ON NORMAL FORMS OF DIFFERENTIAL EQUATIONS*

BY

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Klein† has treated the question of obtaining invariant forms for the differential equation

$$y''+py'+qy = 0, \tag{1}$$

or the resolvent equation of the third order,

$$[\eta] = 2q-p^2-\frac{dp}{dx}, \tag{2}$$

where

$$[\eta] = \frac{\eta'''}{\eta'}-\frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$$

is the Schwarzian derivative and the coefficients are single-valued functions on a given algebraic configuration; and he has given the solution for the hyperelliptic case, and the case of a canonical Riemann surface.

He raised the question of what the form would be in the case of a plane non-singular quartic, $p = 3$, considered in the projective plane, when one imposes the further condition that the answer shall be given in terms of invariant expressions which bear symmetrically on the three ternary homogeneous variables x_1, x_2, x_3, and the question was answered by Gordan.‡

This last restriction, though doubtless interesting, is not prescribed by the nature of the problem, which admits an altogether satisfactory projective treatment for the case that one assumes the given algebraic configuration in the form of Noether's normal C_{p-1}. Let the projective homogeneous coördinates of a point of this curve be denoted by $(x_1, \cdots, x_p)$, and let the curve be projected on a pencil of hyperplanes,

$$u_x-zv_x = 0, \tag{3}$$

* Presented to the Society, February 28, 1925.

† *Über lineare Differentialgleichungen der zweiten Ordnung,* Göttingen, 1894 (lithographed), pp. 90–105.

‡ Mathematische Annalen, vol. 46 (1895), p. 606. For further references to the literature cf. Fricke in the Encyklopädie der mathematischen Wissenschaften, vol. 2_2, pp. 437–8.

where

$$u_x = u_1 x_1 + \cdots + u_p x_p = 0, \qquad v_x = v_1 x_1 + \cdots + v_p x_p = 0$$

denote two non-specialized hyperplanes. Let F be the corresponding Riemann surface spread out over the z-plane, where

$$z = \frac{u_x}{v_x}.$$

Then F has $2p-2$ leaves, connected by $6p-6$ simple branch points.

The key to the solution of the problem, so far as the η-equation is concerned, consists in the identity*

$$(4) \qquad [\eta]_z = \left(\frac{dt}{dz}\right)^2 [\eta]_t + [t]_z,$$

where

$$(5) \qquad z = \varphi(t)$$

may be any analytic function whatever. Now choose, in particular, as the function $\varphi(t)$ the automorphic function with limiting circle, which maps F on a fundamental domain $\mathfrak{F}$ of the automorphic group. Then the form of (4) which is useful in what follows is

$$(6) \qquad [\eta]_t = \varphi'(t)^2 \{[\eta]_z - [t]_z\}.$$

1. **The η-Differential Equation.** We wish to consider such differential equations

$$(A) \qquad [\eta] = \text{single valued function on } C_{p-1}$$

as have only *regular* singular points on C_{p-1}. These points, n in number, are given, and the difference of the exponents in each is given. In particular, there are ∞^{3p-3} equations (A) having no singular points. (Here, $p > 2$, since we will assume the C_{p-1} to be simple, and thus exclude the hyperelliptic case.)

The Manifold S_p. Let S_p denote the real four-dimensional manifold of the points $(x_1, \cdots, x_p)$ corresponding to Noether's C_{p-1}. It will be convenient to uniformize S_p as follows.† Let

* Klein, loc. cit., p. 59.

† Cf. *The Madison Colloquium*, p. 224.

$$w_k = \int \Phi_k(t)\, dt \qquad (k = 1, \cdots, p) \tag{7}$$

denote the normal integrals of the first kind, and let C_{p-1} be assumed in the form

$$C_{p-1}: \qquad x_1 : x_2 : \cdots : x_p = \Phi_1(t) : \Phi_2(t) : \cdots : \Phi_p(t).$$

Then we may set

$$x_k = \varrho\, \Phi_k(t) \qquad (k = 1, \cdots, p). \tag{8}$$

If t be restricted to the fundamental domain $\mathfrak{F}$ and ϱ be allowed to take on any value but 0, then, not only will each pair of values (ϱ, t) lead to one point of S_p, but conversely each point of S_p will lead to just one such pair of values (ϱ, t). Furthermore, to an arbitrary point (x^0) of S_p will correspond at least one pair of integers (α, β)—which may be different for a second point (x^1) of S_p—such that the equations

$$x_\alpha = \varrho\, \Phi_\alpha(t), \qquad x_\beta = \varrho\, \Phi_\beta(t),$$

when solved for ϱ and t, yield functions

$$\varrho = \varrho(x_\alpha, x_\beta), \qquad t = t(x_\alpha, x_\beta)$$

both analytic in (x_α, x_β) in the neighborhood of the point (x_α^0, x_β^0).

