conflict with ABC , by C and A. Therefore $ABX \sim XBC$.

THEOREM 9j. *Proof of 9 from A, B, 3², 4², 7.*

To prove: $ABC \cdot X \cdot \supset \cdot ABX \sim XBC$.

By A, CBA . By B, $AXC \sim XAC \sim XCA$.

Case 1. If AXC , then also, by A, CXA . Then by 4, $AXC \cdot ABC \cdot \supset \cdot AXB \sim ABX$, and also, by 4, $CXA \cdot CBA \cdot \supset \cdot CXB \sim CBX$, whence, by A, $CXB \sim XBC$. Then if ABX and XBC are both false, we must have AXB and CXB . Hence, by 7, $CXB \cdot AXB \cdot \supset \cdot CAX \sim ACX$, whence, by A, $XAC \sim XCA$.

Case 2. If XAC , then, by 3, $XAC \cdot ABC \cdot \supset \cdot XBC$.

Case 3. If XCA , then, by 3, $XCA \cdot CBA \cdot \supset \cdot XBA$, whence by A, ABX . Therefore, in any case, $ABX \sim XBC$.

THEOREM 9k. *Proof of 9 from A, B², C³, 3², 4, 8².*

Proof same as for Theorem 9i down to

Case 4. Suppose AXB and CXB . Then, by 8, $AXB \cdot CXB \cdot \supset \cdot ACX \sim CAB$, and by 8, $CXB \cdot AXB \cdot \supset \cdot CAX \sim ACB$. But CAB and ACB conflict with ABC , by C and A, and ACX and CAX conflict with each other, by C and A; so that Case 4 is impossible. Also, Cases 1, 2, and 3 conflict with ABC , by C and A.

Hence $ABX \sim XBC$ must be true.

THEOREM 9l. *Proof of 9 from A, B, 2, 3², 4.*

To prove: $ABC \cdot X \cdot \supset \cdot ABX \sim XBC$.

By B, $XAC \sim XCA \sim AXC$.

Case 1. If XAC , then by 3, $XAC \cdot ABC \cdot \supset \cdot XBC$.

Case 2. If XCA , then by 3 and A, $XCA \cdot CBA \cdot \supset \cdot XBA$, whence, by A, ABX .

Case 3. If AXC , then by 4, $ABC \cdot AXC \cdot \supset \cdot ABX \sim AXB$. But if AXB , then by 2 and A, $CBA \cdot BXA \cdot \supset \cdot CBX$, whence, by A, XBC .

Therefore, $ABX \sim XBC$.

THEOREM 9m. *Proof of 9 from A, B², 1, 3², 7.*

To prove: $ABC \cdot X \cdot \supset \cdot ABX \sim XBC$. Suppose both ABX and XBC are false.

By B, $XAC \sim XCA \sim AXC$. But if XAC , then by 3, $XAC \cdot ABC \cdot \supset \cdot XBC$, which is false; and if XCA , then by 3 and A, $XCA \cdot CBA \cdot \supset \cdot XBA$, whence, by A, ABX , which is false. Therefore AXC .

Now by B, $XBC \sim BCX \sim CXB$. But XBC is false; and if BCX , then by 1, $ABC \cdot BCX \cdot \supset \cdot ABX$, which is false. Therefore CXB , whence, by A, BXC .

Then by 7, $BXC \cdot AXC \cdot \supset \cdot BAX \sim ABX$. But ABX is false; and if BAX , then by 1 and A, $CBA \cdot BAX \cdot \supset \cdot CBX$, whence, by A, XBC , which is false.

Therefore $ABX \sim XBC$.

These 45 new theorems, together with the 71 theorems proved in the earlier paper, complete the list of 116 theorems on deducibility among the thirteen postulates of our revised basic list.

The results are collected for reference in Table I'.

TABLE I'. 116 THEOREMS ON DEDUCIBILITY

| Theorem | Postulate | follows from | Set in which used |
|---------|-----------|--------------|-------------------|
| 1a | 1 | ABC 2 4 | 6 |
| 1b | 1 | ABC 8 4 | 9,10,11 |
| 1c | 1 | ABC 2 5 | 7 |
| 1d | 1 | ABC 3 5 | 8 |
| 1e | 1 | A C 9 | 12 |
| 2a | 2 | ABC 1 7 | 4 |
| 2b | 2 | ABC 1 6 | 8 |
| 2c | 2 | ABC 3 6 | 9 |
| 2d | 2 | A C 3 7 | 10 |
| 2e | 2 | A C 8 4 6 | 9 |
| 2f | 2 | ABC 1 8 | 5 |
| 2g | 2 | ABC 1 5 | 2 |
| 2h | 2 | A C 3 8 | 11 |
| 2i | 2 | A C 3 5 | 8 |
| 2j | 2 | A C 9 | 12 |
| 3a | 3 | ABC 1 | 1,2,3,4,5 |
| 3b | 3 | ABC 2 | 1,6,7 |
| 3c | 3 | A C 2 6 | |
| 3d | 3 | A 1 2 | 1 |
| 3e | 3 | A C 2 8 | |
| 3f | 3 | A C 9 | 12 |
| 3g | 3 | A 1 | 9 |
| 3h | 3 | AB 2 | 9 |
| 3i | 3 | A 2 6 | 9 |
| 3j | 3 | A 2 7 | 9 |
| 3k | 3 | A 2 8 9 | |
| 4a | 4 | ABC 1 | 1,2,3,4,5 |
| 4b | 4 | AB 1 2 | 1 |
| 4c | 4 | AB 1 7 | 4 |
| 4d | 4 | A C 5 | 2,7,8 |
| 4e | 4 | A 3 5 7 | |
| 4f | 4 | A 5 7 8 | |
| 4g | 4 | A 2 5 | 7 |
| 4h | 4 | A 1 5 7 | |
| 4i | 4 | C 5 7 8 | |
| 4j | 4 | C 1 5 7 | |
| 4k | 4 | A C 9 | 12 |
| 4l | 4 | A 1 7 9 | |
| 4m | 4 | A 8 7 9 | |
| 4n | 4 | A 7 8 9 | |
| 4o | 4 | C 7 8 9 | |
| 4p | 4 | A 2 9 | |
| 5a | 5 | AB 1 2 | 1 |
| 5b | 5 | AB 1 7 | 4 |
| 5c | 5 | ABC 1 8 | 5 |
| 5d | 5 | ABC 1 6 | 8 |
| 5e | 5 | A 2 4 | 6 |
| 5f | 5 | A C 4 7 | 10 |
| 5g | 5 | A C 4 6 | 9 |
| 5h | 5 | A 1 4 7 | |
| 5i | 5 | A C 4 8 | 11 |
| 5j | 5 | A 3 4 7 | 10 |
| 5k | 5 | A 9 | 12 |

