ON EXTENDING A CONTINUOUS (1-1) CORRESPONDENCE OF TWO PLANE CONTINUOUS CURVES TO A CORRESPONDENCE OF THEIR PLANES*

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Various authors have considered the following problem: given two sets of points, M and M', lying in planes S and S' respectively, and a continuous (1-1) correspondence† T, such that $T(M)=M'$, under what conditions can the correspondence be extended to the planes? That is, under what conditions does there exist a continuous (1-1) correspondence U, such that $U(S)=S'$, and such that for points of M, U is identical with T?

A. Schoenflies‡ has shown that in case M is a simple closed curve the correspondence can be extended to the planes, without any conditions being imposed.

R. L. Moore§ has shown that if M and M' are subsets of arcs, the correspondence can always be extended to the planes. If we consider only the case where M is a connected set, Moore's theorem applies only to the case where M is an arc.

Moore and Schoenflies, then, have proved that if M is an arc or a simple closed curve, the correspondence can be extended to the planes, without any conditions being imposed on the correspondence. In this paper, we show that if M is any plane continuous curve,|| the correspondence can

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† A correspondence T which sends M into $T(M)$ is said to be *continuous*, if in case the point P of M is a limit point of N, a subset of M, then $T(P)$ is a limit point of $T(N)$. See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), p. 289. We shall refer to this paper hereafter as "Report."

‡ A. Schoenflies, *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, vol. 62 (1906), p. 324. See also J. R. Kline, *A new proof of a theorem due to Schoenflies*, Proceedings of the National Academy of Sciences, vol. 6 (1920), p. 529. A *simple closed curve* is a set which is in continuous (1-1) correspondence with a circle.

§ R. L. Moore, *Conditions under which one of two given closed linear point sets may be thrown into the other one by a continuous transformation of a plane into itself*, American Journal of Mathematics, vol. 48 (1926), p. 67. An *arc* is a set which is in continuous (1-1) correspondence with an interval of a straight line.

|| For the various definitions of a continuous curve see Report, pp. 289–295.

be extended to the planes, provided that we impose the condition that sides of arcs be preserved under the correspondence. Our theorem includes Schoenflies's theorem as a special case, since our condition is evidently satisfied in this special case.

The following example shows the necessity for imposing some condition on the correspondence T. The continuous curve M in the XY plane consists of the straight line intervals from (0, 0) to (3, 0), from (1, 0) to (1, 1) and from (2, 0) to (2, 1). The continuous curve M' in the $X'Y'$ plane is given by subjecting the points of M to the following transformation: if $x \neq 2$, $x = x'$ and $y = y'$; if $x = 2$, $x = x'$ and $y = -y'$. Here T is a continuous (1-1) correspondence and $T(M) = M'$, but the correspondence evidently cannot be extended to the entire planes.

Definition. If M and M' are continuous curves lying in planes S and S' respectively, and T is a continuous (1-1) correspondence such that $T(M) = M'$, we say that *sides are preserved under* T, if, given any arc AB of M, and any simple closed curve J in S containing AB as a subset, then there exists a simple closed curve J' in S' containing $T(AB) = A'B'$ as a subset, and such that if N designates the points of M interior to J, then the interior of J' contains $T(N) = N'$; and also, if given any simple closed curve J_1' in S' containing $A'B'$ as a subset, then there exists a simple closed curve J_1 in S containing AB as a subset, and such that if N_1' designates the points of M' interior to J_1', then the interior of J_1 contains $T^{-1}(N_1') = N_1$.

In the following we shall frequently use this notation: if X is any subset of M, we shall denote $T(X)$ by X'; if Y' is any subset of M', we shall denote $T^{-1}(Y')$ by Y.

Theorem I. *If M and M' are continuous curves containing no simple closed curve* and lying in planes S and S' respectively, and if there exists a continuous* (1-1) *correspondence T, such that $T(M) = M'$, and such that sides are preserved under T, then there exists a continuous* (1-1) *correspondence U, such that $U(S) = S'$, and such that if, for any point P of M, $T(P) = P'$, then $U(P) = P'$.*

Before proceeding with the proof of Theorem I, we shall discuss the definition of "sides preserved under T" for the case where M is a continuous

* For a discussion of this type of continuous curve, see S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), p. 119; R. L. Wilder, abstracts in the Bulletin of the American Mathematical Society, vol. 29 (1923), p. 118, and *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 340; and R. L. Moore, *Concerning the cut-points of continuous curves*, Proceedings of the National Academy of Sciences, vol. 9 (1923), p. 101.

curve containing no simple closed curve. In this discussion we have need of the following lemma.

