A GENERAL THEORY OF LINEAR SETS*

BY

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INTRODUCTION

Section I of the following paper, though using the postulational method, is motivated by the consideration of classes of vectors, on a finite range \( P^n = (1, 2, \ldots, n) \), whose elements belong to a general division algebra or, as we shall say, number system.

Section II deals only with vectors on a finite range.

Section I is also of use as giving a general basis preliminary to the more intensive study of
(a) classes of vectors on a general range,
(b) number systems over a division number system; that is, to the initiation of a theory analogous to that of an "algebra over a field," where the field is replaced by an associative division number system.

Notation. Throughout the paper certain logical notations† will be used as follows:

\( = \) logical identity
\( \neq \) logical diversity
\( \equiv \) definitional identity
\( ::=:: \) definitional identity between statements
\( .) \) implies
\( \sim\) is equivalent to
\( .\) such that
\( \exists \) there exists
\( | \) is unique, used before the element which is unique: thus, \( |a \) means \( a \) is unique.

and

\( \lor \) or

\( \not\) not

\( .: : .:: \) etc. punctuation signs; the principal implication of a sentence has its sign accompanied by the largest number of periods, thus \( a .) b .) c \) is a statement that \( a \) implies that \( (b \) implies \( c) \) whereas \( a .) b :: c \) states that the implication \( a \) implies \( b \), implies the fact \( c \). We may also use punctuation to show continued implication, thus \( a .) b .) c \) means \( a .) b \) and \( b .) c \).

* Presented to the Society, April 11, 1925.

† These signs are mostly taken from E. H. Moore's Introduction to a Form of General Analysis. Yale University Press, 1910, p. 150.
\[ \sim \] corresponds to
\[ \ast (a) \] the statement \( \ast \) holds for every \( a \)
\[ [\ ] \] class of elements. A non-vacuous class we call a set.
\[ \cap [P] \] the greatest common subclass of the classes \( P \) of the class \([P]\) of classes
\[ \cup [P] \] the least common superclass of the classes \( P \) of the class \([P]\) of classes
\[ \subseteq \] inclusion. In speaking of classes \( M \) and \( N \), \( M \subseteq N \) means \( M \) includes \( N \), in the sense that every element of \( N \) is an element of \( M \). This may also be written \( N \subseteq M \).

The principal results will frequently be stated both in logical notation and in the written form; proofs, however, will as far as practical be given in logical notation only.

In dealing with subsets of the fundamental classes \( \mathfrak{A}, U, V, \) etc.,

\[ \mathfrak{A}_0 \equiv [a_0], \, \mathfrak{A}_1 \equiv [a_1], \, \ldots, \, U_0 \equiv [u_0], \, \ldots, \, V_0 \equiv [v_0], \, \ldots \] etc.

as subsets of \( \mathfrak{A}, U, V \) etc. respectively.

We shall use exponents to denote properties of an entity \( a \); for example \( \mathfrak{A}^A \) denotes that \( \mathfrak{A} \) is of type \( A \). When we use the notation for a class as an exponent of an element we shall mean that the element belongs to the given class; thus \( u_0^U \) means that \( u_0 \) is a member of the class \( U \).

If we have a single valued function, \( f \), of \( n \) independent variables whose values belong to the ranges \( P_1, \ldots, P_n \), and the functional values of \( f \) belong to a class \( M \) then we say that the function \( f \) is on \( P_1 \ldots P_n \) to \( M \), that is \( f^{on P_1 \ldots P_n} to M \).

**Number systems of type \( A \).** We will consider a number system which is a generalization of a "division algebra." We will define what we mean by a number system, \( \mathfrak{A} \), being of type \( A \), in such a way that whenever multiplication between every two elements of \( \mathfrak{A} \) is commutative, it follows that \( \mathfrak{A} \) is a field. If \( \mathfrak{A} \) is a field we will say \( \mathfrak{A}^F \). However as we do not assume that multiplication is commutative, there is introduced both a right and left distributive law.

We say that a system \( \mathfrak{A} \) is a number system of type \( A \) or symbolically \( \mathfrak{A}^A \) if it is of the following type:†

† This definition of a number system of type \( A \) is based directly on that used by E. H. Moore in his course in General Analysis. A number system of type \( A \) has properties 1–11 as given for real numbers by D. Hilbert but does not necessarily fulfill conditions 12–17. See D. Hilbert, *Grundlagen der Geometrie*, p. 35, 3d edition, 1909, or the translation by E. J. Townsend, *The Foundations of Geometry*, Open Court Publishing Company, 1902, p. 37.
\( \mathfrak{A} \) contains at least two distinct elements;

There is an addition function, \(+\), on \( \mathfrak{A} \mathfrak{A} \) to \( \mathfrak{A} \), which forms a commutative group, with identity of addition \( = 0 \);

There is a multiplication function, \( \times \), on \( \mathfrak{A} \mathfrak{A} \) to \( \mathfrak{A} \) which obeys the following restrictions:

1. \( 0 \times a = 0 = a \times 0 \) \((a)\);
2. \( \times \) on \( \mathfrak{A} \) except 0 forms a group (not supposed commutative);
3. \( a_1 \times a_2 = 0 \Rightarrow a_1 = 0 \lor a_2 = 0 \);
4. The identity of multiplication \( = 1 \);
5. \( a_1 \times (a_2 + a_3) = (a_1 \times a_2) + (a_1 \times a_3) \) \((a_1, a_2, a_3)\),
\[ (a_2 + a_3) \times a_1 = (a_2 \times a_1) + (a_3 \times a_1) \] \((a_1, a_2, a_3)\).

We will call such a system an associative division number system.

For simplicity of notation we will write \( a_1 \times a_2 = a_1 a_2 \). We will use the exponential method of denoting reciprocals thus:

\[ a_1 a_2 = a_3 a_1 = 1 \Leftrightarrow a_2^{-1} = a_1 \cdot a_1^{-1} = a_2. \]

For sake of clarity a few examples of number systems of type \( \mathfrak{A} \) will be given.

Ex. 1. All real rational numbers.
Ex. 2. The system, \( R \), of all real numbers.
Ex. 3. The system, \( C \), of all complex numbers.
Ex. 4. The system, \( Q \), of all real quaternions.
Ex. 5. Any Galois field \( GF[p^n] \);
\( \text{e.g. for } n = 1 \text{ the rational numbers modulo } p \).
Ex. 6.* The Hilbert example of a non-Archimedian Veronesean number system.

Consider \( P = (\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots) \) and a number system \( \mathfrak{A}^A \).
Consider \( F = \{ \text{all single valued functions } f: P \to \mathfrak{A} : f(a) = 0 \} \).
For a definiton of addition we have
\[ f = f_1 + f_2 \Leftrightarrow f(n) = f_1(n) + f_2(n) \] \((n)\).

For a definition of multiplication we have
\[ f = f_1 f_2 \Leftrightarrow f(n) = \sum_{j+k=n} f_1(j)f_2(k) \] \((n)\).

* This example is developed by E. H. Moore in his course in General Analysis. For \( \mathfrak{A} = R \) this is Hilbert's example of a non-Archimedian Veronesean number system. Loc. cit., p. 31, or trans., p. 34.
The identity of addition is \[ f_0(n) = 0 \ (n) \].
The identity of multiplication is \[ f_1(0) = 1, f_1(n) = 0 \ (n \neq 0) \].

\( F \) is of type \( A \).

Ex. 7.* Consider \( P = (\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots) \) and a number system \( \mathbb{A} \) of type \( A \).

Consider \( F[^{\text{on PP to } \mathbb{A}}_{\ldots \cdot \varepsilon: \cdot n_f \cdot \varepsilon: n_1 \leq n_f \cdot n_2 \cdot}. f(n_1 \cdot n_2) = 0 \cdot n \cdot \ldots \cdot \varepsilon: n_f \cdot n_1 \cdot f(n_1 \cdot n_2) = 0 \].

For a definition of addition we have \[ f = f_1 + f_2 : \equiv: f(n_1 \cdot n_2) = f_1(n_1 \cdot n_2) + f_2(n_1 \cdot n_2) \ (n_1 \cdot n_2). \]

For a definition of multiplication we have

\[
\begin{align*}
\alpha & = f_1 \cdot f_2 : \equiv: f(n_1 \cdot n_2) = \sum_{k_1 \cdot k_2 \cdot m_1 \cdot m_2} \alpha_{k_1 \cdot m_1} \cdot f_1(k_1 \cdot k_2) \cdot f_2(m_1 \cdot m_2) \ (n_1 \cdot n_2),
\end{align*}
\]

where \( \alpha \) is a number of \( \mathbb{A} \).

Our identity of addition is \[ f_0(n_1 \cdot n_2) = 0 \ (n_1 \cdot n_2) \].

Our identity of multiplication is \[ f_1(0 \cdot 0) = 1, f_1(n_1 \cdot n_2) = 0(n_1 \neq 0 \cdot n_2 \neq 0) \].

\( F \) is of type \( A \).†

**Theorem 1.** If \( \mathbb{A} \) is of type \( A \), and \( \mathbb{A}_0 \) is a subset of \( A \), then the totality \( \mathbb{A}_\infty \) of numbers of \( \mathbb{A} \) which are commutative as to multiplication with all the numbers of \( \mathbb{A}_0 \), forms with the original addition and multiplication a number system of type \( A \):

\[ \begin{align*}
\mathbb{A}_A \cdot \mathbb{A}_0 & = [a_0]_A, \mathbb{A}_\infty = [\text{all } a : a_0 \cdot]. a a_0 = a_0 a \cdot \cdot ;) \mathbb{A}_\infty A.
\end{align*} \]

The proof is evident when we note that \( a^{\infty_a \cdot} \cdot (a^{-1})^{\infty_a} = a^{-1} a_0 a \cdot a_0 a^{-1} = a^{-1} a_0 (a_0). \)

**Corollary.** If \( \mathbb{A} \) is of type \( A \), then the totality \( \mathbb{A}' \) of numbers of \( \mathbb{A} \) which are commutative as to multiplication with all the numbers of \( \mathbb{A} \), forms, with the original addition and multiplication, a field:

\[ \mathbb{A}_A \cdot \mathbb{A}' = [\text{all } a : a_1 \cdot]. a a_1 = a_1 a \cdot \cdot ;) \mathbb{A}^{AF}. \]

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* A special case of this example is given by Hilbert, loc. cit., p. 100, or trans., p. 103. J. H. M. Wedderburn brought this example to my attention in a more general form than that of Hilbert.

† The proof of this directly follows Hilbert’s proof. It should be noted that in both Ex. 6 and Ex. 7 Hilbert uses formal power series in one or two symbolic parameters respectively, rather than the functional notation used here.
It should be noted that $\mathcal{A}'$ is not necessarily a maximal field in $\mathcal{A}$. If, for example, $\mathcal{A} = \mathbb{Q}$ then $\mathcal{A}' = \text{scalars}$ but all the quaternions such that the coefficients of $j$ and $k$ are 0 form a field isomorphic with $\mathbb{C}$ and containing the scalars.

$\mathcal{A}$ may be of infinite rank in respect to $\mathcal{A}'$ as is shown by Example 7. Thus a number system of finite rank as regards a number system $\mathcal{A}$ might be of infinite rank as regards the field $\mathcal{A}'$ of which $\mathcal{A}$ is an extension.

I. THE THEORY OF LINEAR SETS

Contents

1. General postulational basis.
2. Sum and intersection of two sets, supplementary sets, additive sets, linear sets.
3. Interrelation of certain extensions of $U_0$.
4. Normal order, rank, difference sets.
5. Linear sets with commutative basis.