| Theorem | Postulate | follows from | Set in which used |
|---------|-----------|--------------|-------------------|
| 6a | 6 | ABC 2 | 1,6,7 |
| 6b | 6 | AB 2 7 | |
| 6c | 6 | A 1 7 | 4 |
| 6d | 6 | A 3 7 | 10 |
| 6e | 6 | A 1 8 | 5 |
| 6f | 6 | A 3 8 | 11 |
| 6g | 6 | AB 2 8 | |
| 6h | 6 | ABC 3 5 | 8 |
| 6i | 6 | A C 8 | 5,11 |
| 6j | 6 | ABC 1 5 | 2 |
| 6k | 6 | ABC 9 | 12 |
| 6l | 6 | AC 7 9 | |
| 6m | 6 | A 2 7 9 | |
| 6n | 6 | A 2 8 9 | |
| 6o | 6 | A 4 8 9 | |
| 6p | 6 | A 7 8 9 | |
| 6q | 6 | C 8 9 | |
| 7a | 7 | ABC 2 | 1,6,7 |
| 7b | 7 | ABC 6 | 3,9 |
| 7c | 7 | A C 4 6 | 9 |
| 7d | 7 | A 2 6 | |
| 7e | 7 | A 2 8 | |
| 7f | 7 | A C 8 | 5,11 |
| 7g | 7 | ABC 5 | 2,7,8 |
| 7h | 7 | A C 5 6 | |
| 7i | 7 | A 4 5 8 | |
| 7j | 7 | A 1 4 5 6 | |
| 7k | 7 | ABC 9 | 12 |
| 7l | 7 | A C 6 9 | |
| 7m | 7 | A 4 8 9 | |
| 7n | 7 | A 1 4 6 9 | |
| 8a | 8 | ABC 2 | 1,6,7 |
| 8b | 8 | ABC 1 5 | 2 |
| 8c | 8 | ABC 3 5 | 8 |
| 8d | 8 | ABC 3 6 | 9 |
| 8e | 8 | A 1 7 | 4 |
| 8f | 8 | ABC 1 6 | 8 |
| 8g | 8 | A 3 7 | 10 |
| 8h | 8 | A 2 6 | |
| 8i | 8 | A C 5 6 | |
| 8j | 8 | A C 4 6 | 9 |
| 8k | 8 | AB 2 7 | |
| 8l | 8 | AB 1 5 6 | |
| 8m | 8 | A 1 4 5 6 | |
| 8n | 8 | ABC 9 | 12 |
| 8o | 8 | AB 1 6 9 | |
| 8p | 8 | A C 6 9 | |
| 8q | 8 | A C 7 9 | |
| 8r | 8 | A 1 4 6 9 | |
| 8s | 8 | A 2 7 9 | |
| 9a | 9 | AB 1 2 | 1 |
| 9b | 9 | ABC 1 5 | 2 |
| 9c | 9 | ABC 1 6 | 8 |
| 9d | 9 | ABC 1 7 | 4 |
| 9e | 9 | ABC 1 8 | 5 |
| 9f | 9 | ABC 2 4 | 6 |
| 9g | 9 | ABC 2 5 | 7 |
| 9h | 9 | AB 3 5 | 8 |
| 9i | 9 | ABC 3 4 6 | 9 |
| 9j | 9 | AB 3 4 7 | 10 |
| 9k | 9 | ABC 3 4 8 | 11 |
| 9l | 9 | AB 2 3 4 | |
| 9m | 9 | AB 1 3 7 | |

EXAMPLES OF PSEUDO-BETWEENNESS.

In order to prove that no other theorems on deducibility are possible except those stated above, we first exhibit 54 examples of pseudo-betweenness, that is, 54 examples of systems K, R , which have some but not all of the properties mentioned in our basic list.

Of these examples, 37 were given in the earlier paper, and 17 are new. In the table following, the numbering of the examples is so arranged as to avoid conflict with the numbering in the earlier paper. (It will be noted that seven examples of the old list, namely, 17, 22, 25, 27, 31, 34, 35, are now omitted, being no longer needed, in view of certain of the new examples.)

In the case of each example, the postulates which are satisfied are mentioned explicitly, while the postulates which are not satisfied are indicated by a minus sign.

The new examples are as follows (the class K consisting of four elements, 1, 2, 3, 4, and the triads explicitly listed in each case being the only triads for which the relation R is supposed to be true):

- Ex. 41. 123, 134, 142, 143, 213, 214, 234, 241, 312, 321, 324, 341,
412, 423, 431, 432.
- Ex. 42. 123, 124, 142, 241, 243, 321, 324, 342, 421, 432.
- Ex. 43. 123, 143, 214, 243, 314, 321, 324, 412, 413, 423.
- Ex. 44. 123, 143, 214, 231, 243, 312, 314, 412, 423, 431.
- Ex. 45. 123, 132, 134, 142, 231, 241, 243, 321, 324, 342, 423, 431.
- Ex. 46. 123, 124, 132, 134, 142, 213, 214, 231, 234, 241, 243, 312,
321, 324, 342, 412, 421, 423, 431, 432.
- Ex. 47. 123, 142, 312, 314, 341, 342, 412, 423.
- Ex. 48. 123, 321.
- Ex. 49. 123, 142, 324, 341.
- Ex. 50. 123, 124, 312, 412, 431, 432.
- Ex. 51. 123, 124, 231, 234, 241, 243, 341.
- Ex. 52. 123, 231, 312, 412, 423, 431.
- Ex. 53. 123, 134, 421, 423.
- Ex. 54. 123, 124, 132, 134, 143, 213, 214, 231, 243, 312, 321, 324,
341, 342, 412, 421, 423, 431.
- Ex. 55. 123, 132, 142, 143, 213, 231, 241, 243, 312, 321, 341, 342.
- Ex. 56. 123, 143, 214, 243, 321, 324, 341, 342, 412, 423.
- Ex. 57. 123, 124, 143, 243, 312, 341, 342, 412, 423.