LEMMA A. *If I_1 and I_2 are two simply connected domains, whose boundaries are B_1 and B_2, and whose outer boundaries* C_1 and C_2 have in common an arc UV such that no point of $\underset{\smile}{UV}$*† *is a limit point of $B_1+B_2-\underset{\smile}{UV}$, then $\underset{\smile}{UV}$ is in the boundary of two different domains complementary to B_1+B_2, such that either* (1) *one domain is a subset of I_1, the other of I_2, or* (2) *one domain is a subset of both I_1 and I_2, and the other has no points in common with either I_1 or I_2.*

Proof. About each point X of $\underset{\smile}{UV}$, let us construct a circle C_X whose exterior contains $B_1+B_2-\underset{\smile}{UV}$. Then corresponding to each point X, we can construct a simple closed curve J_X, formed of an arc X_1XX_3 of $\underset{\smile}{UV}$ and an arc $\underset{\smile}{X_1X_2X_3}$ in I_1, and whose interior is in I_1 and the interior of C_X.‡ The sum of the interiors of the simple closed curves J_X is a domain D, because if the boundaries of any pair have an arc of $\underset{\smile}{UV}$ in common, their interiors have a point in common. The domain D contains no points of B_1+B_2 by construction. If we add to D all points which can be joined to a point of D by an arc having no points in common with B_1+B_2, we obtain a domain D_1 complementary to B_1+B_2, and such that $\underset{\smile}{UV}$ forms part of the boundary of D_1. Evidently D_1 is a subset of I_1.

By a construction similar to the above, but taking in this case the arc $\underset{\smile}{X_1X_2X_3}$ exterior to I_1, we obtain a domain D_2 complementary to B_1+B_2, whose boundary contains $\underset{\smile}{UV}$, and which has no points in common with I_1.

Since $\underset{\smile}{UV}$ is part of the boundary of I_2, either D_1 or D_2 contains a point of I_2. Since a point of I_2 cannot be joined to a point exterior to I_2 by an arc having no points in common with B_2, it is evident that either D_1 or D_2 is a subset of I_2, and the other has no points in common with I_2. We have accordingly, the two possibilities mentioned in Lemma A.

We shall now consider some consequences of the definition of "sides are preserved under T" for a continuous curve M containing no simple closed curve. Let AB be a maximal arc§ in M. Let J be a simple closed curve in the plane S, containing AB as a subset and containing no other points of M.

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* R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 254.

† If UV is an arc, $\underset{\smile}{UV}$ denotes $UV-U-V$.

‡ R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916) p. 131. See especially Theorem 28.

§ A *maximal arc* in a set M is an arc which is not a proper subset of any other arc in M. See S. Mazurkiewicz, loc. cit., Lemma 13, p. 129.

Let N be the points of M interior to J. Since sides are preserved under T, there exists in S' a simple closed curve J', containing $A'B'$ as a subset, and enclosing N'. We shall now show that in case J' contains or encloses any points of $M'-(A'B'+N')$, J' can be replaced by a simple closed curve J'' which contains $A'B'$ as a subset, and which encloses N', but which neither contains nor encloses any other points of M'.

We shall first show that A' and B' are on the boundary of the same domain of $(I'+M')-M'$, where I' designates the interior of J'. For, if not, there is an arc $C'D'$ in M', such that (a) C' is on $J'-A'B'$; (b) D' is on $\underset{\smile}{A'B'}$; (c) $\underset{\smile}{C'D'}$ is in I'. Since N' is entirely in I', C' is not a point of N'. Since $N+AB$ is closed, $N'+A'B'$ is closed, and therefore no point of $\underset{\smile}{C'D'}$ can be in the set N'.

Now let J' be the J_1' of the definition. The curve J' encloses N' and $\underset{\smile}{C'D'}$. The corresponding simple closed curve J_1 in S encloses N and $\underset{\smile}{CD}$, and contains the arc AB as a subset. The simple closed curves J and J_1 in S have AB in common, and therefore satisfy the conditions of Lemma A. Since their interiors have N in common, and since limit points of N are on $\underset{\smile}{AB}$, it follows that one of the domains (complementary to $J+J_1$), whose boundary contains $\underset{\smile}{AB}$, consists entirely of points common to the interiors of J and J_1. Since D is a point of $\underset{\smile}{AB}$, and since $\underset{\smile}{CD}$ is interior to J_1, it follows that at least part of $\underset{\smile}{CD}$ is interior to J and therefore in the set N. In that case, the corresponding part of $\underset{\smile}{C'D'}$ lies in N', contrary to a previous statement. Therefore A' and B' are on the boundary of the same connected domain of $(I'+M')-M'$.