1. General postulational basis. In Section I we consider a system

$$\Sigma \equiv (\mathcal{A}, U \equiv [u], \oplus, \ominus, \otimes, \odot),$$

viz., a number system $\mathcal{A}$ of type $A$, a class $U$ of (abstract vectors or) elements $u$ and three processes or functions $\oplus, \ominus, \otimes$, serving to connect numbers $a$ and elements, $u$; as follows:

1. $U$ has at least two distinct elements.
2. $\oplus$ is a function on $UU$ to $U$ which forms a commutative group with identity element $0_U$; $u = u_1 \oplus u_2$, $u$ is the sum of $u_1$ and $u_2$.
3. $\ominus$ is a function on $U\mathcal{A}$ to $U$; $u = u_1 \ominus a$; $u$ is the product of $u_1$ by $a$ (on the right).
4. $\otimes$ is a function on $\mathcal{A}U$ to $U$; $u = a \otimes u_1$; $u$ is the product by $a$ (on the left) of $u_1$.
5.  $a \otimes 0_U = 0_U = 0_U \ominus a$.
6. $u \otimes 0 = 0_U = u \ominus 0$.
7. $u \otimes 1 = u = 1 \otimes u$.
8. Associative law of addition,

$$u_1 u_2 u_3 \cdot (u_1 \oplus u_2) \oplus u_3 = u_1 \oplus (u_2 \oplus u_3).$$

9. Distributive law,

$$u_1 u_2 a_1 a_2 \cdot (u_1 \ominus a_1) \oplus (a_1 \otimes a_2) = u_1 \ominus (a_1 + a_2),$$

$$a_1 \otimes (u_1 \ominus a_1) \oplus (a_2 \otimes u_1) = (a_1 + a_2) \otimes u_1,$$

$$(u_1 \ominus a_1) \oplus (u_2 \ominus a_1) = (u_1 \oplus u_2) \ominus a_1,$$

$$a_1 \otimes (u_1 \ominus a_2) = a_1 \otimes (u_1 \ominus u_2).$$
10. Associative law of multiplication,

\[ u \circ_r a_1 \circ_r a_2 = u \circ_r (a_1 a_2), \]
\[ u \circ_l (a_2 \circ_l u) = (a_1 a_2) \circ_l u, \]
\[ (a_1 \circ_l u) \circ_r a_2 = a_1 \circ_l (u \circ_r a_2). \]

From 9 and 10 the general distributive and associative laws follow. As it
will not lead to ambiguity we will simplify the notations as follows:

\[ u_1 \circ u_2 = u_1 + u_2; \]
\[ u \circ_r a = ua; \]
\[ a \circ_l u = au; \]
\[ (ua_1) a_2 = u(a_1 a_2) = ua_1 a_2; \]
\[ a_1 (a_2 u) = (a_1 a_2) u = a_1 a_2 u; \]
\[ a_1 (ua_2) = (a_1 u) a_2 = a_1 u a_2. \]

There follow two examples of a system \( \Sigma \).

Ex. 1.* Any algebra over a field.

It should be noted that our system \( \Sigma \) is more general in that we have
not limited \( \mathcal{A} \) to be a field and we have not required the existence of
a multiplication process between the elements of \( U \) (\( \circ \) on \( U \times U \) to \( U \)).

Ex. 2. If we are given a general range \( P \) and a number system \( \mathcal{A} \) of
type \( A \), then we may take as \( U \) the set \( F \) of all vectors \( f \) (single valued
functions), on \( P \) to \( \mathcal{A} \),—

\[ F = \{ \text{all } f^\text{on } P \text{to } \mathcal{A} \}, \quad \text{with} \]
\[ f = f_1 + f_2 \quad \text{:=} \quad f(p) = f_1(p) + f_2(p) \quad (p), \]
\[ f = af_1 \quad \text{:=} \quad f(p) = af_1(p) \quad (p), \]
\[ f = f_1 a \quad \text{:=} \quad f(p) = f_1(p)a \quad (p). \]

The symmetry between \( \circ_r \) and \( \circ_l \), and the symmetry between right and
left multiplication in \( \mathcal{A} \) should be noted. Each theorem will carry with
it a theorem by parity (not always different). As a convention, theorems
involving only one type of multiplication will be stated in terms of right
hand multiplication.

From Postulate 2 we know that

\[ u : \mathcal{A} | u_1 \cdot u + u_1 = u_1 + u = 0_U; \]

* See L. E. Dickson, *Algebras and their Arithmetics*, University of Chicago Press,
1923, p. 9.
The uniquely existing $u_1$ we designate the negative of $u$, in notation, $-u$, so that

$$u + u_1 = 0_U : \implies u_1 = -u.$$  

**Theorem 1.** The negative of any element of $U$ is that element multiplied on either right or left by the number $-1$:

**Th. 1.** $u \cdot (-1)u = -u = u(-1).$

**Proof.** $u = 1u$; then by Postulates 7, 8 and 9,

$$u + (-1)u = 1u + (-1)u = (1-1)u = 0u = 0_U.$$

**Theorem 2.** If the product of an element $u$ of $U$ by a number $a$ of $\mathbb{A}$ is $0_U$ then either $a$ is $0$ or $u$ is $0_U$ or both:

**Th. 2.** $au = 0_U := a = 0_U, u = 0_U.$

**Proof.** $a \neq 0 : \implies au = 0_U \therefore u = a^{-1}au = 0_U.$

**Theorem 3.** Relative to a subset $U_0$ of $U$, the totality $\mathbb{A}_0$ of numbers $a$ of $\mathbb{A}$ which are commutative as to multiplication with all the elements of $U_0$ forms, with the original addition and multiplication, a number system $\mathbb{A}_0$ of type $A$:

**Th. 3.** $U_0 : (\mathbb{A}_0 \equiv \text{all } a : 2, u_0 \text{ }, au_0 = u_0a)A.$

**Proof.**

1. $a_1u = ua_1, a_2u = ua_2 : (a_1 + a_2)u = n(a_1 + a_2), a_1a_2u = ua_1a_2,$

   for

   $$(a_1 + a_2)u = a_1u + a_2u = ua_1 + ua_2 = u(a_1 + a_2),$$

   $$(a_1a_2)u = a_1(a_2u) = a_1(na_2) = (a_1u)a_2 = (na_1)a_2 = u(a_1a_2).$$

2. $au = ua . \therefore a^{-1}u = ua^{-1}$

   for

   $$u = a^{-1}ua \text{ and therefore } a^{-1}u = ua^{-1}.$$

3. The set $\mathbb{A}_0$ contains at least two numbers, for it contains $0$ and $1$.

Other properties of $\mathbb{A}^A$ may be readily checked.

**Definition.** A set $U_0$ is said to be commutative, $U_0^c$, in case every number $a$ is commutative with every element $u_0$ of $U_0$:

**Def.** $U_0^c : \equiv au_0 \therefore au_0 = u_0a.$
2. Sum and intersection of two sets, supplementary sets, additive sets, linear sets.

Definition of the sum of two sets $U_1, U_2$.

$$U_1 + U_2 \equiv \{ u \text{ : there exists } u_1 \in U_1, u_2 \in U_2 \text{ such that } u = u_1 + u_2 \}.$$ 

Definition of the intersection of two sets $U_1, U_2$. The intersection of $U_1, U_2, \cap [U_1 U_2]$, is the greatest common subset of $U_1$ and $U_2$.

Definition of supplementary sets. Two sets $U_1$ and $U_2$ are supplementary, $(U_1 U_2)_{\text{sup}}$, if their sum is $U$ and their intersection is the set whose only element is $0_U$.

Definitions* of right linear (rl), left linear (ll), properly linear (l), and additive* (ad) sets.

$$U_0^{rl} := \{ a_1 a_2 u_{01} u_{02} \text{ : } (u_{01} a_1 + u_{02} a_2) \text{ belongs to } U_0 \};$$

$$U_0^{ll} := \{ a_1 a_2 u_{01} u_{02} \text{ : } (a_1 u_{01} + a_2 u_{02}) \text{ belongs to } U_0 \};$$

$$U_0^{l} := \{ U_0^{rl}, U_0^{ll} \};$$

$$U_0^{ad} := \{ u_{01} u_{02} \} \cdot (u_{01} + u_{02}) \text{ belongs to } U_0 \text{ and } u_0, (-u_0) \text{ belongs to } U_0.$$ 

It should be noted that any properly linear subset $U_0$ of $U$, other than the set consisting of the single element $0_U$, together with the number system $\mathfrak{A}$ and the original definition of addition and multiplication forms a system $\Sigma$ satisfying the postulates of § 1. In the sequel we shall often make use of this fact by applying theorems stated in terms of $U$ to a properly linear subset $U_0$ of $U$.

* We have not included in the text a definition of a linear set. A satisfactory definition of linearity would be such that any right (left) linear set is a linear set, and in case $U$ is commutative should reduce to the definition of right (left) linearity. In arriving at such a definition we make use of the number system $\mathfrak{A}$ consisting of all numbers $a$ of $\mathfrak{A}$ which are commutative with every element $u$ of $U$. $\mathfrak{A}$ is a field contained in $\mathfrak{A}'$. for by § 1, Theorem 3, $\mathfrak{A}$ is of type $\mathfrak{A}'$, and

$$a_e a u, (a_e a) u = a_e (au) = (au) a_e = a (ua_e) = a (a_e u) = (aa_e) u \text{ : } a_e a = aa_e.$$ 

We will say that a subset $U_0$ of $U$ is linear in case for every pair of numbers $a_{e1}$ and $a_{e2}$ in $\mathfrak{A}$, and for every pair of elements $u_{01}$ and $u_{02}$ of $U_0$, the element $u_{01} a_{e1} + u_{02} a_{e2}$ belongs to $U_0$.

† Note the symmetry of right and left linearity. When a theorem concerns only one of the two we will state it in terms of right linearity and omit the parity theorem in terms of left linearity.

‡ This is a strong form of the definition. One might use the first condition alone as a weaker form.
Theorem 1. U is properly linear.

Theorem 2. A right (or left) linear set is additive.

Theorem 3. Every additive set contains $0_U$; hence, the intersection of two additive sets is non-vacuous, and each of two additive sets is contained in their sum.

Theorem 4. If two sets $U_1$ and $U_2$ are additive (right linear, left linear or properly linear) then their sum and intersection are additive (right linear, left linear or properly linear).

Theorem 5. If $U_1$ and $U_2$ are additive sets and their intersection is the set whose only element is $0_U$, then any element in their sum can be expressed in one and only one way as the sum of one element of $U_1$ and one element of $U_2$:

\[ U_1^{\text{ad}} \cap U_2^{\text{ad}} = \{U_1 \cup U_2\} \]

\[ u_{11} = u_{12} = u_{21} = u_{22} . \]

Proof. $u_{11} - u_{12} = u_{22} - u_{21}$. Then, since $U_1$ and $U_2$ are additive, $u_{11} - u_{12}$ belongs to $U_1$ and $u_{22} - u_{21}$ belongs to $U_2$, hence $u_{11} - u_{12}$ belongs to $U_1 \cup U_2$. Therefore $u_{11} - u_{12} = 0_U$ and $u_{11} = u_{12}$ and hence $u_{21} = u_{22}$.

Theorem 6. Relative to a subset $\mathcal{A}_0$ of $\mathcal{A}$, the class $U_0$ of all elements $u_0$ of $U$ which are commutative with every number $a_0$ of $\mathcal{A}_0$ is additive and is properly linear in respect to the set $\mathcal{A}_0$ of all numbers of $\mathcal{A}$ which are commutative as to multiplication with every number of $\mathcal{A}_0$.

Proof. From the distributive and associative laws it follows that

\[ u_1 a_0 = u_2 a_0 = a_0 (u_1 + u_2) \]

\[ : (u_1 a_1 + u_2 a_2) = (a_0 + a_1 + a_2) . \]

Hence, since $\mathcal{A}_0$ contains the numbers 1 and $-1$, $U_0$ is additive and is properly linear in respect to $\mathcal{A}_0$.

In his Introduction to a Form of General Analysis,* E. H. Moore introduces the notion of the extensional attainability of a property $P$, defined for the subclasses $M_0$ of $M$. Considering a class $M$ and a property $P$ defined for the subclasses $M_0$ of $M$, we say that the property $P$ is extensionally attainable in case "for every subclass $M_0$ of $M$ there exists a class $M_{0P}$ containing $M_0$ and contained in $M$, having the property $P$ and such that every subclass of $M$ which contains $M_0$ and has the property $P$ contains $M_{0P}$.”

* Loc. cit., p. 54.
or, what is equivalent, (1) $M$ has the property $P$, and (2) for every class $M_0$ the greatest common subclass of all classes containing $M_0$ and having the property $P$ has the property $P$. For such a property $P$, the $P$-extension of $M_0$ is the class $M_0P$ of the first definition and also the greatest common subclass etc., of the second definition: In notation

$$M_0P = \bigcap \{\text{all } M^P \models M_1 \models M_0\}.$$ 

It is important to note that $M_0P \subseteq M_1 \subseteq M_0$ :): $M_0P = M_1P$.