The Manifold Σ_p. We proceed to introduce a new four-dimensional manifold Σ_p corresponding to the Riemann surface F spread out over the z-plane, where

$$z = \frac{u_x}{v_x} = \frac{u_\Phi}{v_\Phi}. \tag{9}$$

Let

$$z_1 = u_x, \qquad z_2 = v_x. \tag{10}$$

Then Σ_p is the Riemann manifold whose points are (z_1, z_2), and it is uniformized by the equations

$$\begin{aligned} z_1 &= \varrho u_\Phi = \varrho[u_1 \Phi_1(t) + \cdots + u_p \Phi_p(t)], \\ z_2 &= \varrho v_\Phi = \varrho[v_1 \Phi_1(t) + \cdots + v_p \Phi_p(t)]. \end{aligned} \tag{11}$$

The branch function, s, on F is given by the formula

$$s = \frac{dz}{dW}, \qquad W = \int v_\Phi \, dt.$$

Thus

$$s = \frac{v_\Phi u_{\Phi'} - u_\Phi v_{\Phi'}}{v_\Phi^3}. \tag{12}$$

The branch form, σ, is now defined by the equation

$$\sigma = \varrho^3 (v_\Phi u_{\Phi'} - u_\Phi v_{\Phi'}). \tag{13}$$

Let the last factor be denoted by D, or $D(t)$. Then

$$\sigma = \varrho^3 D. \tag{14}$$

We now have the material in hand for writing the η-differential equation in the desired form. The function $\varphi(t)$ which we chose in equation (6) is no other than the function defined by (9):

$$\varphi(t) = \frac{u_\Phi}{v_\Phi}.$$

Thus

$$\varphi'(t) = \frac{D}{v_\Phi^2} = \frac{\sigma}{\varrho z_2^2}.$$

Moreover, Klein has pointed out that the form

$$\frac{1}{z_2^4} [\eta]_z$$

is invariant under a linear transformation of the binary homogeneous variables z_1, z_2. Hence we write (6) in the form

$$\sigma^2 \left\{ \frac{1}{z_2^4} [\eta]_z - \frac{1}{z_2^4} [t]_z \right\} = \varrho^2 [\eta]_t. \tag{15}$$

From this identity follow the two normal forms of the η-equation, which we set out to obtain, namely

$$(A_1) \qquad \sigma^2\left\{\frac{1}{z_2^4}[\eta]_z - \frac{1}{z_2^4}[t]_z\right\} = \Gamma(x_1, x_2, \cdots, x_p),$$

where Γ denotes a homogeneous rational function of the second dimension, whose singularities form precisely the singular points of the η-equation

$$(A_2) \qquad [\eta]_t = Q(t),$$

where

$$Q(t) = \frac{\Gamma(\varrho\Phi_1(t), \cdots, \varrho\Phi_p(t))}{\varrho^2}.$$

Thus $Q(t)$ is single-valued and meromorphic. It is not an absolute invariant of the automorphic group, but takes on a factor for each transformation

$$(16) \qquad t_\alpha = L_\alpha(t)$$

of this group. Since each x_k is invariant under (16), we have

$$\varrho_\alpha \Phi_k(t_\alpha) = \varrho\Phi_k(t).$$

Moreover,

$$du_k = \Phi_k(t_\alpha)\,dt_\alpha = \Phi_k(t)\,dt.$$

Hence

$$(17) \qquad \varrho_\alpha = L_\alpha'(t)\varrho, \qquad \Phi_k(t_\alpha) = L_\alpha'^{-1}(t)\,\Phi_k(t).$$

Thus it appears that

$$(18) \qquad Q(t_\alpha) = L_\alpha'(t)^{-2}\,Q(t).$$

2. Discussion of (A_1) and (A_2). Equation (A_1), which may be called the *algebraic* form of the η-differential equation, is based on the algebraic manifold assumed in the form of Noether's C_{p-1}. This real two-dimensional manifold is replaced by the four-dimensional manifold S_p of the homogeneous variables $(x_1, \cdots, x_p)$. The latter manifold, like the former, has no singular points whatever. Let (a) be a point of S_p in which the η-differential equation is to have a regular singular point, and let the difference of the exponents there be denoted by α. Now let u_x and v_x be so chosen that

(i) the hyperplane $v_x = 0$ does not pass through (a);

(ii) the branch form σ does not vanish in (a).