The points A' and B' can therefore be connected by an arc* in $I'-M'$, which forms with the arc $A'B'$ of M' a simple closed curve J''. The arc $J''-\underset{\smile}{A'B'}$ separates I' into two parts, and the part enclosed by J'' contains all of N'. The supposition that J'' encloses other points of M' in addition to N' leads to a contradiction similar to that obtained above.

Therefore in case M contains no simple closed curve, and AB is a maximal arc of M, and J contains AB but no other points of M, then we can add to our definition of "sides are preserved under T," that J' contains $A'B'$ but no other points of M', and that the interior of J' contains N' and no other points of M'; and similarly for J_1' and J_1.

Lemma A and the previous discussion show also that if any two simple closed curves have a maximal arc AB of M in common and contain no other points of M, then either their interiors contain the same subset of M, or

* Report, pp. 290–291.

their interiors have no points of M in common, in which case, since $M-A-B$ is connected, they contain all of M, save the arc AB.

DEFINITION. If M is a continuous curve containing no simple closed curve, and N is a closed and connected subset of M, we shall call a maximal connected subset of $M-N$ a *tree with respect to* N, or a *tree in* $M-N$. A tree has one and only one limit point in N, which point we shall call the *foot* of the tree. A tree plus its foot forms a closed set.

R. L. Wilder* has proved that the number of trees is countable, and that given any positive number ϵ, there are at most a finite number of trees of diameter greater than ϵ.

Proof of Theorem I. Since M and M' are bounded we can construct in the plane S a circle C containing M in its interior I, and in the plane S' a circle C' containing M' in its interior I'. If AB is a maximal arc of M, we can join A to any point D of C by an arc in $(I-M)+A+D$, and we can join B to any other point E of C by an arc in $(I-M)+B+E-AD$. The arc $A'B'$ in M' is also a maximal arc, and if we select any arbitrary points D' and E' of C', we can join A' to D' and B' to E' by arcs in $(I'-M')+A'+D'$ and $(I'-M')+B'+E'-A'D'$, respectively.

If X and Y are points of C separating D and E, the arcs EXD (of C), DA, AB (of M), and BE form a simple closed curve J, and the arcs EYD (of C), DA, AB and BE form a simple closed curve J_1. The interiors of J and J_1 have no points in common, and the sum of their interiors contains all points of M save AB.

If X' and Y' are any two points of C' separating D' and E', there exist likewise in S' two simple closed curves $J'=E'X'D'A'B'E'$ and $J_1'=E'Y'D'A'B'E'$, whose interiors have no points in common, and the sum of whose interiors contains all points of M' save $A'B'$.

In our previous discussion of "sides are preserved under T," we have shown that under the above conditions, one of the simple closed curves J' or J_1' (suppose the former) will enclose all the points of M' which correspond under T to the points of M which J encloses, and the other, J_1', will enclose the points of M' which correspond to the points of M which J_1 encloses.

Let us select an arbitrary positive number ϵ. Suppose either $M-AB$ or $M'-A'B'$ contains a tree of diameter greater than $\frac{1}{9}\epsilon$. Let T be a tree in $M-AB$ which is interior to J, and let T' be the corresponding tree in $M'-A'B'$ which is interior to J', these trees being such that the diameter of either T or T' is greater than $\frac{1}{9}\epsilon$. Let the foot of T be F, and let FG be

* R. L. Wilder, loc. cit., first paper, Theorem II.

a maximal arc of $(T+F)$ such that the diameter of either FG or $F'G'$ is greater than $\frac{1}{18}\epsilon$. If H is any point of $\underline{DXE}$, H can be joined to G by an arc interior to J, save for H, and having only G in common with M. If H' is any point of $\underline{D'X'E'}$ a similar arc $H'G'$ can be drawn. The arc FGH separates the interior of J into two domains, and the arc $F'G'H'$ similarly separates the interior of J'. It is evident that the points of M interior to the simple closed curve $FGHDAF$ have their corresponding points in M' interior to the simple closed curve $F'G'H'D'A'F'$. Similarly for points interior to $FGHEBF$.