LEMMAS ON NON-DEDUCIBILITY

We are now in position to prove 84 lemmas on non-deducibility, which, taken together, establish the fact that no other theorems on deducibility are possible besides the 116 theorems listed above.

TABLE II'. LIST OF 54 EXAMPLES OF PSEUDO-BETWEENNESS

| Ex. | has properties | | | | | | | | | | | Lemma in which example is used | | |
|-----|----------------|---|---|---|---|---|---|---|---|---|---|--------------------------------|---|------------------------------|
| A | — | B | C | D | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A. 1 |
| B | A | — | C | D | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | B. 1 |
| C | A | B | — | D | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | C. 1 |
| D | A | B | C | — | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. 1 |
| 1 | A | B | C | D | — | 2 | 3 | — | — | 6 | 7 | 8 | — | 1.1, 4.1, 5.2, 9.2 |
| 2 | A | B | C | D | — | — | — | 4 | 5 | 6 | 7 | 8 | — | 1.2, 2.2, 3.1, 9.3 |
| 3 | A | B | C | D | 1 | — | 3 | 4 | — | — | — | — | — | 2.1, 5.1, 6.1, 7.1, 8.1, 9.1 |
| 4 | A | B | C | D | — | — | — | 4 | 5 | — | 7 | — | — | 6.2, 8.3 |
| 5 | A | B | C | D | — | — | — | — | — | 6 | 7 | — | — | 8.2 |
| 6 | A | — | C | D | — | 2 | 3 | 4 | 5 | 6 | 7 | 8 | — | 1.4 |
| 7 | A | — | C | D | 1 | — | 3 | — | — | 6 | — | — | — | 2.4, 7.2, 8.6 |
| 8 | A | — | C | D | 1 | — | — | 4 | 5 | 6 | 7 | 8 | — | 2.5, 3.5 |
| 9 | A | — | C | D | — | 2 | — | 4 | 5 | — | 7 | — | — | 3.6, 6.8, 8.5 |
| 10 | A | — | C | D | 1 | 2 | 3 | — | — | 6 | 7 | 8 | — | 4.4, 5.5 |
| 11 | A | — | C | D | 1 | 2 | 3 | 4 | 5 | — | — | — | 9 | 6.7, 7.3, 8.4 |
| 12 | A | B | — | D | — | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1.3 |
| 13 | A | B | — | D | 1 | — | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 2.3 |
| 14 | A | B | — | D | 1 | — | — | 4 | 5 | 6 | 7 | 8 | — | 3.2, 9.5 |
| 15 | A | B | — | D | — | 2 | — | 4 | 5 | 6 | 7 | 8 | — | 3.3, 9.6 |
| 16 | A | B | — | D | 1 | — | 3 | — | 5 | 6 | — | 8 | 9 | 4.2, 7.4 |
| 18 | A | B | — | D | 1 | — | 3 | 4 | — | 6 | — | 8 | — | 5.3, 7.5 |
| 19 | A | B | — | D | 1 | 2 | 3 | 4 | 5 | — | — | — | 9 | 6.3, 7.6, 8.7 |
| 20 | A | B | — | D | — | — | 3 | 4 | 5 | 6 | — | — | 9 | 7.7, 8.8 |
| 21 | A | B | — | D | — | — | — | 4 | — | 6 | 7 | 8 | — | 5.4 |
| 23 | A | B | — | D | 1 | — | 3 | 4 | — | 6 | — | — | — | 8.10 |
| 24 | A | B | — | D | — | — | — | 4 | 5 | — | 7 | 8 | — | 6.4 |
| 26 | — | B | C | D | 1 | — | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 2.6 |
| 28 | — | B | C | D | 1 | 2 | 3 | — | — | 6 | 7 | 8 | — | 4.5 |
| 29 | — | B | C | D | 1 | — | 3 | — | 5 | 6 | — | 8 | 9 | 4.6 |
| 30 | — | B | C | D | 1 | — | 3 | 4 | 5 | 6 | — | 8 | 9 | 7.8 |
| 32 | — | B | C | D | — | 2 | 3 | 4 | 5 | — | 7 | 8 | — | 6.11 |
| 33 | — | B | C | D | 1 | 2 | 3 | 4 | — | 6 | 7 | 8 | — | 5.6 |
| 36 | — | B | C | D | 1 | — | 3 | 4 | 5 | 6 | — | — | 9 | 8.14 |
| 37 | — | B | C | D | — | — | 3 | — | 5 | 6 | 7 | — | — | 4.8 |
| 38 | — | B | — | D | 1 | — | 3 | — | 5 | 6 | 7 | 8 | 9 | 4.10 |
| 39 | A | — | — | D | — | 2 | — | 4 | 5 | — | 7 | 8 | — | 6.9 |
| 40 | A | — | — | D | 1 | — | 3 | — | 5 | 6 | — | — | 9 | 8.11 |
| 41 | A | B | — | D | — | — | — | 4 | 5 | 6 | 7 | 8 | 9 | 3.4 |
| 42 | A | — | — | D | — | 2 | — | 4 | 5 | — | — | — | 9 | 3.7 |
| 43 | — | B | C | D | 1 | — | — | — | 5 | 6 | 7 | — | 9 | 4.7 |
| 44 | — | B | C | D | — | — | 3 | — | 5 | — | 7 | — | 9 | 4.9 |
| 45 | A | B | — | D | — | — | — | 4 | 5 | — | 7 | — | 9 | 6.5 |
| 46 | A | B | — | D | — | — | — | — | 5 | — | — | 8 | 9 | 6.6 |
| 47 | — | B | — | D | — | 2 | 3 | 4 | 5 | — | 7 | 8 | 9 | 6.13 |
| 48 | A | — | C | D | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | — | 9.7 |
| 49 | — | B | C | D | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | — | 9.8 |
| 50 | — | B | C | D | — | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1.5 |
| 51 | — | B | C | D | 1 | 2 | — | 4 | 5 | 6 | 7 | 8 | 9 | 3.8 |
| 52 | — | B | C | D | — | 2 | 3 | 4 | 5 | — | 7 | — | 9 | 6.12, 8.12 |
| 53 | — | B | C | D | 1 | 2 | 3 | 4 | 5 | — | — | — | 9 | 6.10, 7.9, 8.15 |
| 54 | A | B | — | D | — | — | — | 4 | 5 | 6 | 7 | — | 9 | 8.9 |
| 55 | A | B | — | D | 1 | — | 3 | 4 | — | 6 | — | 8 | — | 9.4 |
| 56 | A | B | — | D | — | — | — | — | 5 | 6 | 7 | — | 9 | 4.3 |
| 57 | — | B | C | D | — | — | 3 | 4 | 5 | 6 | 7 | — | 9 | 8.13 |