**Theorem 7.** The properties of additivity, right linearity, left linearity and proper linearity are extensionally attainable in $U$.

**Proof.** (The proof is given for right linearity only.)

(1) $U$ is right linear by Theorem 1.

(2) $U_0 : U_0r = \bigcap \{\text{all } U^r \models U_1 \supset U_0\}$ :): $U_0r$ by definition of right linearity.

Accordingly we introduce notations, for the various extensions of $U_0$ as follows:

- $A_dU_0$ = the additive extension of $U_0$;
- $L_rU_0$ = the right linear extension of $U_0$;
- $L_lU_0$ = the left linear extension of $U_0$;
- $LU_0$ = the properly linear extension of $U_0$.

**Theorem 8.** The right linear extension of any subset $U_0$ of $U$ is the totality of all right linear combinations of elements of $U_0$:

**Th. 8.** $U_0 : U_0r = \left[\text{all } u : a_1 u_1, \ldots, a_n u_1, \ldots, a_1 u_1, \ldots, a_n u_1. \models u = \sum_i a_i u_i \right]$ :): $U_0r = L_rU_0$.

**Proof.** Obviously $L_rU_0 \supset U_0r$, and $U_0r \supset U_0$ and is right linear, and therefore $U_0 \supset L_rU_0$.

Due to the symmetry between right and left linearity we will in general state theorems involving only $L_r$ or $L_l$ in terms of $L_r$ and omit the theorem that follows by parity.

3. Interrelation of certain extensions of $U_0$. In this section we consider the iteration of the four processes $A_d; L_r; L_l; L$ and a new process $L_0$; where $L_0U_0$ is defined as the (provably existent) maximal properly linear subset of the intersection of the right and left linear extensions of $U_0$. Moreover, it is shown that these processes along with the iterations $L_0L_r$ and $L_0L_l$ are closed under further iteration.
We also consider how far the seven sets $AdU_0$, $L_rU_0$ etc. are determined from a knowledge of certain of them.

**Theorem 1.** The properly linear extension of any subset $U_0$ of $U$ is the right linear extension of the left linear extension of $U_0$ and by parity the left linear extension of the right linear extension of $U_0$:

**Th. 1.** $U_0 \land L_rL_lU_0 = LU_0 = L_lL_rU_0$.

Proof. (1) $LU_0 \sqsubset L_rL_lU_0$, for $LU_0 \sqsubset L_lU_0 \sqsubset U_0$ and therefore $LU_0 = LL_lU_0 \sqsubset L_rL_lU_0$.

(2) $L_rL_lU_0$ is properly linear, for by definition $(L_rU_1)^{i_1}$ and $L_rL_lU_0$ is left linear, for according to § 2, Theorem 8, every element of $L_rL_lU_0$ is of the form

$$\sum_{i}^{1,n_0} \left( \sum_{j}^{1,n_1} a_{ij} u_{0ij} \right) a_i,$$

where $n_0$ and $n_1$ ($i$) are positive integers and $u_{0ij}$ belongs to $U_0$ ($i,j$) and conversely, and the distributive and associative laws are holding.

Therefore $L_rL_lU_0 = LU_0$ and similarly $L_lL_rU_0 = LU_0$.

**Theorem 2.** $U_1 U_2$ :\): (1) $L_r[U_1 U_2] = L_r U_1 + L_r U_2$;
(2) $LU[U_1 U_2] = LU_1 + LU_2$;
(3) $Ad[U_1 U_2] = AdU_1 + AdU_2$.

The proof follows directly from § 2, Theorem 8 and Theorem 1.

**Theorem 3.** $U_1^{ad} U_2^{ad}$ :\): (1) $L_r[U_1 + U_2] = L_r U_1 + L_r U_2$;
(2) $LU[U_1 + U_2] = LU_1 + LU_2$;
(3) $Ad[U_1 + U_2] = AdU_1 + AdU_2$.

**Theorem 4.** Relative to an additive subset $U_0$ of $U$, there exists a unique maximal properly linear subset $U_{00}$ of $U_0$ such that all properly linear subsets of $U_0$ are contained in $U_{00}$:

**Th. 4.** $U_0^{ad}$ :\): $\forall [U_{00} \sqsubset U_0 : s: U_{01} \sqsubset U_0 \quad U_{00} \sqsubset U_{01}$.

Proof. $U_1 \equiv [\forall u. Lu \sqsubset U_0]$ is effective as $U_{00}$; for
(1) $U_1$ contains the proper linear class $[0_U]$;
(2) $U_{01} \sqsubset U_0 \quad U_1 \sqsupset U_{01}$;

This may also be written in the form

$$\sum_{i}^{1,n} a_{1i} u_{0i} a_{0i}.$$

In this case we may not assume as above that the elements $u_{0i}$ are distinct.
(3) \(U_1\) is properly linear, for a right or left linear combination of elements of \(U_1\) is the sum of elements of \(U_1\) and therefore \(LU_1 = AdU_1\) and \(U_0 \supseteq LU_1\), hence by (2) \(U_1 \supseteq LU_1\) and is properly linear.

Since relative to a subset \(U_0\) of \(U\), \(\cap [Lr U_0 Lr U_0]\) is additive, it follows from Theorem 4 that the following definition has content:

**Definition of \(L_0 U_0\).** Relative to a subset \(U_0\) of \(U\) we define \(L_0 U_0\) as the maximal properly linear subset contained in the intersection of the right and left linear extensions of \(U_0\).

**Theorem 5.** Relative to a right linear subset \(U_0\) of \(U\) the maximal properly linear subset \(U_{00}\) of \(U_0\) is \(L_0 U_0\).

**Proof.** \(U_{00} = Lr U_0 \cap Lr U_0 = U_0\).

**Theorem 6.** \(U_0 = \cap [L_0 Lr U_0, L_0 Lr U_0]\).

The following table shows the sets generated from a set \(U_0\) by iterated processes of the types \(Ad, Lr, Ll, L_0\). For example, column 3, row 6 shows us that \(U_0 \supseteq L_0 Lr (L_4 U_0) = LU_0\).

<table>
<thead>
<tr>
<th>(Ad)</th>
<th>(Lr)</th>
<th>(Ll)</th>
<th>(L)</th>
<th>(L_0)</th>
<th>(L_0 Lr)</th>
<th>(L_0 Ll)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Ad)</td>
<td>(Ad)</td>
<td>(Lr)</td>
<td>(Ll)</td>
<td>(L)</td>
<td>(L_0)</td>
<td>(L_0 Lr)</td>
</tr>
<tr>
<td>(Lr)</td>
<td>(Lr)</td>
<td>(Lr)</td>
<td>(L)</td>
<td>(L_0)</td>
<td>(L_0 Lr)</td>
<td>(L_0 Ll)</td>
</tr>
<tr>
<td>(Ll)</td>
<td>(Ll)</td>
<td>(Ll)</td>
<td>(L)</td>
<td>(L_0)</td>
<td>(L_0 Lr)</td>
<td>(L_0 Ll)</td>
</tr>
<tr>
<td>(L)</td>
<td>(L)</td>
<td>(L)</td>
<td>(L)</td>
<td>(L_0)</td>
<td>(L_0 Lr)</td>
<td>(L_0 Ll)</td>
</tr>
<tr>
<td>(L_0)</td>
<td>(L_0)</td>
<td>(L_0 Lr)</td>
<td>(L_0 Ll)</td>
<td>(L_0)</td>
<td>(L_0 Lr)</td>
<td>(L_0 Ll)</td>
</tr>
<tr>
<td>(L_0 Lr)</td>
<td>(L_0 Lr)</td>
<td>(L_0 Lr)</td>
<td>(L_0)</td>
<td>(L_0 Lr)</td>
<td>(L_0 Ll)</td>
<td>(L_0 Ll)</td>
</tr>
<tr>
<td>(L_0 Ll)</td>
<td>(L_0 Ll)</td>
<td>(L_0 Ll)</td>
<td>(L_0)</td>
<td>(L_0 Lr)</td>
<td>(L_0 Ll)</td>
<td>(L_0 Ll)</td>
</tr>
</tbody>
</table>

The proof of the above table is readily obtained when we bear in mind that \(U_0 \supseteq \cap [Lr U_0 Lr U_0] \supseteq AdU_0 \supseteq U_0\).

The fact that relative to a given \(U_0\) the seven sets \(AdU_0\) etc. may be distinct is shown by an example following Table I in II, § 2.

From an examination of Table I it is seen that the iteration of the seven processes \(Ad\) etc., is closed and associative. Moreover, the iteration of the four processes \(Ad, Lr, Ll,\) and \(L\) is closed.

Table II shows which of the seven sets \(AdU_0\) etc., previously introduced, are determined when any particular combination of them is given. We do not list all the \(2^7 - 1\) different combinations, but we give certain combinations, into which all of the \(2^7 - 1\) can be decomposed, and such that no combination will determine, in general, more than could be determined from the component parts as listed in the table.
The proof that the table is correct follows at once from Table I and Theorem 6.

The following considerations and examples show that Table II is complete:

Whenever the determination of certain sets determines certain others uniquely it follows that no extra knowledge is gained by adding these others to the original sets in the first column of Table II. Thus, since the determination of $L_r U_0$ determines $L_0 L_r U_0$ it follows that we need not add $L_r, L_0 L_r$ to the combinations in the first column.

In Table III we give examples showing the completeness of Table II. In each of these examples we consider a finite range $P^n$, where $n$ is a positive integer, $\mathbb{A} \equiv Q$ (quaternions) and $U \equiv V$ the class of all vectors $v$ on $P$ to $Q$. We have a system satisfying the postulates of § 1 when multiplication and addition are defined as in § 1, Ex. 2 of a system $\Sigma$. We will display the vectors $v$ as rows of ordered elements thus: $(a_1 (i = 1, \ldots, 5)) \equiv (a_1 a_2 a_3 a_4 a_5)$.

In these examples we display subsets $V_1$ and $V_2$ of such a nature that the sets arising from $V_1$ by processes listed in column 2 of Table III are equal respectively to those arising from $V_2$ by the same processes; but the sets arising from $V_1$ differ respectively from those arising from $V_2$ except for sets as shown by Table II to be uniquely determinable from those we have assumed to be equal. Thus, in Example 1, $L_r V_1, LV_1, L_0 L_r V_1$ are respectively equal to $L_r V_2, LV_2, L_0 L_r V_2$, but $Ad V_1, L_1 V_1, L_0 V_1, L_0 L_1 V_1$ differ respectively from $Ad V_2, L_1 V_2, L_0 V_2, L_0 L_1 V_2$.

When an example in Table III is given showing the completeness of Table II for any particular combination of sets $Ad V_0$ etc. the example for

### Table II

<table>
<thead>
<tr>
<th>Combinations of sets given</th>
<th>Sets determined uniquely</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ad$</td>
<td>$Ad, L_r, L_l, L, L_0, L_0 L_r, L_0 L_l$</td>
</tr>
<tr>
<td>$L_r$</td>
<td>$L_r, L, L_0 L_r$</td>
</tr>
<tr>
<td>$L_l$</td>
<td>$L_l, L, L_0 L_l$</td>
</tr>
<tr>
<td>$L$</td>
<td>$L$</td>
</tr>
<tr>
<td>$L_0$</td>
<td>$L_0$</td>
</tr>
<tr>
<td>$L_0 L_r$</td>
<td>$L_0 L_r$</td>
</tr>
<tr>
<td>$L_0 L_l$</td>
<td>$L_0 L_l$</td>
</tr>
<tr>
<td>$L_r, L_l$</td>
<td>$L_r, L_l, L, L_0, L_0 L_r, L_0 L_l$</td>
</tr>
<tr>
<td>$L_r, L_0 L_l$</td>
<td>$L_r, L_0 L_l, L, L_0 L_r, L_0$</td>
</tr>
<tr>
<td>$L_l, L_0 L_r$</td>
<td>$L_l, L_0 L_r, L, L_0 L_l, L_0$</td>
</tr>
<tr>
<td>$L_0 L_r, L_0 L_l$</td>
<td>$L_0 L_r, L_0 L_l, L_0$</td>
</tr>
</tbody>
</table>
the same sets with left and right interchanged is immediately securable by parity.