If $M-(AB+FG)$ or $M'-(A'B'+F'G')$ contains a tree of diameter greater than $\frac{1}{9}\epsilon$, the simple closed curves enclosing it and the corresponding tree in the other set can both be separated by arcs in the manner indicated above. After a finite number of steps $M-(AB+FG+\cdots)$ and $M'-(A'B'+F'G'+\cdots)$ will contain no trees of diameter greater than $\frac{1}{9}\epsilon$, otherwise a theorem* due to R. L. Wilder is contradicted. At this stage, this state of affairs exists: the interior of the circle C is divided into a finite number of domains plus boundary points of these domains, where the domains are such that they are bounded by simple closed curves, each of which consists of a single maximal arc of M, and a single arc having no points in common with M save its end points; the interior of C' is divided into the same number of domains plus boundary points, where the domains are bounded by simple closed curves, each of which consists of a single maximal arc of M' and a single arc having no points in common with M' save its end points; if N represents the set of points of M contained and enclosed by one of the simple closed curves in the plane S, the set $T(N)=N'$ will be contained and enclosed by one of the simple closed curves in the plane S', and this simple closed curve will contain or enclose no points of M' which are not in N'; no tree in N (or N') with respect to the maximal arc of M (or M') belonging to the simple closed curve which contains and encloses N (or N'), is of diameter greater than $\frac{1}{9}\epsilon$.

Let K be any one of the simple closed curves in S; PQ the maximal arc of M on K; N the subset of M contained and enclosed by K. Let K' denote the corresponding simple closed curve in S'. We can select a finite set of points $P_1, P_2, \cdots, P_n$, and $P_1', P_2', \cdots, P_n'$, such that (a) $P_1=P$ and $P_1'=P'$; (b) $P_n=Q$ and $P_n'=Q'$; (c) P_i precedes P_{i+1} on PQ, and P_i' precedes P_{i+1}' on $P'Q'$, for $i=1, 2, \cdots, n-1$; (d) the diameter of each of the arcs P_iP_{i+1} and $P_i'P_{i+1}'$ is less than $\frac{1}{9}\epsilon$; (e) no tree in $N-PQ$ or $N'-P'Q'$ has its foot at any of the points P_i or P_i'; (f) $T(P_i)=P_i'$. Let the set K_i

* R. L. Wilder, loc. cit., second paper, Theorem II.

$(i=1, 2, \cdots, n-1)$ be P_iP_{i+1} plus all trees in $N-PQ$ with feet on P_iP_{i+1}; and let K_i' be $P_i'P_{i+1}'$ plus all trees in $N'-P'Q'$ with feet on $P_i'P_{i+1}'$. Evidently $T(K_i)=K_i'$, and the diameter of each of the sets $K_1, K_2, \cdots, K_{n-1}$, $K_1', K_2', \cdots, K_{n-1}'$ is less than $\frac{1}{3}\epsilon$.

The point P_1 can be joined to P_2 by an arc in the interior of K, having only its end points in common with M. We shall show that under our given conditions P_1 can be joined to P_2 by such an arc whose diameter is less than $\frac{2}{3}\epsilon$. Suppose an arc P_1RP_2 has been constructed whose diameter is greater than $\frac{2}{3}\epsilon$. This arc forms with P_1P_2 a simple closed curve H which encloses $K_1-P_1P_2$, but encloses no other points of M. There also exists a simple closed curve L, enclosing K_1, and such that every point of L plus its interior is at a distance less than $\frac{1}{12}\epsilon$ from K_1,* and therefore such that the diameter of L is less than $\frac{1}{2}\epsilon$. Since the diameter of the arc P_1RP_2 is greater than $\frac{2}{3}\epsilon$, it must contain points exterior to L. The simple closed curves H and L satisfy the conditions† under which there exists a simple closed curve Q which is a subset of $H+L$, contains the arc P_1P_2 of M, and every point of whose interior is interior to both H and L. The arc from P_1 to P_2 in Q which has only its end points in common with M, has no points exterior to L, and is therefore of diameter less than $\frac{2}{3}\epsilon$; it has no points exterior to H and except for its end points, has no points in common with K_1, and therefore has no points in common with M, and lies entirely in the interior of K. This is an arc satisfying the conditions stated.