TABLE III'. 84 LEMMAS ON NON-DEDUCIBILITY

| Lem- ma | Post- ulate | is not deducible from | Proof by Ex. |
|------------|----------------|------------------------|--------------------|
| A.1 | A | BCD 1 2 3 4 5 6 7 8 9 | A |
| B.1 | B | A CD 1 2 3 4 5 6 7 8 9 | B |
| C.1 | C | AB D 1 2 3 4 5 6 7 8 9 | C |
| D.1 | D | ABC 1 2 3 4 5 6 7 8 9 | D |
| 1.1 | 1 | ABCD 2 3 6 7 8 | 1 |
| 1.2 | 1 | ABCD 4 5 6 7 8 | 2 |
| 1.3 | 1 | AB D 2 3 4 5 6 7 8 9 | 12 |
| 1.4 | 1 | A CD 2 3 4 5 6 7 8 | 6 |
| 1.5 | 1 | BCD 2 3 4 5 6 7 8 9 | 50 |
| 2.1 | 2 | ABCD 1 3 4 | 3 |
| 2.2 | 2 | ABCD 4 5 6 7 8 | 2 |
| 2.3 | 2 | AB D 1 3 4 5 6 7 8 9 | 13 |
| 2.4 | 2 | A CD 1 3 6 | 7 |
| 2.5 | 2 | A CD 1 4 5 6 7 8 | 8 |
| 2.6 | 2 | BCD 1 3 4 5 6 7 8 9 | 26 |
| 3.1 | 3 | ABCD 4 5 6 7 8 | 2 |
| 3.2 | 3 | AB D 1 4 5 6 7 8 | 14 |
| 3.3 | 3 | AB D 2 4 5 6 7 8 | 15 |
| 3.4 | 3 | AB D 4 5 6 7 8 9 | 41 |
| 3.5 | 3 | A CD 1 4 5 6 7 8 | 8 |
| 3.6 | 3 | A CD 2 4 5 7 | 9 |
| 3.7 | 3 | A D 2 4 5 9 | 42 |
| 3.8 | 3 | BCD 1 2 4 5 6 7 8 9 | 51 |
| 4.1 | 4 | ABCD 2 3 6 7 8 | 1 |
| 4.2 | 4 | AB D 1 3 5 6 8 9 | 16 |
| 4.3 | 4 | AB D 5 6 7 9 | 56 |
| 4.4 | 4 | A CD 1 2 3 6 7 8 | 10 |
| 4.5 | 4 | BCD 1 2 3 6 7 8 | 28 |
| 4.6 | 4 | BCD 1 3 5 6 8 9 | 29 |
| 4.7 | 4 | BCD 1 5 6 7 9 | 43 |
| 4.8 | 4 | BCD 3 5 6 7 | 37 |
| 4.9 | 4 | BCD 3 5 7 9 | 44 |
| 4.10 | 4 | B D 1 3 5 6 7 8 9 | 38 |
| 5.1 | 5 | ABCD 1 3 4 | 3 |
| 5.2 | 5 | ABCD 2 3 6 7 8 | 1 |
| 5.3 | 5 | AB D 1 3 4 6 8 | 18 |
| 5.4 | 5 | AB D 4 6 7 8 | 21 |
| 5.5 | 5 | A CD 1 2 3 6 7 8 | 10 |
| 5.6 | 5 | BCD 1 2 3 4 6 7 8 | 33 |