<table>
<thead>
<tr>
<th>Ex. No.</th>
<th>Example shows completeness of Table II for</th>
<th>$P_n^m$</th>
<th>$V_1$</th>
<th>$V_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$L_r (L_t)$</td>
<td>2</td>
<td>(1 0)</td>
<td>(0 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>$L$</td>
<td>2</td>
<td>(1 0)</td>
<td>(0 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>$L_0$</td>
<td>3</td>
<td>(0 0 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>$L_0 L_r$ $(L_0 L_t)$</td>
<td>5</td>
<td>(1 0 0 0</td>
<td>(1 i 0 0 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0 0 1 0)</td>
<td>(0 0 1 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0 0 0 1)</td>
<td>(0 0 j 0)</td>
</tr>
<tr>
<td>5.</td>
<td>$L_r, L_t$</td>
<td>1</td>
<td>(1)</td>
<td>(i)</td>
</tr>
<tr>
<td>6.</td>
<td>$L_r, L_0 (L_4, L_0)$</td>
<td>2</td>
<td>(1 i)</td>
<td>(j k)</td>
</tr>
<tr>
<td>7.</td>
<td>$L_r, L_0 L_t$ $(L_4, L_0 L_r)$</td>
<td>2</td>
<td>(1 i)</td>
<td>(j k)</td>
</tr>
<tr>
<td>8.</td>
<td>$L, L_0$</td>
<td>2</td>
<td>(1 i)</td>
<td>(j k)</td>
</tr>
<tr>
<td>9.</td>
<td>$L, L_0 L_r$ $(L, L_0 L_t)$</td>
<td>4</td>
<td>(1 0 0 0)</td>
<td>(1 i 0 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0 1 0 0)</td>
<td>(j 0 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0 0 1)</td>
<td>(0 0 1)</td>
</tr>
<tr>
<td>10.</td>
<td>$L_0, L_0 L_r$ $(L_0, L_0 L_t)$</td>
<td>3</td>
<td>(1 0 0)</td>
<td>(j 0 0)</td>
</tr>
<tr>
<td>11.</td>
<td>$L_0 L_r, L_0 L_t$</td>
<td>3</td>
<td>(1 0 0)</td>
<td>(1 0 0)</td>
</tr>
<tr>
<td>12.</td>
<td>$L, L_0, L_0 L_r$ $(L, L_0, L_0 L_t)$</td>
<td>2</td>
<td>(1 i)</td>
<td>(j k)</td>
</tr>
<tr>
<td>13.</td>
<td>$L, L_0 L_v, L_0 L_t$</td>
<td>2</td>
<td>(1 i)</td>
<td>(j k)</td>
</tr>
</tbody>
</table>

4. Right (left) linear independence, normal order, basis, rank, supplementary and difference sets. In this section we consider a system $\Sigma$ such that the set $U$ is normally ordered. In this case we show that a right linear subset $U_0$ of $U$ contains a right linearly independent base, and that every such base has the same cardinal number, which we call the right rank of $U_0$. Moreover we show that relative to a right linear subset $U_0$ of $U$ there exists a supplementary right linear set; and that relative to a properly linear subset $U_0$ of $U$ all properly linear supplementary sets are isomorphic with the difference set $\bar{U} -- U_0$ which in definition is analogous to a difference algebra.
We say a subset $U_0$ of $U$ is right linearly independent* as to $\mathfrak{A}_0$ in case

$U_0 \mathfrak{A}_0 \ldots \mathfrak{A}_n (u_{01} \ldots u_{0n})^{\text{distinct}} a_{01} \ldots a_{0n}$

\[ \vdash: \sum_{i=1}^{n} u_{0i} a_{0i} = 0_U . \ldots a_{0i} = 0 \quad (i = 1, \ldots, n). \]

In case $U_0$ is right linearly independent as to $\mathfrak{A}$ we say $U_0$ is right linearly independent ($U_0^{\text{rl}}$). If $U_0$ is right linearly dependent we use the notation $U_0^{\text{ld}}$. We say $U_0$ is right linearly independent as to $U_1$ in case $L_U U_1$ does not contain any members of $U_0$. Definitions of left linear independence as to $\mathfrak{A}_0$ etc. follow at once by parity.

**Theorem 1.** If a subset $U_0$ of $U$ is commutative and right linearly independent as to $\mathfrak{A}'$, $U_0$ is right linearly independent.

**Proof.** 1. $U_0$ does not contain $0_U$.

2. If the theorem were not true we would have $\sum_i u_{0i} a_i = 0_U$, where the elements $u_{0i}$ are distinct and the numbers $a_i$ are $\neq 0$. Then

\[ \sum_i u_{0i} a_i a_i^{-1} + u_{0n} = 0_U; \]

moreover since $u_{01}, \ldots, u_{0n}$ are right linearly independent as to $\mathfrak{A}'$ $\mathfrak{A} (j a \neq 0) . \forall. (a a_j a_i a_i^{-1} a_i a_i^{-1}) \neq 0$ and since $U_0$ is commutative

\[ \sum_i u_{0i} a a_i a_i a_i^{-1} a_i a_i^{-1} + u_{0n} = 0_U; \]

hence from (1) and (2) it follows that

\[ \sum_i u_{0i} (a a_i a_i a_i^{-1} a_i a_i^{-1}) = 0_U; \]

and therefore

\[ \mathfrak{A} n_1 : n \geq n_1 > 0 \quad (u_{01} \ldots u_{0n})^{\text{distinct} . \text{ld}}. \]

By repetition of the above reasoning we conclude that

\[ \mathfrak{A} (u_0 \neq 0_U a \neq 0) . \forall. u_0 a = 0_U, \]

and since this is impossible our theorem is valid.

In the remainder of this section we will make use of the notion of normal order. "Normally ordered" is here used synonymously with well ordered

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*This could more exactly be called finite right linear independence but as we do not consider the infinite case in this paper we shall use the shorter term.*
Thus we say a subset $U_0$ of $U$ is normally ordered ($U_0^{no}$) in case $U_0$ is linearly ordered in such a way that every subset $U_{01}$ has a first element. We shall use the notation $U_{0u}$, where $u$ is an element of a normally ordered set containing $U_0$, to denote all the elements of $U_0$ which precede $u$.

We recognize that many accept the Zermelo principle of selection or the multiplicative axiom and therefore feel that the Zermelo demonstration of the normal orderability of any aggregate warrants the assumption of such normal orderability without further stipulation; yet as many do not agree with this point of view we will introduce normal order explicitly whenever we wish to make use of it.

**Definition.** We say that $U_1$ is a right base for $U_0$ in case $L_r U_1 = U_0$. If, besides, $U_1$ is right linearly independent we say that $U_1$ is a proper right base for $U_0$. The definitions of a left base and a proper left base follow at once by parity.

**Theorem 2.** If $U$ is normally ordered and if $U_*$ is the set of all elements $u$ of $U$ which are right linearly independent of the preceding elements then $U_*$ is a proper right base for $U$.

**Proof.**
1. $U_*$ is non-vacuous, for $U$ has a first element different from $0_U$.
2. $L_r U_* = U$. We use the indirect method of proof for this. If it is not so, then $\exists u . L_r U_*$ does not contain $u$, and hence there exists a first such $u$, say $u_0$. Since $u_0$ is not a member of $U_*$ it is a right linear combination of elements of $U_{u_0}$, all of which are themselves right linear combinations of elements of $U_*$. Hence $u_0$ is a right linear combination of elements of $U_*$, in contradiction to our hypothesis that $L_r U_*$ did not contain $u_0$.
3. $U_*$ is right linearly independent, for, if not,

$$\exists ((u_1 \ldots u_n)^{distinct} u_\ast a_1 \neq 0 \ldots a_n \neq 0) . \sum_i u_i a_i = 0_U,$$

which is in contradiction to the definition of $U_\ast$.

**Theorem 3.** If there exists a normally ordered right base $U_*$ for $U$, then there exists a normally ordered proper right base $U_{0*}$ for every right linear subset $U_0 = [0_U]$ of $U$.

**Proof.** By reasoning analogous to that used in the proof of Theorem 2 we see that $[u_* \ast . u_* is right linearly independent as to U_{u_*}] \equiv U'$ is a proper right base for $U$.

From § 2, Theorem 5, it follows that every $u$ is expressible uniquely in the form $u = \sum_i u'_i a_i$, where $a_i \neq 0 (i)$ and $i < \ast$. $u_i$ precedes $u'_j$.

Consider $U_1 = [u_0 \ast . a_1 (for u_0) = 1]$; obviously $L_r U_1 = U_0$. 

...
Consider the class \([U_p']\) of all sets \(U_p'\) consisting of a finite number of elements of \(U'\). We can normally order this class as follows:

- \(U_1'\) precedes \(U_2'\) if
  1. \(U_1'\) has fewer members than \(U_2'\);
  2. \(U_1'\) has the same number of elements as \(U_2'\), but the first member of \(U_1'\) not in \(U_2'\) precedes the first member of \(U_2'\) not in \(U_1'\).

Corresponding to every set \(U_p'\) there exists a class \(U_{1p}\) consisting of all elements of \(U_1\) which are right linear combinations with non-zero coefficients of all elements of \(U_p'\), and no other elements. The sets \(U_{1p}\) (i.e., the non-vacuous classes \(U_{1p}\)) are in 1-1 correspondence with the sets \(U_p'\) from which they arise. Every element \(u_i\) of \(U_1\) falls in one and only one such set. We say that the set \(U_{11}\) precedes \(U_{12}\) provided \(U_1'\) precedes \(U_2'\).

No element \(u_p\) is right linearly independent of the elements in the sets preceding the set to which it belongs unless it is the only element in this set. For consider a set \(U_{11}\) containing two distinct elements \(u_{111}\) and \(u_{112}\) where

\[
\begin{align*}
  u_{111} &= u_{11} + \sum_{i=1}^{n} u_{1i} a_{1i}, \\
  u_{112} &= u_{11} + \sum_{i=1}^{n} u_{1i} a_{2i},
\end{align*}
\]

where \(n\) is the number of elements in \(U'\) and no multiplier \(a_{1i}\) or \(a_{2i}\) is zero. Hence \(L_r[u_{111}, u_{112}]\) contains \(u_{111} - u_{112} = u_{03}\) and \(u_{03} \neq 0\), and hence there exists a number \(a\) such that \(u_{03} a\) belongs to \(U_1\) and is in a set which precedes \(U_{11}\). Moreover there exists a number \(a_1\) such that \(u_{111} - u_{03} a_1 = u_{04}\) is in a set preceding \(U_{11}\). However, it is evident that \(L_r[u_{03} a, u_{04}]\) contains both \(u_{111}\) and \(u_{112}\).

Denote by \(U_2\) the totality of elements of \(U_1\) which are the only members of the sets to which they respectively belong. These elements may be normally ordered as were the sets to which they belonged and constitute a right base for \(U_0\), and hence by reasoning analogous to that used in the proof of Theorem 2 it can be readily shown that \([\text{all } u_2 : : u_2\text{ is right linearly independent as to } U_{2n}, u_1]\) is a normally ordered proper right base for \(U_0\).

**Theorem 4.** If a subset \(U_0\) of \(U\) is right linear and if \(U_1\) and \(U_2\) are normally ordered proper right bases for \(U_0\), then \(U_1\) and \(U_2\) have the same cardinal number.

**Proof.** For every \(u_1\) consider

\[
\begin{align*}
  U_{3u} &= \text{[all } u_2 \text{ right linearly independent as to } U[U_{2n}, U_{1n}],} \\
  U_{4u} &= \text{[all } u_2 \text{ right linearly independent as to } U[U_{2n}, U_{1n}, u_1].}
\end{align*}
\]
obviously, for every $u_1$, $U_{3u} \supset U_{4u}$ and $L_r U \cup [U_{1u}, U_{3u}] = U_0$, but $L_r U_{1u}$ does not contain the element $u_1$. Hence every element $u_1$ of $U_1$ is a right linear combination with non-zero coefficients of at least one element of $U_{3u}$ and elements of $U_{1u}$. Therefore $U_{3u}$ contains at least one element not in $U_{4u}$. Let $u_1$ correspond to the first $u_2$ belonging to $U_{3u}$ but not in $U_{4u}$.