The point P_1 can therefore be joined to P_2 by an arc $P_1W_1P_2$ in the interior of K which forms with the arc P_1P_2 of M a simple closed curve C_1, enclosing $K_1-P_1P_2$ but no other points of M, and containing P_1P_2 but no other points of M, and such that the diameter of C_1 is less than ϵ. In the same way an arc $P_2W_2P_3$ can be constructed in the interior of K (save for P_2 and P_3) and exterior to C_1 (save for P_2), which forms with P_2P_3 of M a simple closed curve C_2, enclosing and containing the set K_2 and no other points of M and such that the diameter of C_2 is less than ϵ. In this way we construct the simple closed curves $C_1, C_2, \cdots, C_{n-1}$, all of diameter less than ϵ. In the same way we construct in S' the simple closed curves $C_1', C_2', \cdots, C_{n-1}'$, all of diameter less than ϵ.

Note that if from the interior of K we remove the simple closed curves $C_1+C_2+\cdots+C_{n-1}$ and their interiors, there remains a domain whose

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* R. L. Moore, *Concerning the separation of point-sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), p. 469.

† R. L. Moore, *On the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry*, American Journal of Mathematics, vol. 41 (1919), p. 299. See especially Theorem 26.

boundary is the simple closed curve consisting of the arcs $K-\underbrace{P_1P_n}$, $P_1W_1P_2$, $P_2W_2P_3, \cdots, P_{n-1}W_{n-1}P_n$; similarly for K'.

If the interior of any of the simple closed curves $C_1, \cdots, C_{n-1}$ contains points of M, we shall separate its interior as we have separated the interiors of J and J' into a finite set of domains free from points of M, plus a finite set of simple closed curves (such as C_1) of diameter less than $\frac{1}{2}\epsilon$, the same being true of M'. We shall then continue this process with such of these simple closed curves as enclose points of M.

We shall now define a continuous (1-1) correspondence U, such that $U(S)=S'$, and we shall show that if, for any point P of M, $T(P)=P'$, then $U(P)=P'$.

For points of the circles C and C', U is any continuous (1-1) correspondence between C and C' subject only to these conditions: (1) $U(D)=D'$; $U(E)=E'$; (2) if P is a point of $\underbrace{EXD}$, $U(P)$ is on $\underbrace{E'X'D'}$; (3) if P is a point of $\underbrace{EYD}$, $U(P)$ is on $\underbrace{E'Y'D'}$; (4) $U(H)=H'$; and similarly for points of C other than H that were joined by arcs to points on trees of M of diameter greater than $\frac{1}{9}\epsilon$, or such that the corresponding trees in M' were of diameter greater than $\frac{1}{9}\epsilon$.

Having defined U for points of the circles C and C', we define U for points exterior to these circles, as being any continuous (1-1) correspondence between S and S' subject only to the condition that for points of C and C' the correspondence be the one defined in the preceding paragraph. That such a correspondence exists, follows from Schoenflies's theorem.

For points interior to C, and on one of the simple closed curves K, consisting of an arc PQ of M, an arc QX interior to C save for X, an arc XY on C, and an arc YP interior to C save for Y, we define U as being any continuous (1-1) correspondence between K and K' subject only to the following conditions: (1) for points of PQ, U is identical with T; (2) for points of XY, U is identical with the correspondence previously defined for points of C.

For points interior to one of the simple closed curves K, and on the arc $P_1W_1P_2W_2P_3 \cdots P_{n-1}W_{n-1}P_n$, we define U as being any continuous (1-1) correspondence between $P_1W_1 \cdots P_n$ and $P_1'W_1' \cdots P_n'$ subject only to the following conditions: $U(P_1)=P_1' \cdots, U(P_n)=P_n'$; and if C_i encloses points of M, the correspondence between $\underbrace{P_iW_iP_{i+1}}$ and $\underbrace{P_i'W_i'P_{i+1}'}$ must be such that $U(H_1)=H_1'$, where H_1 is a point of $\underbrace{P_iW_iP_{i+1}}$ which was joined, by an arc interior to C_i, to a point of a tree, in $M-P_iP_{i+1}$, of diameter greater than $\frac{1}{18}\epsilon$.

For points interior to the simple closed curve formed by $K-\underbrace{P_1P_n}$, and $P_1W_1P_2, \cdots, P_{n-1}W_{n-1}P_n$, U is defined as being any continuous (1-1) correspondence between S and S' subject only to the condition that for points of the simple closed curves, the correspondence be the one defined in the preceding paragraphs.