Then (a) will go over into a finite point $z = a$ of F, which is not a branch point. The function $[t]_z$ will be analytic at this point. On the other hand,

$$[\eta]_z \doteq \frac{(1-\alpha^2)/2}{(z-a)^2} + \frac{C}{z-a} + \mathfrak{A}(z),$$

where $\mathfrak{A}(z)$ is a generic notation for a function analytic at $z = a$.

It follows, then, from (A_1) that $\Gamma(x_1, \cdots, x_p)$ must have a pole of the second order in (a), and that the coefficient of the term of the second order in the principal part of this pole is determined.

Since the number of singular points of the differential equation is finite, u_x and v_x can be so chosen that conditions (i) and (ii) will be satisfied for every singular point of the differential equation. Similarly, u_x and v_x can be so chosen that an arbitrary point (b) of S_p will go over into a finite point of F, not a branch point; and since both $[\eta]_z$ and $[t]_z$ will be analytic there if (b) is not a singular point of the differential equation, it follows that $\Gamma(x_1, \cdots, x_p)$ is analytic at (b).

But u_x and v_x cannot be so chosen once for all that all of the above conditions will be fulfilled for every point of S_p. It is like the case of the normal differential on a non-singular curve of the projective (x_1, x_2, x_3)-plane:

$$d\omega = \frac{\begin{vmatrix} c_1 & x_1 & dx_1 \\ c_2 & x_2 & dx_2 \\ c_3 & x_3 & dx_3 \end{vmatrix}}{c_1 f_1 + c_2 f_2 + c_3 f_3}.$$

If a point (a) of the curve be chosen in advance, then (c) can be so taken that the denominator does not vanish in (a). But (c) cannot be so taken once for all that this condition is fulfilled for every point of the curve.

Equation (A_2). This form, which may be called the *automorphic* form of the η-differential equation, meets completely the difficulty just discussed, for it holds without let or hindrance for *every* point in $\mathfrak{F}$ and its analytic continuations. The function $Q(t)$ is single-valued and analytic in every point t except the singular points of the differential equation. It is the form which serves as the definition of η when this function is studied on the basis of $\mathfrak{F}$ as the defining element of the given algebraic equation.

3. **The Linear Differential Equation.** It would seem to be a simple matter to pass from equation (A_2) to the linear differential equation corresponding to (1), since one need only set

$$(19) \qquad y_1 = \frac{\eta}{\sqrt{\dfrac{d\eta}{dt}}}, \qquad y_2 = \frac{1}{\sqrt{\dfrac{d\eta}{dt}}},$$

and these functions are linearly independent solutions of the equation

$$\frac{d^2 y}{dt^2} + \tfrac{1}{2} Q(t) y = 0. \tag{20}$$

And, indeed, for the automorphic treatment this is the whole story.

When, however, these functions y_1 and y_2 are transplanted to the algebraic form C_{p-1} or S_p of the algebraic configuration, they do not satisfy a differential equation of the form (1),—namely, one whose coefficients are single-valued on C_{p-1} or S_p. In fact,

$$\frac{d\eta}{dt} \quad \text{and} \quad \frac{d\eta}{dt_\alpha} = \frac{1}{L'_\alpha(t)} \frac{d\eta}{dt}$$

are two different functions, although t and $t_\alpha = L_\alpha(t)$ correspond to the same point of C_{p-1}.*

On the other hand, if the η-equation be assumed in the form arising from (A_1),

$$[\eta]_z - [t]_z = R(z, s), \tag{21}$$

and if now we set

$$Y_1 = \frac{\eta}{\sqrt{\dfrac{d\eta}{dz}}}, \qquad Y_2 = \frac{1}{\sqrt{\dfrac{d\eta}{dz}}}, \tag{22}$$

the points of the given configuration for which $z = \infty$, and also the branch points, assume an exceptional rôle.

