A UNIQUENESS THEOREM FOR THE LEGENDRE AND HERMITE POLYNOMIALS*

BY

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1. If we replace \( y \) in the expansion of \((1+y)^{-v}\) by \( 2xz+z^2 \), the coefficient of \( z^n \) will, when \( x \) is replaced by \(-x\), be the generalized polynomial \( L_n(x) \) of Legendre. It is also easy to show that the Hermitian polynomial \( H_n(x) \), usually defined by

\[
e^{xz} \frac{d^n}{dz^n} e^{-z^2} = H_n(x),
\]

is the coefficient of \( z^n/n! \) in the series obtained on replacing \( y \) in the expansion of \( e^{-y} \) by the same expression \( 2xz+z^2 \). Furthermore, there is a simple recursion formula between three successive Legendre polynomials and between three successive Hermitian polynomials. These facts suggest the following problem.

Let

\[
\phi(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \cdots
\]

and put

\[
\phi(2xz + z^2) = P_0 + P_1(x) z + P_2(x) z^2 + \cdots.
\]

To what extent is the generating function \( \phi(y) \) determined if it is known that a simple recursion relation exists between three of the successive polynomials \( P_0, P_1(x), P_2(x), \cdots \)? We shall find that the generalized Legendre polynomials and those of Hermite possess a certain uniqueness in this regard.

2. We have

\[
P_n(x) = \frac{1}{n!} \left. \frac{d^n}{dz^n} \phi(2xz + z^2) \right|_{z=0}.
\]

When we make use of the formula for the \( n \)th derivative of a function of a function given by Faà de Bruno,† we find without difficulty

\[
P_n(x) = \sum_{i+j=n} \frac{a_{n-i}}{i!j!} (2x)^i,
\]

* Presented to the Society, October 25, 1924.
where the summation extends to all values of \( i \) and \( j \) subject to the relation

\[ i + 2j = n. \]

When developed, the expression is

\[
P_n(x) = \sum_{0}^{n} \frac{a_n}{n!} (2x)^n + \sum_{0}^{n-2} \frac{a_{n-1}}{(n-2)!} (2x)^{n-2} + \sum_{0}^{n-4} \frac{a_{n-2}}{(n-4)!} (2x)^{n-4} + \ldots.
\]

It is seen that while \( P_n \) is an even or an odd function, the coefficients of the generating function that enter into it form a certain consecutive group, a fact which has important consequences.

3. Let us denote by \( A_n^m \) the term in \( P_n(x) \) that is of degree \( m \) in \( x \). We see that

\[
A_n^{n-2j} = \frac{a_{n-j}}{(n-2j)!} (2x)^{n-2j},
\]

\[
A_{n+1}^{n-2j-1} = \frac{a_{n-j-1}}{(n-2j-1)!} (2x)^{n-2j-1},
\]

\[
A_{n+2}^{n-2j} = \frac{a_{n-j+1}}{(n-2j)!} (2x)^{n-2j},
\]

the expressions being valid for \( j = -1, 0, 1, 2, \ldots \) if we agree that \( A_n^m = 0 \), when \( m > n \). The notable fact is that \( A_n^{n-2j}, A_{n+1}^{n-2j-1} \) both contain \( a_{n-j} \), but \( A_{n+2}^{n-2j} \) contains \( a_{n-j+1} \).

Let \( k \) and \( l \) be multipliers, which we shall assume to be polynomials in \( n \) to be determined; then

\[
2x A_{n+1}^{n-2j-1} + k A_n^{n-2j} = [ln + (k-2l)j + k] \frac{a_{n-j}}{a_{n-j+1}} A_{n+2}^{n-2j},
\]

a formula valid for \( j = 0, 1, 2, \ldots \). Let

\[
\psi(j) = ln + (k-2l)j + k.
\]

We see that

\[
\psi(-1) = (n + 2)l,
\]

and when \( n \) is even, that

\[
\psi\left(\frac{n}{2}\right) = \left(\frac{n}{2} + 1\right)k.
\]
This shows that, \( h \) being another polynomial in \( n \) to be determined,

\[
h P_{n+2} - 2x_l P_{n+1} - k P_n = \sum_{j=-1}^{n'} \left\{ h - \psi(j) \frac{a_{n-j}}{a_{n-j+1}} \right\} A_{n+2}^{n-j},
\]

where \( n' = n/2 \), if \( n \) is even, and \( n' = (n-1)/2 \) if \( n \) is odd.