Moreover, it is obvious that

$$u_{11} \sim u_{21}, u_{12} \sim u_{22}, u_{11} \neq u_{12} \iff u_{21} \neq u_{22},$$

and hence there exists a one to one correspondence between $U_1$ and a part of $U_2$ and similarly between $U_2$ and a part of $U_1$.

**Theorem 5.** If a right linear subset $U_0$ of $U$ has a normally ordered proper right base $U_1$, then any other proper right base $U_2$ for $U_0$ can be normally ordered.

**Proof.** Consider the class $[L_p]$ of all finite sets of elements of $U_1$. This class may be normally ordered. Corresponding to every such set $U_{1p}$ consider the totality $U_{2p}$ of elements of $U_2$ which are right linear combinations with non-zero coefficients of all the elements of $U_{1p}$. Every element of $U_2$ falls in one and only one such set. Since $U_2$ is right linearly independent the number of elements in any set $U_{2p}$ can not exceed the number of elements in $U_{1p}$. Hence $U_2$ consists of the members of a normally orderable class of finite sets of elements and is therefore normally orderable.

**Definitions of right (left) rank.** If a subset $U_0$ of $U$ is right linear and there exists a normally ordered proper right base for $U_0$ we say that the cardinal number of such a right base is the right rank of $U_0$ ($rk_r (U_0)$).

**Theorem 6.** If the subsets $U_1$ and $U_2$ of $U$ are right linear, have normally ordered right bases, and $U_1 \supset U_2$, then

$$rk_r (U_1) \geq rk_r (U_2).$$

**Corollary.** Relative to a subset $U_0$ of $U$ of such a nature that there exists a normally ordered right base for $LU_0$

$$rk_r (LU_0) \geq rk_r (L_r U_0) \geq rk_r (L_0 L_r U_0) \geq rk_r (L_0 U_0).$$

**Note:** According to Theorem 3 all of these ranks exist.

**Theorem 7.** If there exists a normally ordered right base $U_u$ for $U$, then relative to a right linear subset $U_0$ of $U$ there exists a right linear subset $U_1$ of $U$ such that $U_0$ and $U_1$ are supplementary.

**Proof.** Case 1. $U_0 = U$. In this case $[0_U]$ is effective as $U_1$ of the theorem.
Case 2. \( U_0 \not\subseteq U \). Consider \( U_{1*} = \{ \text{all } u_\ast : u \text{ belongs to } U_1 \} \); \( L_r U_{1*} \) is effective as \( U_1 \) of the theorem.

1. By proof analogous to Theorem 2 and by § 3, Theorem 3, we see that

\[
U = L_r U_1 + L_r U_0 = L_r U_{1*} + U_0;
\]

2. \( \cap \{ U_0, L_r, U_{1*} \} \) contains only \( U_0 \), for if it contained an element \( u_0 \) of \( U_0 \) such that \( u_0 \not\in U_0 \), then \( u_0 \) would be a right linear combination with non-zero coefficients of elements of \( U_{1*} \) and hence there would exist an element of \( U_{1*} \) not right linearly independent as to \( U_0 \) and the preceding elements of \( U_{1*} \).

**Definition of a difference set.** Relative to a properly linear subset \( U_0 \) of \( U \) we define the difference of \( U \) and \( U_0 \) \( (U - U_0) \) as follows:

\[
U - U_0 = \{ \text{all } u : u \text{ belongs to } U_1, \} + U_0 = U_1 - \{ u \}.
\]

It is seen that \( U - U_0 \) is not itself a set of elements of \( U \) but a class of sets of elements such that the members of any one set differ from each other by an element of \( U_0 \) and all elements of \( U \) which differ by an element of \( U_0 \) belong to the same set.

**Lemma 1.**

\[
U_0 \cdot \{ u \} = \{ u + u_0 \} \quad (u)
\]

**Lemma 2.**

\[
U_0 \cdot \{ u \} = \{ u + u_0 \} \quad (u)
\]

From the lemmas and definitions above it follows at once that relative to a properly linear subset \( U_0 \) of \( U \) where \( U_0 \not\subseteq U \) the difference, \( U - U_0 \), together with the number system \( \mathbb{A} \) and with addition defined as in Lemma 2 and multiplication as in Lemma 1, forms a system \( \Sigma \) satisfying the postulates of I, § 1.

**Theorem 8.** If two properly linear subsets \( U_0 \) and \( U_1 \) of \( U \) are supplementary then \( U_1 \) is isomorphic with \( U - U_0 \) under the correspondence \( u_1 \sim \{ u \} \).

**Proof.** 1. \( u_{11} \sim u_{12} . \quad \{ u_{11} \} \sim \{ u_{12} \}, \) for if not \( u_{11} - u_{12} \) would belong both to \( U_0 \) and \( U_1 \), which is impossible since \( U_0 \) and \( U_1 \) are supplementary.

2. In every set \( \{ u \} \) there exists an element of \( U_1 \), for \( \mathbb{A} u_0, u_1 . \quad u = u_0 + u_1 \) and therefore \( \{ u \} \sim \{ u_1 \} \).

† For an abstract definition of a difference algebra see L. E. Dickson, *Algebras and their Arithmetics*, p. 36 ff. Our definition could be made more general by not requiring that \( U_0 \) be properly linear, but many of the most useful properties would not be preserved. We therefore limit ourselves to this case.
3. The preservation of the correspondence under addition and multiplication follows from the fact that $U_0$ and $U_1$ are properly linear and from Lemmas 1 and 2.

5. Systems with a commutative base. In this section we will consider a system $\Sigma$ of such a nature that there exists a proper right base $U_*$ of $U$ which is commutative with $\mathfrak{A}$; hence $U_*$ is a proper left base for $U$. We show that in this case $U$ is isomorphic with the set of all finitely non-zero vectors on a certain range $P$ and that any properly linear subset $U_0$ of $U$ has a commutative right base and conversely.

If $\Sigma$ is such that $U$ has a commutative proper right base then we say that $\Sigma$ is of type 1.

Note. That not all systems $\Sigma$ are of type 1 is seen from the following example.

Consider both $\mathfrak{A}$ and $U$ as the Hilbert example† of a Veronesean number system. Associate with every number $a \equiv (a(i) \mid i = -\infty \ldots + \infty)$ the number $a'$ where $a' \equiv (a'(2i) = a(i) \cdot a'(2i+1) = 0 \mid i = -\infty \ldots + \infty)$. We define the processes $\oplus$, $\ominus$ and $\otimes$ as follows:

$$
\begin{align*}
    u_1 u_2, \; u_1 \oplus u_2 &\equiv u_1 + u_2, \\
    a u, \; u \ominus a &\equiv u a, \\
    a u, \; a \otimes u &\equiv u \ominus a = u a',
\end{align*}
$$

where the addition and multiplication in the right hand members of the above definitions are the ordinary addition and multiplication for numbers of such a system. Then $0_U(=0)$ is the only element of $U$ which is commutative with every number of $\mathfrak{A}$, for consider $a_1 \equiv (a_1(1) = 1, a_1(i) = 0 \mid i \neq 1)$; then $a'_1 = (a'_1(2) = 1, a'_1(i) = 0 \mid i \neq 2)$, and it follows that $u \neq 0_U$. $a_1 \otimes u = u a'_1 \neq u a_1 = u \ominus a_1$. This is also an example of a properly linear set with a right rank different from the left rank, for $u_0 \equiv (u_0(0) = 1, u_0(i) = 0 \mid i \neq 0)$ is a proper right base for $U$; but $U$ has left rank 2 for $u_0$ and $u_1 \equiv (u_1(1) = 1, u_1(i) = 0 \mid i \neq 1)$ form a proper left base for $U$, for $a u_0 = u \cdot u(i) = 0$ for $i$ odd and $a u_1 = u \cdot u(i)$ for $i$ even.

Theorem 1. If $\Sigma$ is of type 1 and if we consider $P \equiv [p] \equiv U_*$ and $U' \equiv \{\text{all vectors } u' \text{ on } P \text{ to } \mathfrak{A} \text{ finitely non-zero}\}$ and $U'_* \equiv [\delta_\cdot(p)]$ where $\delta_\cdot(p) = 1$ and $\delta\cdot(p_1) = 0$ for every $p_1 \neq p$, then $U$ is isomorphic with $U'$ under the correspondence

$$
u = \sum_{i=1}^{n} u_{*i} a_i \sim u' = \sum_{i=1}^{n} \delta_{u_i} a.$$

† Ex. 6 of a system $\mathfrak{A}$ of type $A$.  


Proof. The theorem follows at once from the fact that $U_*$ is commutative.

Relative to a general range $P$ and any number system $\mathfrak{A}$ of type $A$ the class of all vectors $u$ on $P$ to $\mathfrak{A}$ finitely non-zero is a set $U$ belonging to a system $\Sigma$ of type 1. In the remainder of this article we will therefore consider only systems of finitely non-zero vectors on a range $P$.

Relative to $P^1$ and the numbers of $\mathfrak{A}$ as vectors on $P^1$ to $\mathfrak{A}$ obviously $a \neq 0)$. $L(a) = L(1)$.

Lemma 1. If $P$ is finite and $u$ is a vector on $P$ to $\mathfrak{A}$ nowhere zero, then either there exist a vector $u_1$ and a number $a \neq 0$ such that $u_1$ is commutative and $u_1 a = u$ or there exist $L(u)$ two vectors $u_2$ and $u_3$ and elements $p_2$ and $p_3$ of the range $P$ such that $u_2(p_2) = 0$, $u_3(p_3) = 0$ and $L(u) = L[u_2 u_3]$.

Proof. It is sufficient to prove this for the special case $P = P^2$. Then $u = (a_1, a_2)$ with $a_1 a_2 \neq 0$ and $L(u) = L(u_1)$ where $u_1 = (1, a_1 a_2^{-1})$. If $u$ is not commutative, $a_1 a_2 a_1^{-1} a^{-1} \neq a_2 a_1^{-1}$, and since $L(u)$ contains $(1, a a_2 a_1^{-1} a^{-1})$ it also contains $(0, a_2 a_1^{-1} - a a_2 a_1^{-1} a^{-1})$; therefore $L(u)$ contains $(0, 1)$ and $(1, 0)$.

Lemma 2. Relative to a general range $P$ and the set $U$ of all vectors $u$ on $P$ to $\mathfrak{A}$ finitely non-zero, it is true that $u : A U_0 \rightarrow L U_0 = Lu$.

Proof. This lemma follows by the repeated application of Lemma 1.

Theorem 2 follows at once from Lemma 2.

Theorem 2. Relative to a general range $P$, $U$ the set of all finitely non-zero vectors on $P$ to $\mathfrak{A}$ and a properly linear subset $U_0$ of $U$, there exists a commutative right base $U_0^*$ for $U_0$ which therefore is also a left base for $U_0$.

Theorem 3. Relative to a normally ordered range $P$, $U$ the set of all finitely non-zero vectors on $P$ to $\mathfrak{A}$ and a properly linear subset $U_0$ of $U$, there exists a normally ordered commutative proper right base $U_0^*$ for $U_0$ which therefore is also a proper left base for $U_0$.

Proof. Since by Theorem 2 $U$ and $U_0$ are the linear extensions respectively of their commutative subsets $U'$ and $U'_0$, it follows from I, § 4, Theorem 1, and I, § 2, Theorem 6, that we need only prove the theorem relative to the system $\Sigma_1 = (\mathfrak{A'} U' \oplus \mathfrak{O} \mathfrak{O})$. In this form, however, the theorem is merely a special case of I, § 4, Theorem 3.

Relative to a normally ordered range $P$, $U$ the set of all finitely non-zero vectors on $P$ to $\mathfrak{A}$ and a properly linear subset $U_0$ of $U$, the right rank of $U_0$ is equal to the left rank of $U_0$. In this case we will speak of either the right or left rank of $U_0$ as the rank of $U_0(rk U_0)$. 


II. Sets of Vectors on a Finite Range

Contents

Introduction.
1. Normal forms for bases.
2. Orthogonal sets.
3. Applications to the case where \( \mathfrak{A} \) is real, complex or quaternionic.
4. Identity matrices for properly linear sets.

Introduction. In this section we will consider a system composed of a number system \( \mathfrak{A} \) of type \( A \), the totality \( V \equiv [v] \) of all vectors on a finite range \( P^n \equiv [1, 2, 3, \ldots, n] \) to \( \mathfrak{A} \) and addition and multiplication defined as in Example 2 of I. § 1, for a system \( \Sigma \). Thus we are dealing with a special case of a system \( \Sigma \) of type 1.