How shall these two classes of difficulties be avoided? Klein answers the question by the use of homogeneous variables and transcendental forms. He sets

$$\Pi_1 = \frac{\eta}{\sqrt{\dfrac{d\eta}{d\omega}}}, \qquad \Pi_2 = \frac{1}{\sqrt{\dfrac{d\eta}{d\omega}}}, \tag{23}$$

where $d\omega$ is the *normal differential,* a so-called *differential form,* which, for the C_{p-1}—or rather for the S_p—is defined as follows:

$$d\omega = \frac{|z\,dz|}{\sigma}, \qquad |z\,dz| = z_2\,dz_1 - z_1\,dz_2. \tag{24}$$

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* The same conclusion may be reached by observing that the coefficients of (20) are not single-valued on F, C_{p-1}, or S_p, whereas this differential equation is uniquely determined from (A_2).

The Normal Differential. No matter what the independent variable or variables may be, equations (11) give in all cases

$$\begin{vmatrix} dz_1 & dz_2 \\ z_1 & z_2 \end{vmatrix} = \begin{vmatrix} \varrho u_{\Phi'}\,dt + u_{\Phi}\,d\varrho & \varrho v_{\Phi'}\,dt + v_{\Phi}\,d\varrho \\ \varrho u_{\Phi} & \varrho v_{\Phi} \end{vmatrix} = \varrho^2 D\,dt.$$

Hence

$$\frac{|z\,dz|}{\sigma} = \frac{\varrho^2 D\,dt}{\varrho^3 D}$$

and

(25) $$d\omega = \frac{dt}{\varrho}.$$

Thus*

$$\frac{d\eta}{d\omega} = \varrho\frac{d\eta}{dt},$$

and

(26) $$\Pi_1 = \varrho^{-1/2}\frac{\eta}{\sqrt{\dfrac{d\eta}{dt}}}, \qquad \Pi_2 = \varrho^{-1/2}\frac{1}{\sqrt{\dfrac{d\eta}{dt}}}.$$

The Differential Equation for Π. The expressions Π_1, Π_2 are *transcendental forms* (i. e. homogeneous functions) of dimension $-\frac{1}{2}$, in z_1, z_2. They satisfy a differential equation of the following form:†

(B) $$(\Pi, \sigma^2)_2 + 15\left\{\frac{\tau}{\sigma} + \Gamma\right\}\Pi = 0,$$

where the first term denotes the second transvectant of Π and σ^2, and τ is an integral algebraic form of the fifth dimension, belonging to S_p. The proof follows.

4. **Deduction of (B).** The second transvectant $(\Pi, \varphi)_2$ of two binary forms, Π and φ, can be expressed as follows:

(27) $$(\Pi, \varphi)_2 = \varphi_{22}\,\Pi_{11} - 2\varphi_{12}\,\Pi_{12} + \varphi_{11}\,\Pi_{22}.$$

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* The expression $d^2\eta/d\omega^2$ could be defined for a thread, or for the case that ρ and t are both analytic functions of a third complex variable. It appears to be useless,—I know, at least, of no place in which Klein considers it.

† Klein, loc. cit., p. 98.

The partial derivatives of a form ψ of dimension k, with respect to z_2, are given by the formulas (Klein, loc. cit., pp. 23, 24)

$$\psi_2 = \frac{k\psi - z_1\psi_1}{z_2}, \qquad \psi_{12} = \frac{(k-1)\psi_1 - z_1\psi_{11}}{z_2^2},$$

(28)

$$\psi_{22} = \frac{k(k-1)\psi - 2(k-1)z_1\psi_1 + z_1^2\psi_{11}}{z_2^2}.$$

Since, in (27), Π and $\varphi = \sigma^2$ are of dimension $-\frac{1}{2}$ and 6 respectively, we have

(29) $$z_2^2(\Pi, \sigma^2)_2 = 30\varphi\Pi_{11} + 15\varphi_1\Pi_1 + \tfrac{3}{4}\varphi_{11}\Pi.$$

We proceed to compute the right hand side of (29) in terms of ϱ and t, where Π denotes either one of the functions (23), and y, the corresponding function (19). Thus

(30) $$\Pi = \varrho^{-1/2}y \qquad \text{and} \qquad \varphi = \sigma^2 = \varrho^6 D^2.$$

For brevity, we write

$$u_\Phi = u, \qquad u_{\Phi'} = u', \qquad \text{etc.}$$

Thus

$$z_1 = \varrho u, \qquad z_2 = \varrho v.$$

Hence

(31) $$\frac{\partial\varrho}{\partial z_1} = -\frac{v'}{D}, \qquad \frac{\partial t}{\partial z_1} = \frac{v}{\varrho D}.$$