| Lem- ma | Post- ulate | is not deducible from | Proof by Ex. |
|------------|----------------|-----------------------|--------------------|
| 6.1 | 6 | ABCD 1 3 4 | 3 |
| 6.2 | 6 | ABCD 4 5 7 | 4 |
| 6.3 | 6 | AB D 1 2 3 4 5 9 | 19 |
| 6.4 | 6 | AB D 4 5 7 8 | 24 |
| 6.5 | 6 | AB D 4 5 7 9 | 15 |
| 6.6 | 6 | AB D 5 8 9 | 46 |
| 6.7 | 6 | A CD 1 2 3 4 5 9 | 11 |
| 6.8 | 6 | A CD 2 4 5 7 8 | 9 |
| 6.9 | 6 | A D 2 4 5 7 8 | 39 |
| 6.10 | 6 | BCD 1 2 3 4 5 9 | 53 |
| 6.11 | 6 | BCD 2 3 4 5 7 8 | 32 |
| 6.12 | 6 | BCD 2 3 4 5 7 9 | 52 |
| 6.13 | 6 | B D 2 3 4 5 7 8 9 | 47 |
| 7.1 | 7 | ABCD 1 3 4 | 3 |
| 7.2 | 7 | A CD 1 3 6 | 7 |
| 7.3 | 7 | A CD 1 2 3 4 5 9 | 11 |
| 7.4 | 7 | AB D 1 3 5 6 8 9 | 16 |
| 7.5 | 7 | AB D 1 3 4 6 8 | 18 |
| 7.6 | 7 | AB D 1 2 3 4 5 9 | 19 |
| 7.7 | 7 | AB D 3 4 5 6 9 | 20 |
| 7.8 | 7 | BCD 1 3 4 5 6 8 9 | 30 |
| 7.9 | 7 | BCD 1 2 3 4 5 9 | 53 |
| 8.1 | 8 | ABCD 1 3 4 | 3 |
| 8.2 | 8 | ABCD 6 7 | 5 |
| 8.3 | 8 | ABCD 4 5 7 | 4 |
| 8.4 | 8 | A CD 1 2 3 4 5 9 | 11 |
| 8.5 | 8 | A CD 2 4 5 7 | 9 |
| 8.6 | 8 | A CD 1 3 6 | 7 |
| 8.7 | 8 | AB D 1 2 3 4 5 9 | 19 |
| 8.8 | 8 | AB D 3 4 5 6 9 | 20 |
| 8.9 | 8 | AB D 4 5 6 7 9 | 54 |
| 8.10 | 8 | AB D 1 3 4 6 | 23 |
| 8.11 | 8 | A D 1 3 5 6 9 | 40 |
| 8.12 | 8 | BCD 2 3 4 5 7 9 | 52 |
| 8.13 | 8 | BCD 3 4 5 6 7 9 | 57 |
| 8.14 | 8 | BCD 1 3 4 5 6 9 | 36 |
| 8.15 | 8 | BCD 1 2 3 4 5 9 | 53 |
| 9.1 | 9 | ABCD 1 3 4 | 3 |
| 9.2 | 9 | ABCD 2 3 6 7 8 | 1 |
| 9.3 | 9 | ABCD 4 5 6 7 8 | 2 |
| 9.4 | 9 | AB D 1 3 4 6 8 | 55 |
| 9.5 | 9 | AB D 1 4 5 6 7 8 | 14 |
| 9.6 | 9 | AB D 2 4 5 6 7 8 | 15 |
| 9.7 | 9 | A CD 1 2 3 4 5 6 7 8 | 48 |
| 9.8 | 9 | BCD 1 2 3 4 5 6 7 8 | 49 |

Many of these lemmas were given in the earlier paper; but the new lemmas made necessary by the introduction of postulate 9 so often include certain of the old lemmas, that it is convenient to write out the whole list afresh, using a decimal notation instead of the letters of the alphabet, to avoid all possible confusion. This is done in Table III', above.

It will be noticed that postulate D plays a peculiar rôle. Although it is strictly independent and therefore cannot be omitted, yet it is not used in proving any of the theorems on deducibility, and it may always be made to hold or fail without affecting the holding or failing of any other postulate. It may therefore be called not only independent but altogether "*detached*".

COMPLETE INDEPENDENCE OF POSTULATES A, B, C, D, 9

To establish the complete independence* of the five postulates A, B, C, D, 9, we exhibit $2^5 = 32$ examples, which we number 000—031 inclusive, in Table IV. In this table, a plus sign (+) indicates that a postulate is satisfied, a minus sign (—) that it fails.

TABLE IV. LIST OF 32 EXAMPLES FOR POSTULATES A, B, C, D, 9

| Ex. | A | B | C | D | 9 |
|-----|---|---|---|---|---|
| 000 | + | + | + | + | + |
| 001 | — | + | + | + | + |
| 002 | + | — | + | + | + |
| 003 | + | + | — | + | + |
| 004 | + | + | + | — | + |
| 005 | + | + | + | + | — |
| 006 | — | — | + | + | + |
| 007 | — | + | — | + | + |
| 008 | — | + | + | — | + |
| 009 | — | + | + | + | — |
| 010 | + | — | — | + | + |
| 011 | + | — | + | — | + |
| 012 | + | — | + | + | — |
| 013 | + | + | — | — | + |
| 014 | + | + | — | + | — |
| 015 | + | + | + | — | — |
| 016 | — | — | — | + | + |
| 017 | — | — | + | — | + |
| 018 | — | — | + | + | — |
| 019 | — | + | — | — | + |
| 020 | — | + | — | + | — |
| 021 | — | + | + | — | — |
| 022 | + | — | — | — | + |
| 023 | + | — | — | + | — |
| 024 | + | — | + | — | — |
| 025 | + | + | — | — | — |
| 026 | — | — | — | — | + |
| 027 | — | — | — | + | — |
| 028 | — | — | + | — | — |
| 029 | — | + | — | — | — |
| 030 | + | — | — | — | — |
| 031 | — | — | — | — | — |

Example 000 shows that the five postulates are *consistent*.

Examples 001—005 show that the five postulates are *independent* in the ordinary sense; that is, no one of them is deducible from the other four.

* E. H. Moore, *Introduction to a form of general analysis*, New Haven Colloquium, 1906, published by the Yale University Press, New Haven, 1910; p. 82.

Examples 001—005 may be called “near-betweenness” systems, since they possess all but one of the five properties of betweenness. Examples 006—015 fail on two postulates; examples 016—025 fail on three, and examples 026—030 on four; while example 031 fails to have any one of the properties characteristic of betweenness.

Ex. 000. 123, 124, 134, 234, 321, 421, 431, 432.

Ex. 001. 123, 124, 134, 234.

Ex. 002. 123, 124, 321, 421.

Ex. 003. 123, 124, 134, 234, 321, 324, 421, 423, 431, 432.

Ex. 004. 123, 124, 134, 234, 321, 421, 431, 432; 444.

Ex. 005. 123, 143, 214, 234, 321, 341, 412, 432.

Ex. 006. 123, 124.

Ex. 007. 123, 124, 324, 341, 342.

Ex. 008. 123, 124, 134, 234; 444.

Ex. 009. 123, 142, 324, 341.

Ex. 010. 123, 124, 142, 143, 241, 321, 341, 421.

Ex. 011. 123, 124, 321, 421; 444.

Ex. 012. 123, 234, 321, 432.

Ex. 013. 123, 124, 134, 234, 321, 324, 421, 423, 431, 432; 444.

Ex. 014. 123, 214, 243, 314, 321, 324, 342, 412, 413, 423.

Ex. 015. 123, 143, 214, 234, 321, 341, 412, 432; 444.

Ex. 016. 123, 124, 132, 134.

Ex. 018. 123, 241.