We introduce notations as follows for matrices, vectors and their composition:

\[
W = \{ \text{all matrices } w \text{ on } PP \to \mathfrak{A} \} \\
w = w(i,j) \quad (i = 1, \ldots, n, j = 1, \ldots, n).
\]

We say that a matrix \( w \) is commutative, \( w^c \), in case every element of \( w \) belongs to \( \mathfrak{A} \).

Composition of matrices: S notation:

\[
w_3 = S w_1 w_2 : \equiv: w_3(j,k) = \sum_i w_1(j,i) w_2(i,k) \quad (j,k).
\]

Composition of a vector and a matrix.

\[
v_1 = S w v : \equiv: v_1(i) = \sum_j w(i,j) v(j) \quad (i).
\]

\[
v_1 = S v w : \equiv: v_1(i) = \sum_j v(j) w(j,i) \quad (i).
\]

Inner product of two vectors:

\[
a = S v_1 v_2 : \equiv: a = \sum_i v_1(i) v_2(i).
\]

We say that a matrix \( w \) is non-singular (\( w^{ns} \)) in case it has a right and left reciprocal, which is equivalent to the rows of \( w \) being left linearly independent and the columns right linearly independent. We will use the notation \( \delta \) for the identity matrix.

\footnote{This notation is that used by E. H. Moore in his course in General Analysis.}
1. *Normal forms for bases.* In this article we define what we mean by the base of a right (left) linear set being in semi-normal or normal form. We show that two right linear sets are equal if and only if the normal forms of their bases are equal. We show also that a right base for a properly linear set which is in semi-normal form is also a left base for the same set and is composed of commutative vectors.

**Theorem 1.** If a right linear subset $V_0$ of $V$ has right rank $r$, and $\sigma$ is a set of distinct elements $p_1, \ldots, p_r$ of the range $P$ such that $V$ as on $\sigma$ is of right rank $r$, then there exists one and only one set of vectors $V_{\sigma} = (v_0, \ldots, v_r)$ of such a nature that

1. $L_r V_{\sigma} = V_0$;
2. $v_0 (p_i) = 1$ (for $i = 1, \ldots, r$);
3. $v_0 (p_j) = 0$ (for $i \neq j$, $i = 1, \ldots, r$, $j = 1, \ldots, r$).

Proof. 1. $\exists V_1 = (v_1, \ldots, v_r) : V_0 \supset V_1$ and on $\sigma$ is identical with the set $\delta_{p_1}, \ldots, \delta_{p_r}$. $V_1$ as on $\sigma$ has right rank $r$. Therefore aside from uniqueness $V_1$ is effective as the $V_{\sigma}$ of the theorem.

2. $V_{\sigma}$ is unique, for consider $i \leq r$ and $v'_i$ in $V_0$ of such a nature that $v'_i (p_i) = 1$ and $v'_i (p_j) = 0$ for $j \leq r$ and unequal to $i$; hence $v'_i = v_{0i}$ since it belongs to $V_0$ and therefore to $L_r V_{\sigma}$.

**Theorem 2.**

$V_{\sigma}^* = r V_0 = r$.

1. $V_0$ as on $\sigma$ is of right rank $r$;
2. $p_1 < p_2 \cdots < p_r$;
3. $\sigma' = (p_1' < p_2' \cdots < p_r')$. $V_0$ as on $\sigma'$ is of right rank $r$.

The proof is obvious.

We say that a right base $V_{01}$ of a right linear subset $V_0$ of $V$ is in semi-normal form in case there exists a $\sigma$ satisfying the conditions of Theorem 1 for which $V_{01}$ is effective as the $V_0$ of the theorem. In such a case we say that $V_{01}$ is in normal form provided $\sigma$ is effective as $\sigma^*$ of Theorem 2.

**Theorem 3.** Two right linear sets are equal if and only if the normal forms of their bases are equal.

**Theorem 4.** Relative to a properly linear subset $V_0$ of $V$ a right base $V_{01}$ for $V_0$ in semi-normal form is also a left base for $V_0$ in semi-normal form and is commutative.

Proof. Since $rk V_0 = rk V_0$, $V_{01}$ is a left base for $V_0$ in semi-normal form. Hence the right hand multiples of the vectors of $V_{01}$ must be equal to the left hand multiples with the same coefficients and therefore $V_{01}$ is commutative.
Corollary. The normal form of the right base of a properly linear subset $V_0$ of $V$ is equal to the normal form of its left base and is composed of commutative vectors.

2. Orthogonal sets. In this article we give a definition of the right (left) orthogonality of one vector to another in respect to a commutative non-singular matrix $w$, and in terms of these relations we define the right (left) orthogonal complements $O_{re}V_0 (O_{lw}V_0)$ of a subset $V_0$ of $V$ in respect to $w$. We then study the iteration of the processes $O_{re}, O_{lw}$ together with $L_r, L_l$, etc. and give the iteration table of the resulting twelve distinct processes, one set of whose generators are $O_{re}, O_{lw}$ and $L_0$; and show that these are closed under further iteration. We then make a generalization of these processes such that the resulting iteration table is abstractly equivalent to that obtained from $O_{re}, O_{lw}$, and $L_0$ and applies to the more general situation of Section I.

The statements we make in the remainder of this article will be relative to a commutative symmetric non-singular matrix $w$.

We say that $v_1$ is left orthogonal to $v_2$ and $v_2$ is right orthogonal to $v_1$ in respect to $w$ in case $S^2v_1wv_2 = 0$. We define the right (left) orthogonal complement of a subset $V_0$ of $V$ in respect to $w$, $O_{re}V_0 (O_{lw}V_0)$, and the sets $O_{re}V_0$ and $O_{lw}V_0$ as follows:

$$O_{re}V_0 = \{v : S^2v_0wv = 0\},$$
$$O_{lw}V_0 = \{v : S^2vwv_0 = 0\},$$
$$O_{re}V_0 = O_{lw}L_0V_0 = O_{lw}L_0V_0 = 0 \text{ (see Lemma 3)},$$
$$O_{lw}V_0 = O_{re}L_0V_0 = O_{re}L_0V_0 = 0 \text{ (see Lemma 3)}.$$

In case $w = \delta$ the orthogonality condition reduces to the vanishing of the inner product $Sv_1v_2$, and we say $v_1$ is left orthogonal to $v_2$ etc., and we use the notations $O_{re}V_0$ for $O_{re}V_0$ etc.

Lemma 1. $V_0 \cup (1) \quad O_{lw}V_0 = O_{lw}L_rV_0$,
$$O_{re}V_0 = O_{re}L_lV_0.$$

Lemma 2. $V_1 \cup V_2 \cup (1) \quad O_{re}V_2 \cup O_{re}V_1$,
$$O_{lw}V_2 \cup O_{lw}V_1.$$

Lemma 3. $V_0 \cup \cup (1) \quad 0_{re}V_0 = O_{re}V_0$.

Proof. There exists a commutative base $V_{01}$ for $V_0$ and hence $O_{lw}V_0 = O_{lw}V_{01} = O_{re}V_{01} = O_{re}V_0$.

Lemma 4. $V_0 \cup (1) \quad (O_{re}V_0)^r \cup (O_{lw}V_0)^l$. 
A GENERAL THEORY OF LINEAR SETS

187

Lemma 5.\( V_0 \). (1) \( 0_{\text{re}} 0_{\text{te}} V_0 = L_r V_0 \),
(2) \( 0_{\text{te}} 0_{\text{re}} V_0 = L_t V_0 \).

Proof: Obviously \( 0_{\text{re}} 0_{\text{te}} V_0 \supset L_r V_0 \), and \( 0_{\text{re}} 0_{\text{te}} V_0 \) is right linear, hence
the lemma is true provided \( r k_r 0_{\text{re}} 0_{\text{te}} V_0 = r k_r L_r V_0 \). This follows if we show that

(I) \( V_0 \). (1) \( r k_l 0_{\text{te}} V_0 + r k_r L_r V_0 = n \),
(2) \( r k_r 0_{\text{re}} V_0 + r k_l L_t V_0 = n \).

However, since \( w \) is non-singular we need only prove (I) for the special case in which \( w = \delta \). Let \( r = r k_r L_r V_0 \). Consider \( (p_1 \cdots p_r) \equiv \sigma \) as the
effective \( \sigma \) of § 1, Theorem 2, for \( L_r V_0 \), and \( (v_1 \cdots v_r) \equiv \) the normal
form of the right base for \( L_r V_0 \). Let \( \sigma' = (p_i' \cdots p_{n-r}) \) be the set of
elements of \( P \) not in \( \sigma \). Then consider \( V_0^* = (v_1' \cdots v_{n-r}') \), where, for
every \( i \), \( v_i'(p_i') = 1 \), \( v_i'(p_j') = 0 \) \((i \neq j)\), \( v_i'(p_k) = -v_k(p_i') \) \((k = 1 \cdots r)\),
\( V_0^* \) is a left base in semi-normal form for the left linear set \( V' = L_t V_0^* \).
which is of left rank \( n - r \). Moreover, \( 0_t V_0 \supset V' \). Hence the \( r k_l 0_l V_0 \geq n - r \). However,
if \( r k_l 0_l V_0 > n - r \) there would exist a vector \( v \) in \( 0_t V_0 \) and
an element \( p_i (i \leq r) \) of the range \( P \) such that \( v(p_i) = 1 \) and \( v(p_j') = 0 \)
for every \( j \leq n - r \). Since this is impossible, \( r k_l 0_l V_0 = n - r \) and our
lemma is proved.

Lemma 6. \( V_0 \). (0\text{re} V_0\text{te})^\text{t} \cdot (0\text{te} V_0\text{re})^\text{t}.

This follows directly from Lemmas 3 and 4.

Lemma 7. \( V_0 \). \( L_r V_0 = L_l V_0 = LV_0 = L_0 V_0 = L_0 L_r V_0 = L_0 L_t V_0 \).

Lemma 8. \( V_0 \). \( 0_{\text{re}} L_r V_0 = 0_{\text{re}} L V_0 = 0_{\text{te}} L V_0 = 0_{\text{te}} L_l V_0 = 0_{\text{te}} V_0 \).

Lemma 9. \( V_0 \). \( (0_{\text{re}} V_0)^t \) and \( V_0 \). \( (0_{\text{te}} V_0)^t \).

Lemma 10. \( V_0 \). \( (0_{\text{re}} V_0)^t \cdot (0_{\text{te}} V_0)^t \).

Lemma 11. \( V_0 \). \( L_0 0_{\text{re}} V_0 = 0_{\text{te}} V_0 = L_0 0_{\text{te}} V_0 \).

Proof. \( L V_0 \supset L_r V_0 \) and hence \( 0_{\text{re}} V_0 = 0_{\text{te}} L V_0 \supset 0_{\text{te}} L_r V_0 = 0_{\text{te}} V_0 \),
and since \( 0_{\text{re}} V_0 \) is properly linear it follows that

(1) \( 0_{\text{re}} V_0 \supset L_0 0_{\text{te}} V_0 \).

† This lemma in its equivalent matricial form is due to E. H. Moore and is given in
his course in General Analysis. The proof is the writer's. It may be stated in the
following form:

\[ P'P''w'''v'_1 \cdots : \equiv v'' \cdot z \cdot S''w'''v'' = v'_1 \cdot z \cdot S'v'w'' = 0 \cdot v'' \cdot v'_1 = 0. \]
\( L_0 L_t V_0 \subset L_t V_0 \) and hence \( 0_{\text{w}} L_0 L_t V_0 = 0_{\text{re}} L_0 L_t V_0 \supset 0_{\text{re}} L_t V_0 = 0_{\text{re}} V_0 \), and since \( 0_{\text{w}} L_0 L_t V_0 \) is properly linear it follows that \( 0_{\text{w}} L_0 L_t V_0 \supset L_0 V_0 \), and hence

\[
(2) \quad 0_{\text{w}}^2 L_0 L_t V_0 = L_0 L_t V_0 \subset 0_{\text{w}} L_0 V_0;
\]

using Lemma 5 and \( 0_{\text{te}} V_0 \) as \( V_0 \) in (2) we obtain

\[
(3) \quad L_0 0_{\text{te}} V_0 \subset 0_{\text{w}} L_0 V_0 0_{\text{te}} V_0 = 0_{\text{w}} L_0 V_0 = 0_{\text{w}} V_0,
\]

hence by (1) and (3) we have \( 0_{\text{w}} V_0 = L_0 0_{\text{te}} V_0 \).