The computation yields the following values:

$$\Pi_1 = \varrho^{-3/2}\left\{\frac{v}{D}y' + \frac{1}{2}\frac{v'}{D}y\right\},$$

(32)

$$\Pi_{11} = \varrho^{-5/2}\left\{\frac{v^2}{D^2}y'' + \frac{v(3v'D - vD')}{D^3}y' + \frac{\frac{3}{4}v'^2D + \frac{1}{2}vv''D - \frac{1}{2}vv'D'}{D^3}y\right\},$$

$$\varphi_1 = \varrho^5\{2vD' - 6v'D\},$$

$$\varphi_{11} = \varrho^4\left\{2v^2\frac{D''}{D} - 14vv'\frac{D'}{D} + 30v'^2 - 6vv''\right\}.$$

On substituting these in (29) and reducing, we have

$$(\Pi, \sigma^2)_2 = 15\varrho^{3/2}\{2y'' - \varrho^2 Ty\}, \tag{33}$$

$$T = -\frac{1}{10}\left\{\frac{D''}{D} + 7\frac{v''}{v} - 7\frac{v'D'}{vD}\right\}.$$

From (20) and (30) it follows, since

$$Q(t) = \frac{\Gamma(x_1, \cdots, x_p)}{\varrho^2},$$

that

$$(\Pi, \sigma^2)_2 + 15\{\varrho^2 T + \Gamma(x)\}\Pi = 0, \tag{34}$$

and it remains to discuss the form T.

5. **The Forms** T, τ. It is to be observed that (33) is an identity in the sense that y may be any analytic function of t whatever, Π being then determined by (30); and that T is independent of y. It is possible, therefore, so to choose y that y and Π will be analytic in each point which corresponds to a root of v, and that y will not vanish there. Hence T must remain finite there, and this fact suggests that the terms of the brace,

$$7\frac{v''}{v} - 7\frac{v'D'}{vD} = 7\frac{v''D - v'D'}{vD},$$

admit an algebraic reduction. In fact, since

$$D = vu' - v'u$$

and

$$D' = vu'' - v''u$$

it appears that

$$v''D - v'D' = v(v''u' - v'u''),$$

and thus we have

$$T = -\frac{1}{10}\frac{D'' + 7(v''u' - v'u'')}{D}. \tag{35}$$

The two-rowed determinants which enter suggest the following notation:

$$\begin{vmatrix} u^{(j)} & v^{(j)} \\ u^{(k)} & v^{(k)} \end{vmatrix} = |u^j v^k|. \tag{36}$$

Thus we have finally

$$T = -\frac{1}{10}\frac{|u^3 v^0|-6|u^2 v^1|}{|u^1 v^0|}. \tag{37}$$

Invariant Property. From (34) we readily conjecture that $\varrho^2 T$, which is a form of dimension 2, is invariant under the automorphic group; i.e., if

$$t_\alpha = L_\alpha(t)$$

be a transformation of that group, then

$$T(t_\alpha) = L_\alpha'^{-2}(t)\,T(t). \tag{38}$$

The correctness of this surmise is readily proved by direct computation. Let

$$L_\alpha(t) = \frac{at+b}{ct+d}.$$

Then

$$L_\alpha'(t) = \frac{\Delta}{(ct+d)^2}, \qquad \Delta = ad-bc.$$

For convenience, let $\Delta = 1$. Furthermore, from (17),

$$v_\alpha = v(t_\alpha) = L_\alpha'^{-1}(t)v(t) = (ct+d)^2 v.$$

Hence

$$v'(t_\alpha)\,dt_\alpha = \frac{d}{dt}\{(ct+d)^2 v\}\,dt,$$

or

$$v_\alpha' = v'(t_\alpha) = (ct+d)^4 v' + 2c(ct+d)^3 v,$$

with like formulas for v_α'' and v_α'''.

On substituting in (37), (38) results.

The Form τ. Let τ be defined by the equation

$$\begin{aligned} \tau &= -\frac{1}{10}\varrho^5\{D'' + 7(v''u' - v'u'')\} \\ &= -\frac{1}{10}\varrho^5\{|u^3 v^0| - 6|u^2 v^1|\}. \end{aligned} \tag{39}$$

Then τ is seen to be an integral algebraic form of dimension 5 on either Σ_p or S_p, and

$$\varrho^2 T = \frac{\tau}{\sigma}. \tag{40}$$

Thus (B) is established.