Ex. 020. 123, 124, 134, 234, 243.

Ex. 023. 123, 124, 142, 241, 321, 421.

Exs. 017, 019, 021, 022, 024, 025. Same as Exs. 006, 007, 009, 010, 012, 014, with 444 added.

Ex. 026. 123, 124, 132, 134; 444.

Ex. 027. 123, 124, 132.

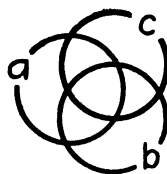
Exs. 028, 029, 030. Same as Exs. 018, 020, 023, with 444 added.

Ex. 031. 123, 213, 234, 243, 423; 444.

This last system (Ex. 031) will be found to violate all the thirteen postulates of our basic list; it is therefore as far removed as possible from a true betweenness system.

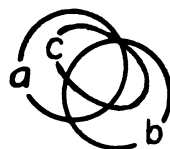
SIGNIFICANCE OF THE NOTION OF COMPLETE INDEPENDENCE*

The significance of the notion of complete independence derives from the fact that every postulate may be stated, at pleasure, in either the positive or the negative form, so that every postulate, a , should be regarded as a pair of coördinate propositions, a and \bar{a} . Thus a set of three postulates, (a, \bar{a}) , (b, \bar{b}) , (c, \bar{c}) , divides the universe of discourse into $2^3 = 8$ compartments, represented by the logical products, abc ; $\bar{a}bc$, $a\bar{b}c$, $ab\bar{c}$; $\bar{a}\bar{b}c$, $\bar{a}b\bar{c}$, $a\bar{b}\bar{c}$; $\bar{a}\bar{b}\bar{c}$, in which the barred and unbarred letters play precisely coördinate rôles.



If now there is no special relation between the postulates, all these compartments will be actually represented in the universe; it is only in the special case when some relation of implication among the propositions a , \bar{a} , b , \bar{b} , c , \bar{c} holds true, that any one of these compartments will be empty.

For example, if $\bar{a}bc$ is empty, then $\bar{a}c$ implies b (and also $\bar{b}c$ implies a , and $\bar{a}\bar{b}$ implies \bar{c}); and, conversely, if any one of these three implications is valid, then the compartment $\bar{a}bc$ will be empty. Similarly for each of the other compartments.



Hence Moore's criterion is a natural one: a set of n postulates is "completely independent" when and only when no one of the 2^n compartments into which the postulates divide the universe is empty.

* Among the many papers on "complete independence" which have appeared in recent years may be mentioned the following:

R. D. Beetle, *On the complete independence of Schimack's postulates for the arithmetic mean*, *Mathematische Annalen*, vol. 76 (1915), pp. 444-446;

L. L. Dines, *Complete existential theory of Sheffer's postulates for Boolean algebras*, *Bulletin of the American Mathematical Society*, vol. 21 (1915), pp. 183-188;

E. V. Huntington, *Complete existential theory of the postulates for serial order*; and *Complete existential theory of the postulates for well ordered sets*, *Bulletin of the American Mathematical Society*, vol. 23 (1917), pp. 276-280 and pp. 280-282;

J. S. Taylor, *Complete existential theory of Bernstein's set of four postulates for Boolean algebras*, *Annals of Mathematics*, ser. 2, vol. 19 (1917), pp. 64-69; and *Sheffer's set of five postulates for Boolean algebras in terms of the operation "rejection" made completely independent*, *Bulletin of the American Mathematical Society*, vol. 26 (1920), pp. 449-454;

B. A. Bernstein, *On the complete independence of Hurwitz's postulates for abelian groups and fields*, *Annals of Mathematics*, ser. 2, vol. 23 (1922), pp. 313-316; and *The complete existential theory of Hurwitz's postulates for abelian groups and fields*, *Bulletin of the American Mathematical Society*, vol. 28 (1922), pp. 397-399, and vol. 29 (1923), p. 33;

E. V. Huntington, *Sets of completely independent postulates for cyclic order*, *Proceedings of the National Academy of Sciences*, February, 1924;

W. E. Van de Walle, *On the complete independence of the postulates for betweenness*, in the present number of these Transactions.

APPENDIX, ON THE RELATION OF BETWEENNESS TO CYCLIC ORDER

The theory of *betweenness* (that is, the order of points along a straight line, without distinction of sense along the line), is closely related to the theory of *cyclic order* (that is, the order of points on a closed curve with a definite sense around the curve).*

Betweenness is characterized by the completely independent postulates A, B, C, D, 9; cyclic order† by the completely independent postulates E, B, C, D, 9.

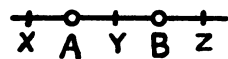
The postulates B, C, D, 9 hold true in both theories, while postulates A and E differ only by the interchange of two letters; thus:

POSTULATE A (*for betweenness*). *If ABC, then CBA.*

POSTULATE E (*for cyclic order*). *If ABC, then CAB.*

The following theorems may serve to bring out the contrast between the two theories.

THEOREM ON BETWEENNESS. (From A, C, 9.) *If A, B are two distinct elements, and if X, Y, Z are three other distinct elements, distinct from A and B, and such that XAB, AYB, ABZ; then XYZ.*

Proof. By 9, $XAB \cdot Y \supset XAY - YAB$; and by 9, $ABZ \cdot Y \supset ABY - YBZ$.
 But YAB and ABY conflict with AYB , by C and A; hence XAY and YBZ . Again, by 9, $AYB \cdot X \supset AYX - XYB$. But AYX conflicts with XAY , by C and A; hence XYB . Then by 9, $XYB \cdot Z \supset XYZ - ZYB$.

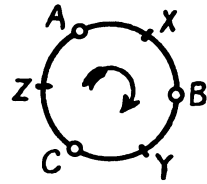
But ZYB conflicts with YBZ , by C and A; hence XYZ .