**Lemma 12.** \( V_0 \). \( L_0 0_{\text{re}} V_0 = 0_{\text{re}} L_0 V_0 \).

**Proof.** Applying Lemmas 11 and 1 to \( 0_{\text{re}} V_0 \) for \( V_0 \) we obtain \( L_0 L_t V_0 = L_0 0_{\text{te}} 0_{\text{re}} V_0 = 0_{\text{w}} 0_{\text{re}} V_0 = 0_{\text{w}} L_0 V_0 \) and the lemma follows at once. This may also be stated in the form

\[
(12_1) \quad V_0 \). \quad 0_{\text{re}} 0_{\text{re}} V_0 = L_0 L_t V_0.
\]

By use of the above lemmas we derive Table 1, which gives the results of iteration of the processes \( L_r, L_t, \ldots, 0_{\text{wte}} \). Thus we find from row 4, column 7, that \( V_0 \). \( L_0 0_{\text{re}} V_0 = 0_{\text{re}} V_0 \). It should be especially noted that the three processes \( 0_{\text{re}}, 0_{\text{te}}, \) and \( L_0 \) are generators of the whole table.

**Table I**

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<th>( L )</th>
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<th>( L_0 L_t )</th>
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<th>( 0_{\text{wte}} )</th>
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<td>( L_0 )</td>
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The proof of Table I may be readily effected by using the lemmas as listed in the following table. Numbers refer to lemmas of this article.
In order to see that there exists a number system $\mathbb{A}$, a finite range $P$, a commutative symmetric non-singular matrix $w$ on $PP$ to $\mathbb{A}$ and a set $V_0$ of vectors on $P$ to $\mathbb{A}$ such that the twelve sets $L_r V_0, \ldots, 0_{0w} V_0$ are distinct, consider $\mathbb{A} = Q$ (real quaternions), the range $P^5$ and $w = \delta$ and

$$V_0 = \begin{pmatrix} 0 & j & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & j - k \end{pmatrix}.$$ 

The twelve sets $L_r V_0, \ldots, 0_{0w} V_0$ are given below with their bases in the normal form:

$$L_r V_0 = L_r \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$L_0 L_r V_0 = L \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$0_r V_0 = L_r \begin{pmatrix} 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$L_0 V_0 = L \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & i \end{pmatrix}.$$
The iteration of the processes represented in Table I is associative. This fact can be readily checked by consideration of the generators $0_{\text{re}}$, $0_{\text{lw}}$, $L_0$.

There are 107 distinct closed sub-tables in Table I. They are listed below by giving the generators, in each case choosing the minimum number necessary. The order chosen is not dependent on the number of generators but on the number of processes generated. This number is shown in Roman numerals. If the two tables include the same number of processes we list first that one whose first process (relative to the order of Table I) not in the second table precedes the first process in the second table but not in the first. We denote by setting two numbers before a list of generators that one may secure by parity a distinct table which therefore is not listed.

I. 1, 2. $L_r$. 3. $L_r$. 4. $L_r$. 5. 6. $L_0$.$L_r$.

II. 7, 8. $L_r$, $L_r$. 9, 10, $L_r$, $L_0$.$L_r$. 11, $L_r$, $L_0$. 12, 13, $L_r$, $L_0$.$L_r$. 14. $0_{\text{re}}$.

15, 16, $L_0$, $L_0$.$L_r$. 17. $0_{\text{lw}}$. 18, $L_0$.$L_r$, $L_0$.$L_r$. 19, 20, $L_0$.$L_r$.

III. 21, $L_r$, $L_r$. 22, 23, $L_r$, $L_0$.$L_r$. 24, 25, $L_r$, $L_0$.$L_r$. 26, 27, $L_r$, $0_{\text{re}}$.

28, 29, $L_r$, $L_0$. 30, 31, $L_r$, $L_0$.$L_r$. 32, 33, $L_r$, $L_0$.$L_r$. 34, $L_r$, $L_0$.$L_r$.

35. $L_0$, $L_0$.$L_r$, $L_0$.$L_r$. 36, 37, $0_{\text{lw}}$.

IV. 38, 39, $L_r$, $L_r$, $L_0$.$L_r$. 40. $L_r$, $L_0$.$L_r$, $0_{\text{re}}$. 41, 42, $L_r$, $L_0$. 43, 44, $L_r$.

$L_r$, $L_0$.$L_r$, $L_0$.$L_r$. 45, 46, $L_r$, $0_{\text{re}}$. 47, $L_r$, $L_0$.$L_r$, $L_0$.$L_r$. 48, $L_0$.$L_r$.

49, 50, $L_r$, $L_0$.$L_r$. 51, 52, $L_0$, $L_0$.$L_r$. 53, $L_0$.$L_r$.

V. 54, $L_r$, $L_r$, $L_0$.$L_r$, $L_0$.$L_r$. 55, 56, $L_r$, $L_0$.$L_r$. 57, 58, $L_r$, $L_0$.$L_r$.

59, 60, $L_r$, $L_0$.$L_r$. 61, 62, $L_0$.$L_r$, $0_{\text{lw}}$. 63, 64, $L_r$, $0_{\text{lw}}$.

VI. 65, $L_r$, $L_0$. 66, 67, $L_r$, $L_0$.$L_r$. 68, 69, $L_r$, $0_{\text{lw}}$. 70, 71, $L_r$, $0_{\text{re}}$.

72, 73, $L_r$, $L_0$, $L_0$.$L_r$. 74, $L_r$, $L_0$.$L_r$, $L_0$.$L_r$.

VII. 75, 76, $L_r$, $L_0$, $0_{\text{re}}$. 77, 78, $L_r$, $L_0$, $L_0$.$L_r$. 79, 80, $L_r$, $L_0$.$L_r$, $L_0$.$L_r$.

81, 82, $L_0$, $0_{\text{re}}$. 83, 84, $L_0$.$L_r$, $0_{\text{re}}$.

VIII. 85, $L_r$, $L_0$.$L_r$, $L_0$.$L_r$. 86, 87, $L_r$, $L_0$. 88, 89, $L_r$, $L_0$.$L_r$. 90, 91, $L_r$, $L_0$.$L_r$. 92, $L$.

$L_0$, $L_0$.$L_r$, $L_0$.$L_r$. 93, 95, $L_r$, $L_0$.$L_r$. 96, $L_r$, $L_0$.$L_r$. 97, 98, $L_0$.$L_r$.

$0_{\text{re}}$. 99, $L_r$, $L_0$.$L_r$, $L_0$.$L_r$. 100, $0_{\text{re}}$, $0_{\text{lw}}$. 101, 102, $L_r$, $L_0$, $0_{\text{re}}$.

103, 104, $L_r$, $L_0$.$L_r$. 105, 106, $L_r$, $L_0$.$L_r$. 107, $0_{\text{re}}$, $0_{\text{lw}}$, $L_0$.

Since relative to a subset $V_0$ of $V$ we may determine the sets $0_{\text{re}}$.$V_0$, $0_{\text{lw}}$.$V_0$, $L_0$.$V_0$, $L_0$.$0_{\text{lw}}$.$V_0$, $0_{\text{re}}$.$V_0$, and $0_{\text{re}}$.$V_0$ from the sets $L_0$.$V_0$, $L_r$.$V_0$,

$L_0$.$L_r$. $V_0$, $L_0$.$L_r$. $V_0$, $L_0$.$L_r$. $V_0$, $L_0$.$0_{\text{lw}}$. $V_0$, respectively, and conversely, we see that
the Table II of I, § 3, shows us which of the twelve sets are determined, in general, when any combination of a number of the sets is given.

Although we have defined the orthogonal complement of a set $V_0$ explicitly our table could be arrived at from a postulational point of view.

Consider a system $\mathfrak{A}$ satisfying the postulates of I, § 1, and further two processes $T_r$ and $T_l$ such that corresponding to every subset $U_0$ of $U$ there exist two subsets $T_r U_0$ and $T_l U_0$ of $U$ and the four following conditions are satisfied:

1. $U_0$. (a) $T_r U_0 = T_r L_l U_0$,
   (b) $T_l U_0 = T_l L_r U_0$;

2. $U_0$. (a) $T_r T_l U_0 = L_r U_0$,
   (b) $T_l T_r U_0 = L_d U_0$;

3. $U_1 \cup U_2$ . (a) $T_r U_1 \subset T_r U_2$,
   (b) $T_l U_1 \subset T_l U_2$;

4. $U_0^l$. (a) $T_r U_0 = T_l U_0$.

From the above conditions we see that

$$U_0_s: (T_r U_0)^l. (T_l U_0)^r,$$

for $L_r T_r U_0 = T_r T_l L_r T_r U_0 = T_r T_l T_r U_0 = T_r L_l U_0 = T_r U_0$. We define $TU_0$ and $T_0 U_0$ as $T_r LU_0$ and $T_r L_0 U_0$ respectively. The necessary lemmas for the construction of an iteration table of the $T$-processes may be readily derived and a table arrived at in terms of the $T$'s of which Table I is a special case.

3. Applications to the case where $\mathfrak{A}$ is real, complex or quaternionic. In case $\mathfrak{A}$ is the real, complex or quaternion number system there exists for every number $a$ its conjugate $\bar{a}$. We define the conjugate of a vector $v = (v(i)\mid i)$ as $\bar{v} = (\bar{v}(i)\mid i)$ and the conjugate of a subset $V_0$ of $V$ as the totality of the conjugates of the vectors of $V_0$, in notation $\bar{V}_0$. We readily verify the following statements:

1. $V_0$. (a) $L_r \bar{V}_0 = \bar{L_l V}_0$,
   (b) $L \bar{V}_0 = \bar{L V}_0$,
   (c) $L_0 \bar{V}_0 = \bar{L_0 V}_0$.

Since in the case of quaternions every properly linear subset of $V$ has a commutative base it follows that

2. $\mathfrak{A} = Q_0 \cdot V_0^l \cdot \sim \cdot V_0 \cup v$. $V_0 \cup \bar{v}$. 
In such a case the notion of what we shall call conjugate orthogonality proves useful. We define the sets $O_i V_0$ etc. as follows:

$$\begin{align*}
o_i V_0 &\equiv \overline{O_i V_0} = \{ \text{all } v \cdot \bar{v} \cdot S \bar{v} v_0 = 0 \} (v_0);
o_i V_0 &\equiv \overline{O_i V_0} = \{ \text{all } v \cdot \bar{v} \cdot S \bar{v} v_0 = 0 \} (v_0);
o' V_0 &\equiv O'_i L V_0 = \overline{O' V_0};
o'_i V_0 &\equiv O'_i L_0 V_0 = \overline{O'_i V_0}.
\end{align*}$$

Since every properly linear subset $V_0$ of $V$ has a commutative base it follows that

$$\begin{align*}
(3) &\quad V_0^{'}. (a) \quad O'_i V_0 = O'_i V_0 = O'_i V_0;
(4) &\quad V_0^{'}. (a) \quad O^{'2} V_0 = O'_i O'_i V_0 = O'_i O'_i V_0 = L_r V_0,
(b) \quad O'_i O'_i V_0 = O'_i O'_i V_0 = O'_i O'_i V_0 = L_0 L_4 V_0.
\end{align*}$$

By use of the above definitions and lemmas in connection with Table I of § 2, we arrive at Table I which gives the iteration of the processes $O'_r$, $O'_i$ and $L_0$ and the processes which they generate.

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The example illustrating the distinctness of the twelve processes of Table I, § 2, may be used to show the fact that the twelve processes of the above table are distinct. In this case,

$$\begin{align*}
o'_i V_0 &= L_r (0 1 -i 0 0),
o'_i V_0 &= L_d (0 0 0 1 -i),
L_0'_i V_0 &= L_0'_i V_0,
L_0'_i V_0 &= L_0'_i V_0,
0' V_0 &= 0 V_0,
o'_i V_0 &= 0_0 V_0.
\end{align*}$$
It should be noted that relative to subset \( V_0 \) of \( V \) we can determine the sets \( 0'_r V_0, 0'_l V_0, L0'_r V_0, L0'_l V_0, 0'_r V_0 \) and \( 0'_l V_0 \) from the sets \( L_r V_0, L_l V_0, L_0 L_r V_0, L_0 L_l V_0, L V_0 \) and \( L_0 V_0 \) respectively, and the converse is true. Thus Table II of I, § 3, shows which of the twelve sets \( L_r V_0, \ldots, 0'_l V_0 \) are determined, in general, when any combination of these sets is known.