The Function $[t]_z$ *in Terms of* t. We append the following formula:

$$\begin{aligned} [t]_z &= \frac{v^4}{D^3}\left\{2(v''u' - v'u'') - D'' + \frac{3}{2}\frac{D'^2}{D}\right\} \\ &= \frac{v^4}{D^3}\left\{\frac{3}{2}\frac{D'^2}{D} - [\,|u^3 v^0| + 3|u^2 v^1|\,]\right\}. \end{aligned} \tag{41}$$

6. **Computation of** τ **in** z_1, z_2. If we set Y equal to either of the functions (22), then

$$Y = \sqrt{\frac{dz}{dt}}\,y, \qquad \frac{dz}{dt} = \frac{D}{v^2},$$

and

$$H = \frac{z_2}{\sqrt{\sigma}}\,Y. \tag{42}$$

By computation similar to that of § 4 an identity analogous to (33) is obtained, namely

$$(H, \sigma^2)_2 = 30\frac{\sigma^{3/2}}{z_2^3}\,Y'' + \frac{9\sigma_1^2 - \frac{27}{2}\sigma\sigma_{11}}{z_2\,\sigma^{1/2}}\,Y. \tag{43}$$

On the other hand from (A_1), written in the form

$$[\eta]_z = [t]_z + \frac{z_2^4}{\sigma^2}\,\Gamma(x),$$

follows the equation (1) for Y, namely,

$$\frac{d^2 Y}{dz^2} + \left\{\frac{1}{2}[t]_z + \frac{1}{2}\frac{z_2^4}{\sigma^2}\,\Gamma(x)\right\}Y = 0. \tag{44}$$

From (42), (43), and (44) we now infer

$$(45)\qquad (\Pi, \sigma^2)_2 + 15\left\{\frac{\sigma^2}{z_2^4}[t]_z - \frac{6\sigma_1^2 - 9\sigma\sigma_{11}}{10 z_2^2} + \Gamma(x)\right\}\Pi = 0.$$

On comparing (45) with (B) we see that τ has the value

$$(46)\qquad \tau = \frac{\sigma^3}{z_2^4}[t]_z - \frac{\sigma(6\sigma_1^2 - 9\sigma\sigma_{11})}{10 z_2^2}.$$

Remark. It appears that τ, like σ, is an integral algebraic form belonging to the manifold Σ_p and uniquely determined by it. In terms of σ and τ the differential equation for t as a function on Σ_p assumes the form

$$(47)\qquad \frac{1}{z_2^4}[t]_z = \frac{\tau}{\sigma^3} + \frac{6\sigma_1^2 - 9\sigma\sigma_{11}}{10 z_2^2 \sigma^2}.$$

7. **The Parameters t and ϱ as Functions on Σ_p.** In the theory of the differential equations (1) and (2) it is a leading question to find conditions of scientific importance which determine uniquely the differential equation. One answer to this question was given by Klein through the method of conformal mapping* and consists in the case before us in the function t,—the inverse of the function (6), $z = \varphi(t)$,—and the differential equation which it satisfies,

$$(48)\qquad [t]_z = \varpi(z, s).$$

For t is uniquely determined save as to a linear transformation. The Schwarzian derivative is invariant of a linear transformation of t. Hence $\varpi(z, s)$ is uniquely determined on F, and thus becomes a function belonging to the surface. This function can be expressed in terms of σ and τ by means of (47):

$$(49)\qquad \varpi(z, s) = \frac{z_2^4}{\sigma^2}\left\{\frac{\tau}{\sigma} + \frac{6\sigma_1^2 - 9\sigma\sigma_{11}}{10 z_2^2}\right\}.$$

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* When the linear differential equation (2) is considered in the real domain, the theorems of oscillation can be used for a similar purpose.

Thus the parameter t admits the interpretation of being a solution of the differential equation on F given by Formula (48).

The other parameter, ϱ, can be interpreted by means of the linear differential equation of the second order for Π, Formula (B). If in (26) we set $\eta = t$, then one of the functions Π reduces to $\varrho^{-1/2}$. On the other hand, if we set $\eta = t$ in (A_2), $Q(t)$ vanishes identically; hence also $\Gamma(x)$. Thus the equation (B) which corresponds to $\eta = t$ reduces to

$$(\Pi, \sigma^2)_2 + 15 \frac{\tau}{\sigma} \Pi = 0, \tag{50}$$

and one solution of this equation is

$$\Pi = \varrho^{-1/2}.$$

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