THEOREM ON CYCLIC ORDER. (From E, C, 9.) *If A, B, C are three distinct elements, such that ABC; and if X, Y, Z are three other distinct elements, distinct from A, B, C and such that AXB, BYC, CZA; then XYZ.*

* Besides (1) *betweenness* and (2) *cyclic order*, both of which are expressed in terms of a triadic relation, there are two other important types of order, namely: (3) *serial order* (that is, the order of points along a straight line with a definite sense along the line), which is expressed in terms of a dyadic relation; and (4) *separation of point pairs* (that is, the order of points on a closed curve without distinction of sense around the curve), which is expressed in terms of a tetradic relation. Sets of completely independent postulates for serial order are well known (*loc. cit.*); similar sets for the separation of point pairs will form the subject of a later paper.

† For the set E, B, C, D, 9, and two equivalent sets, E, B, C, D, 2 and E, B, C, D, 3, see E. V. Huntington, *Sets of completely independent postulates for cyclic order*, Proceedings of the National Academy of Sciences, February, 1924.

Proof. By 9, $ABC.Y \supset .ABY \sim YBC$, whence by E, $YAB \sim BCY$. But BCY conflicts with BYC , by C. Hence YAB . Then by 9, $YAB.X \supset .YAX \sim XAB$. But XAB conflicts with AXB , by E and C. Hence YAX , whence by E, AXY .



By E and 9, $BCA.Y \supset .BCY \sim YCA$. But BCY conflicts with BYC , by C. Hence YCA . Then by E and 9, $CAY.Z \supset .CAZ \sim ZAY$. But CAZ conflicts with CZA , by C. Hence ZAY . Then by E and 9, $AYZ.X \supset .AYX \sim XYZ$.

But AYX conflicts with AXY , by C. Hence XYZ .

The six postulates A, E, B, C, D, 9, taken together, would form, of course, an *inconsistent set*, since no system (K, R) has all these properties. It is interesting, however, to note the following "theorems of deducibility" among these six postulates.

THEOREM 201. *Proof of 9 from A, E, B.*

To prove: $ABC.X \supset .ABX \sim XBC$. By B, at least one of the six permutations of A, B, X will be true; hence, by A and E, all six will be true, so that ABX will be true. Similarly, XBC will be true.

THEOREM 202. *Proof of 9 from A, E, C.*

To prove: $ABC.X \supset .ABX \sim XBC$. Suppose 9 fails; that is, suppose ABX and XBC are both false while ABC is true. Then by A and E, we have CBA and CAB , which conflict with each other, by C. Hence 9 must hold.

THEOREM 203. *Proof of B from not-A, E, C, and 9.*

To prove: If A, B, C are distinct, then at least one of the six permutations, $ABC, ACB, BAC, BCA, CAB, CBA$, is true; or, more briefly: If A, B, C are distinct, then $P(A, B, C)$.*

Since postulate A is violated, there must exist at least one true triad, say XYZ .

Let A be any element distinct from X, Y, Z .

By 9, $XYZ.A \supset .XYA \sim AYZ$. But if XYA , then by E and 9, $YAX.Z \supset .YAZ \sim ZAX$; and if AYZ , then by E and 9, $ZAY.X \supset .ZAX \sim XAY$.

Therefore $YAZ \sim ZAX \sim XAY$.

Case 1. If YAZ , then by E and 9, $AZY.X \supset .AZX \sim XZY$; and by E and 9, $ZYA.X \supset .ZYX \sim XYA$. But XZY and ZYX conflict with XYZ , by E and C; hence, in Case 1, AZX and XYA .

* For essential details of this proof, including the convenient notation $P(A, B, C)$, I am indebted to Mr. C. H. Langford.

Case 2. If ZAX , then by E and 9, $AXZ.Y.\supset.AXY\sim YXZ$; and by E and 9, $XZA.Y.\supset.XZY\sim YZA$. But YXZ and XZY conflict with XYZ , by E and C; hence, in Case 2, AXY and YZA .

Case 3. If XAY , then by E and 9, $AYX.Z.\supset.AYZ\sim ZYX$; and by E and 9, $YXA.Z.\supset.YXZ\sim ZXA$. But ZYX and YXZ conflict with XYZ , by E and C; hence, in Case 3, AYZ and ZXA .

Therefore (making use of E), we have

$$XYZ.A.\supset.(AZY.AZX.AXY)\sim(AZX.AXY.AYZ)\sim(AYX.AYZ.AZX),$$

whence

$$P(X, Y, Z).A.\supset.P(A, Y, Z).P(A, Z, X).P(A, X, Y),$$

where the notation $P(A, X, Y)$, for example, means that at least one of the six possible permutations of the three letters, A, X, Y , forms a true triad.

Now let B be any element distinct from X, Y, Z, A . Then, by the same reasoning,

$$P(A, Y, Z).B.\supset.P(B, Y, Z).P(B, A, Z).P(B, A, Y).$$

Finally, let C be any element distinct from X, Y, Z, A, B . Then

$$P(B, A, Y).C.\supset.P(C, A, Y).P(C, B, Y).P(C, B, A).$$

This last result, $P(C, B, A)$, states that at least one of the permutations of the letters C, B, A forms a true triad, which establishes the theorem.

THEOREM 204. *Proof of not-B from A, E, C.*

To prove: that three elements, A, B, C , exist, such that all six permutations, $ABC, ACB, BAC, BCA, CAB, CBA$, are false.

If the system contains no true triad, then the theorem is clearly true. If the system contains any true triad, say XYZ , then by A and E, we have ZYX and ZXY , which is impossible, by C. Hence the theorem is true.