Except for the \( 0'_r \) and \( 0'_l \) rows and columns, Table I can be obtained from Table I of § 2 by the substitution of \( L0'_r, L0'_l, 0'_r \) and \( 0'_l \) for \( L0''_r, L0''_l, 0''_r \) and \( 0''_l \) respectively. Hence a list of the closed subtables of Table I not involving either \( 0'_r \) or \( 0'_l \) may be obtained by the same substitution in the list of the closed subtables of Table I of § 2 which do not involve \( 0''_r \) and \( 0''_l \). Besides these 73 closed subtables we have the following 18 listed by their generators:

- II. 1, 2. \( 0'_r \).
- VI. 3, 4. \( L, 0'_l \).
- VII. 5, 6. \( L_r, 0'_l \).
- VIII. 7, 8. \( L_0, 0'_r, 9, 10. L_0 L_r, L_0 L_l, 0'_r \).
- IX. 11, 12. \( L_r, L_0 L_r, 0'_l \).
- X. 13. \( 0'_r, 0'_l \). 14, 15. \( L_0, L_0 L_r, 0'_l \).
- XI. 16, 17. \( L_r, L_0, 0'_l \).
- XII. 18. \( L_0, 0'_r, 0'_l \).

We can arrive at a generalization of Table I from a postulational point of view. Consider a system \( \Sigma \) satisfying the conditions of I, § 1, and two processes \( T'_r \) and \( T'_l \) of such a nature that for every subset \( U_0 \) of \( U \) there exist two subsets \( T'_r U_0 \), and \( T'_l U_0 \) and the following conditions are satisfied:

\[
\begin{align*}
(1) & \quad U_0 \vdash (a) \quad T'_r U_0 = T'_r L_r U_0; \\
& \quad (b) \quad T'_l U_0 = T'_l L_l U_0; \\
(2) & \quad U_0 \vdash (a) \quad T'_r^2 U_0 = L_r U_0; \\
& \quad (b) \quad T'_l^2 U_0 = L_l U_0; \\
(3) & \quad U_1 \vdash U_2 \vdash (a) \quad T'_r U_2 \vdash T'_r U_1; \\
& \quad (b) \quad T'_l U_2 \vdash T'_l U_2; \\
(4) & \quad U_0 \vdash T'_r U_0 = T'_l U_0;
\end{align*}
\]

and we make the following definitions:

\[
T'_r U_0 = T'_r L U_0, \quad T'_l U_0 = T'_r L_0 U_0.
\]
The necessary lemmas for the proof of Table I in terms of the $T$'s instead of the $O$'s may be readily derived.

4. **Identity matrices for properly linear sets.** In this article we arrive at a generalization, relative to properly linear subsets $V_0$ of $V$ and certain commutative non-singular matrices, of the notion of an identity matrix. Moreover we show that for every properly linear subset $V_0$ of $V$ there exists a commutative symmetric non-singular matrix $w$ which transforms $V_0$ into a properly linear subset $V_1$ of $V$ which is supplementary to its orthogonal complement.

Throughout we prove theorems by proving them for the case where $A$ is a field, and noting that due to the existence of a commutative base for every properly linear set the theorem follows from the theorem for the special case.

In the case where $A$ is a field the six linear processes $L_r$ etc. and the six orthogonal processes $O_{rw}$ etc. coincide.

**Lemma 1.** If $A$ is a field not modulo 2, $V_0$ is a linear subset of $V$ with rank $r$ greater than zero, and $w$ is an $n$ by $n$ commutative symmetric non-singular matrix such that $\bigcap [O_w V_0, V_0]$ is the set consisting of the single vector $0_V$, then there exists a vector $v_0$ of $V_0$ such that $S^2 v_0 w v_0 \neq 0$.

**Proof.** Let $v_{01} \ldots v_{0r}$ be a base for $V_0$. If for every $i \leq r$, $S^2 v_{0i} w v_{0i} = 0$, there exists an $i$ and a $j$ such that $i \neq j$ and $S^2 v_{0i} w v_{0j} \neq 0$. Since $2 = 1 + 1 \neq 0$ it follows that $S^2(v_{0i} + v_{0j}) w (v_{0i} + v_{0j}) = 2 S^2 v_{0i} w v_{0j} \neq 0$, and hence $v_{0i} + v_{0j}$ is effective as the $v_0$ of the lemma.

That the lemma need not hold for the case of a field with a modulus 2 is shown by the following example. Consider $P^3$, $A = \text{integers modulo } 2$, $w \equiv 0$, and

\[
V_0 = L \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix};
\]

the $O V_0 = L(1 1 1)$ and $V_0$ consists of the four vectors $(0 0 0)$, $(1 1 0)$, $(1 0 1)$, and $(0 1 1)$, but all the elements of $V_0$ are self orthogonal.

**Theorem 1.** If $A$ is a field not modulo 2, $V_0$ is a properly linear subset of $V$ and $w$ is an $n$ by $n$ commutative symmetric non-singular matrix such that $\bigcap [O_w V_0, V_0]$ is the set consisting of the single vector $0_V$, then there exists one and only one $n$ by $n$ matrix $\varepsilon$ of such a nature that

1. $S \varepsilon v_0 = v_0 = S v_0 \varepsilon$ ($v_0$),
2. $V_0 \supset S \varepsilon v$ ($v$),
3. $O_w V_0 \supset (v - S \varepsilon v)$ ($v$).

Moreover $\varepsilon$ is commutative and therefore $S \varepsilon v = S v \varepsilon$ for every $v$. 
Proof. Since \( V_0 \) has a commutative base it is sufficient to prove the theorem for the case in which \( \mathfrak{A} \) is a field. Relative to two vectors \( v_1 \) and \( v_2 \) we define the dyad \( (v_1 v_2) \) as the matrix \( \Phi \) where \( \Phi(i, j) = v_1(i)v_2(j) \).

The proof will be divided into two parts, Part 1 the existence of \( \varepsilon \) and Part 2 the uniqueness of \( \varepsilon \).

Part 1. Existence. Case 1. \( \text{rk} V_0 = 0 \).

In this case the zero matrix is effective as \( \varepsilon \).

Case 2. \( \text{rk} V_0 = r > 0 \).

According to Lemma 1 there exists \( v_{01} \) such that \( S^2 v_{01} w v_{01} = 0 \). Let

\[ \varepsilon_1 = \frac{S(v_{01} v_{01}) w}{S^2 v_{01} w v_{01}}, \]

and \( V_{01} = L(v_{01}) \). \( \varepsilon_1 \) is effective as \( \varepsilon \) for \( V_{01} \). Consider \( V_{02} = [v_0 - S\varepsilon_1 v_0 (v_0)] \).

It follows at once that \( V_{02} \) is linear and \( V_{01} + V_{02} = V_0 \). \( \cap [V_{01}, V_{02}] \) is the set consisting of the single vector \( 0_V \) and \( 0 \in V_{02} \supset V_{01} \). Hence if there exists a matrix \( \varepsilon_2 \) effective as \( \varepsilon \) for \( V_{02} \), \( \varepsilon_1 + \varepsilon_2 \) is effective as \( \varepsilon \). Thus the existential part of our theorem is true for the case when the rank of \( V_0 \) is \( r \) in case it is true when the rank of \( V_0 \) is \( r - 1 \). Hence, since we have found an effective \( \varepsilon \) in case \( \text{rk} V_0 = 0 \), there exists an effective \( \varepsilon \) for the case in which \( \text{rk} V_0 = r \).

Part 2. Uniqueness.

Consider \( \varepsilon' \) and \( \varepsilon'' \) effective as \( \varepsilon \) of the theorem:

\[
\begin{align*}
(1) \quad v \cdot v_0 :& S^2(v - S\varepsilon' v)w v_0 = 0, \quad S^2(v - S\varepsilon'' v)w v_0 = 0 \\
(2) \quad v & \cdot S(\varepsilon' - \varepsilon'')v \subseteq V_0.
\end{align*}
\]

Hence from (1) and (2) and the hypothesis of the theorem it follows that

\[ v \cdot S(\varepsilon' - \varepsilon'')v = 0_V, \]

and hence \( \varepsilon' - \varepsilon'' \) is the zero matrix and \( \varepsilon' = \varepsilon'' \).

Theorem 2. Relative to a properly linear subset \( V_0 \) of \( V \) there exists a commutative non-singular \( n \times n \) matrix \( \Phi \) such that

\[ w = S\tilde{\Phi} \Phi \cdot \cap [0_V, V_0] = [0_V]. \]

Proof. Since \( V_0 \) has a commutative base it is sufficient to prove the theorem for the case in which \( \mathfrak{A} \) is a field.
Case 1. \( \mathcal{A} \) is a field not modulo 2 or 3.

Consider \( V_{0*} = [v_1 \cdots v_r] \) the base for \( V_0 \) of normal form. It is readily seen since order is not involved that we may assume \( \sigma_* = (1 \cdots r) \).

Let \( \Phi_i = \delta \) if \( S v_i v_i \neq 0 \) and \( \Phi_i = \delta + \delta_{11} \) if \( S v_i v_i = 0 \) and \( w_i = S \Phi_i \Phi_i \).

Then

\[
S v_i v_i \neq 0 \), \( S v_i w_i v_i = S v_i v_i,
S v_i v_i = 0 \), \( S v_i w_i v_i = 3 \neq 0.
\]

Let \( V_1 = L(v_1) \) and \( \epsilon_1 \) be the identity matrix of Theorem 2 for \( V_1 \) in respect to \( w_1 \). Then the set \( [v_i - S \epsilon_1 v_i \equiv v_i (i = 2, 3, \ldots, r)] \) is a base in semi-normal form for the \( w_1 \)-orthogonal complement of \( V_1 \) in \( V_0 \).

This process may be repeated by the general recursion formulas for \( j = 1, \ldots, r - 1 \). (Let \( v_{10} = v_1 \).

\[\begin{align*}
V_j &= L[v_1, v_{21}, \ldots, v_{j-1}]; \\
e_j &= \text{the identity matrix of Theorem 2 for } V_j \text{ in respect to } w_j.
\end{align*}\]

\[\left[ v_i - S \epsilon_j v_i = v_j (i = j + 1, \ldots, r) \right] \text{ is a base in semi-normal form for the } w_j \text{-orthogonal complement of } V_j \text{ in } V_0, \text{ and}
\]

\[
\Phi_{j+1} = \Phi_j \text{ if } S v_{j+1} j w_j v_{j+1} j \neq 0,
\]

but

\[
\Phi_{j+1} = \Phi_j + \delta_{j+1} j+1 \text{ if } S v_{j+1} j w_j v_{j+1} j = 0
\]

and

\[
w_{j+1} = S \Phi_{j+1} \Phi_{j+1}.
\]

Hence

\[
S v_i i-1 w_{j+1} v_i i-1 = S v_i i-1 w_i v_i i-1 \neq 0 \quad (i = 1, \ldots, j),
S v_i i-1 w_{j+1} v_{k-1} = 0 \quad (i \neq k, i = 1, \ldots, j, k = 1, \ldots, j+1),
S v_{j+1} j w_{j+1} v_{j+1} j \neq 0.
\]

Hence \( \Phi_r \) is effective as the \( \Phi \) of the theorem.

Case 2. \( \mathcal{A} \) is a field modulo 2 or 3.

In this case we make \( \Phi_{j+1} = \Phi_j + \delta_{j+2} j+1 \) if \( S v_{j+1} j w_j v_{j+1} j = 0 \). Otherwise the proof is analogous to that for Case 1.

We may state Theorem 3 as follows: Relative to a properly linear subset \( V_0 \) of \( V \) there exists an \( n \) by \( n \) commutative non-singular matrix \( \Phi \) which transforms \( V_0 \) into a properly linear subset \( V'_0 \) of such a nature that \( 0V_0 \) is supplementary to \( V'_0 \).

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