4. We see from the last expression what must be the character of the recursion relation,* and that for it to exist we must have

\[
a_{n+1} = \varphi(n) a_n,
\]

where \( \varphi(n) \) is a polynomial in \( n \). In order that the summation on the right vanish, it is necessary that

\[
\psi(j) = \varphi(n-j) \theta(n),
\]

\( \theta(n) \) being a polynomial in \( n \). The polynomial \( h(n) \) is then given at once by

\[
h(n) = \theta(n).
\]

It is easy to determine \( l \) and \( k \), so that \( \psi(j) \) will have the desired form. Since \( \varphi(n-j) \) is of the same degree in \( j \) that \( \varphi(n) \) is in \( n \), and since \( \psi(j) \) is linear in \( j \), we see that \( \varphi(n) \) must be linear in \( n \).

Put

\[
\varphi(n) = \alpha n + \beta.
\]

Then

\[
l n + (k - 2l) j + k = (\alpha n - \alpha j + \beta) \theta(n).
\]

This is to be an identity in both \( n \) and \( j \). Put \( j = -1 \), and we find

\[
(n + 2) l = (\alpha n + \alpha + \beta) \theta(n).
\]

Since \( \alpha \) and \( \beta \) are arbitrary it follows that \( \theta(n) \) must contain \( n+2 \) as a factor, and

\[
l = (\alpha n + \alpha + \beta) \frac{\theta(n)}{n+2}.
\]

* It is evident that a linear recursion relation will not exist unless the factor \( 2x \) is introduced as in the middle term above.
It follows then at once that
\[ k = (\alpha n + 2 \beta) \frac{\theta(n)}{n + 2}, \]

No loss of generality results from putting
\[ h = \theta(n) = (n + 2), \quad l = (\alpha n + \alpha + \beta), \quad k = (\alpha n + 2 \beta). \]

The polynomials will therefore have the recursion relation
\[ (n + 2) P_{n+2}(x) - 2x(\alpha n + \alpha + \beta) P_{n+1}(x) - (\alpha n + 2 \beta) P_n(x) = 0, \]
if
\[ a_{n+1} = (\alpha n + \beta) a_n. \]

Taking \( a_0 = 1 \), we have for generating function
\[ \varphi(y) = F\left(\alpha, \frac{\beta}{\alpha}, a, \alpha y\right) = (1 - \alpha y)^{-\beta/\alpha}, \text{ if } \alpha \neq 0, \]
where \( F \) represents the hypergeometric function, and
\[ \varphi(y) = e^{\beta y}, \text{ if } \alpha = 0. \]

These then are the only types of generating function that will give a recursion relation, with the conditions that \( h, l, \) and \( k \) are polynomials in \( n \).*

5. A further remark might be made about the case \( \alpha \neq 0 \).
We have
\[ 2 \varphi'(2xz + z^2) = P'_1(x) + P'_2(x) z + P'_3(x) z^2 + \cdots. \]

Also we find
\[ \varphi'(y) (1 - \alpha y) = \beta \cdot \varphi(y), \]
and can then deduce
\[ P'_{n+2}(x) - 2\alpha x P'_{n+1}(x) - \alpha P'_n(x) = 2 \beta P_{n+1}(x). \]

When this is combined with the recursion formula we have
\[ x P'_{n+1}(x) + P'_n(x) = (n + 1) P_{n+1}(x), \]

* It would evidently be no more general to take \( h, l, k \) rational in \( n \).
a relation independent of $\alpha$ and $\beta$. From this and the recursion relation we can obtain
\[(1 + \alpha x^2) P_n' + (\alpha + 2\beta) x P_n - n(\alpha n + 2\beta) P_n = 0.\]

Now the differential equation
\[(1 + \alpha x^2) \frac{d^2 y}{dx^2} + (\alpha + 2\beta) x \frac{dy}{dx} - n(\alpha n + 2\beta) y = 0\]
is changed into
\[(n^2 - 1) \frac{d^2 y}{du^2} + (1 + 2\gamma) u \frac{dy}{du} - n(n + 2\gamma) y = 0\]
by putting $u = \sqrt{\alpha - x}$, $\gamma = \beta/\alpha$. But this is the differential equation satisfied by the generalized Legendre polynomials.

6. It is evident that we can now state the following theorem:

Let

$$\varphi(y) = a_0 + a_1 y + \frac{a_2}{2!} y^2 + \cdots,$$

and put

$$\varphi(2xz + z^2) = P_0 + P_1(x) z + P_2(x) z^2 + \cdots.$$

The only cases in which there will be a recursion relation of the form

$$h(n) P_{n+2}(x) - 2l(n) x P_{n+1}(x) - k(n) P_n(x) = 0,$$

where $h(n)$, $l(n)$, and $k(n)$ are polynomials, are essentially where we have the generalized polynomials of Legendre, and the polynomials of Hermite.