We are now prepared to exhibit the "complete existential theory" of these six postulates A, E, B, C, D, 9. The six postulates divide the universe into $2^6 = 64$ compartments, some of which, however, will be "empty." Thus, the four theorems just proved show that examples of the types

$$A, E, B, \bar{9}; \quad A, E, C, \bar{9}; \quad A, E, B, C; \quad \bar{A}, E, \bar{B}, C, 9$$

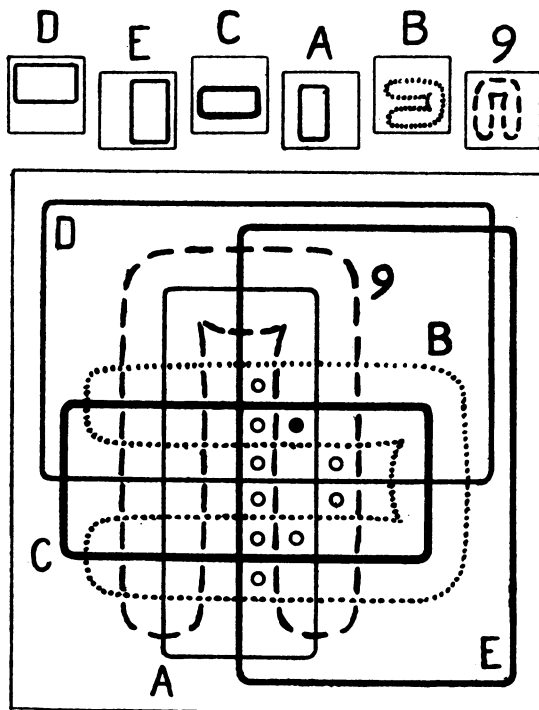
are impossible, so that at least ten of the 64 compartments will be empty (see the list in Table V below). This list shows that all the remaining 54 examples actually exist, so that the "existential theory" is complete.

TABLE V. EXAMPLES FOR THE SIX INCONSISTENT POSTULATES A, E, B, C, D, 9

| Rec. | A | E | B | C | D | 9 | Ex. | Rec. | A | E | B | C | D | 9 | Ex. |
|------|---|---|---|---|---|---|-----|---|---|---|---|---|---|---|-----|
| (1) | + | + | + | + | + | + | — | 19 | — | + | + | — | + | + | 035 |
| (2) | + | + | + | + | + | — | — | 20 | — | + | + | — | + | — | 036 |
| 3 | + | + | + | — | + | + | 037 | (21) | — | + | — | + | + | + | — |
| (4) | + | + | + | — | + | — | — | 22 | — | + | — | + | + | — | 041 |
| 5 | + | + | — | + | + | + | 038 | 23 | — | + | — | — | + | + | 043 |
| (6) | + | + | — | + | + | — | — | 24 | — | + | — | — | + | — | 042 |
| 7 | + | + | — | — | + | + | 039 | 25 | — | — | + | + | + | + | 001 |
| 8 | + | + | — | — | + | — | 040 | 26 | — | — | + | + | + | — | 009 |
| 9 | + | — | + | + | + | + | 000 | 27 | — | — | + | — | + | + | 007 |
| 10 | + | — | + | + | + | — | 005 | 28 | — | — | + | — | + | — | 020 |
| 11 | + | — | + | — | + | + | 003 | 29 | — | — | — | + | + | + | 006 |
| 12 | + | — | + | — | + | — | 014 | 30 | — | — | — | + | + | — | 018 |
| 13 | + | — | — | + | + | + | 002 | 31 | — | — | — | — | + | + | 016 |
| 14 | + | — | — | + | + | — | 012 | 32 | — | — | — | — | + | — | 027 |
| 15 | + | — | — | — | + | + | 010 | Records 33-64 are the same as Records 1-32 with D + changed to D —, and the letter "d" added to each example-number (in so far as these numbers exist). | | | | | | | |
| 16 | + | — | — | — | + | — | 023 | | | | | | | | |
| 17 | — | + | + | + | + | + | 033 | | | | | | | | |
| 18 | — | + | + | + | + | — | 034 | | | | | | | | |

- The requisite examples, not already listed under Table IV, are as follows:
- Ex. 033. 123, 231, 312; 124, 241, 412; 134, 341, 413; 234, 342, 423.
 - Ex. 034. 123, 231, 312; 214, 142, 421; 134, 341, 413; 432, 324, 243.
 - Ex. 035. 123, 231, 312; 312, 213, 132; 421, 214, 142; 431, 314, 143; 234, 342, 423; 432, 324, 243.
 - Ex. 036. 123, 231, 312; 214, 142, 421; 134, 341, 413; 432, 324, 243; 321, 132, 213.
 - Ex. 037. All the twenty-four possible triads are true.
 - Ex. 038. No triads true.
 - Ex. 039. 123, 231, 312; 321, 132, 213; 124, 241, 412; 421, 214, 142; 234, 342, 423; 432, 324, 243.
 - Ex. 040. 123, 231, 312; 321, 213, 132; 124, 241, 412; 421, 214, 142.
 - Ex. 041. 123, 231, 312; 214, 142, 421.
 - Ex. 042. 123, 231, 312; 321, 213, 132; 214, 142, 421.
 - Ex. 043 Here the class K consists of 5 elements, 1, 2, 3, 4, 5; all the sixty possible triads are true *except* the following: 321, 213, 132; 345, 453, 534; 543, 435, 354.
 - Exs.033d, 034d, etc., are the same as Exs. 033, 034, etc., with the addition of the triad 444 (so as to violate postulate D).

Finally, the inter-relations between the six postulates A, E, B, C, D, 9 may be shown diagrammatically as in the accompanying figure.* In this diagram, a zero in any compartment indicates that no example having the properties belonging to that compartment exists. For instance, the fact that no example of the type A, E, B, C, D, 9 exists, shows that the postulates are inconsistent.



* This form of diagram, the possibility of which was vaguely suggested by Venn in 1881, is believed to be an improvement over those in common use. See

John Venn, *On the diagrammatic and mechanical representation of propositions and reasonings*, *Philosophical Magazine*, ser. 5, vol. 10 (July, 1880), pp. 1-18; or his *Symbolic Logic*, 1st edition, 1881, p. 108, 2d edition, 1894, p. 118 (with extensive historical notes);

H. Marquand, *On logical diagrams for n terms*, *Philosophical Magazine*, ser. 5, vol. 12 (October, 1881), pp. 266-270;

C. L. Dodgson ["Lewis Carroll"], *Symbolic Logic*, London, 1896, known to me only through a citation by C. I. Lewis;

W. J. Newlin, *A new logical diagram*, *Journal of Philosophy, Psychology, and Scientific Methods*, vol. 3 (1906), pp. 539-545;

W. E. Hocking, *Two extensions of the use of graphs in elementary logic*, *University of California Publications in Philosophy*, vol. 2 (1909), pp. 31-44; and

C. I. Lewis, *A Survey of Symbolic Logic*, University of California Press, 1918, p. 180.

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