If the interior of any of the simple closed curves $C_1, C_2, \cdots, C_n$, say C_1, is free from points of M, U is defined for points of the interior of C_1 as being any continuous (1-1) correspondence between S and S' subject only to the condition that for points of C_1 and C_1' the correspondence be the one already defined for points of C_1 and C_1'.

In case the interior of any of the simple closed curves $C_1, \cdots, C_n$ contains points of M, we have already indicated the method of separation of its interior into a finite number of domains, and we have indicated above how U is defined for every point of S, save for points interior to simple closed curves of diameter less than ϵ, enclosing points of M. We shall now show that by a continuation of the above process, U is defined for such points.

If P is any point of $S-M$, there is some value of k such that $\epsilon/2^k$ is less than the distance from P to a point of M. Therefore at the step where the diameters of the simple closed curves enclosing points of M are less than $\epsilon/2^k$, P will not lie in the interior of such a simple closed curve, and hence U will have been defined for the point P. Similarly for any point in $S'-M'$.

If P is a point of M, and if U has not already been defined for P, then P lies at each step in the interior of a simple closed curve, consisting of an arc of M and an arc in $S-M$. Let the sequence of simple closed curves be $D_1, D_2, \cdots$, where D_{i+1} lies in D_i plus its interior; let the diameter of D_i be less than $\epsilon/2^i$; and let the arc of M on D_i be X_iY_i. Evidently P is the only limit point of this sequence. The corresponding simple closed curves in S' are $D_1', D_2', \cdots$, and D_{i+1}' lies in the interior of D_i', the diameter of D_i' is less than $\epsilon/2^i$, and the arc of M on D_i' is $X_i'Y_i'$, where $X_i'=T(X_i)$, and $Y_i'=T(Y_i)$. The sequence $D_1', D_2', \cdots$ approaches a single point P' as a sequential limit point, and we define $U(P)$ as P'.

It remains to be shown that $T(P)$ is the point $U(P)=P'$. The points $X_1, X_2, \cdots$ of M approach P as a sequential limit. Since T is continuous, the point in M' approached by $X_1', X_2', \cdots$ is $T(P)$, and we have called this point $U(P)=P'$. Therefore U is identical with T for all points of M.

The correspondence U as defined above satisfies the conditions of Theorem I.

Definition. If M and M' are continuous curves lying in planes S and S' respectively, and T is a continuous (1-1) correspondence such that $T(M)=M'$, we say that *interiors are preserved under* T if, given any simple closed curves J of M and J' of M', such that $T(J)=J'$, and if N is the set of points of M interior to J, and N' is the set of points of M' interior to J', then $T(N)=N'$.

Lemma B. *If sides are preserved under T, interiors are preserved under T.*

Proof. Given $T(M)=M'$ and sides are preserved under T. Suppose P, a point of M interior to J, is such that $T(P)=P'$ is exterior to J'. We shall show that this leads to a contradiction.

Let PQ be an arc in M, having only Q in common with J, and therefore interior to J, save for Q. Then $P'Q'$ will be exterior to J', save for Q'.

Let A and B be two points of J, $A\neq Q\neq B$, and let D be a point of J, such that D and Q separate A and B. In the exterior of J we shall construct an arc $\underbrace{AXB}$, such that BQA (of J) plus $\underbrace{AXB}$ forms a simple closed curve C containing in its interior the interior of J, and therefore $\underbrace{ADB}$ and $\underbrace{QP}+P$.

Since sides are preserved under T, there exists a simple closed curve C' in S' containing $B'Q'A'$ and whose interior contains $\underbrace{A'D'B'}$ and $\underbrace{Q'P'}+P'$. The curve C' is composed of the arcs $B'Q'A'$ of J', and $\underbrace{B'X'A'}$ in the exterior of J'. The interior of C' therefore contains the interior of J', in fact $\underbrace{A'D'B'}$ divides the interior of C' into two parts, the interior of $J'=A'D'B'Q'A'$ and the interior of $A'D'B'X'A'$. Since $\underbrace{Q'P'}+P'$ is exterior to J' and interior to C', it must lie within $A'D'B'X'A'$. Since T is continuous, the limit point Q' of $\underbrace{Q'P'}+P'$ must lie on or within $A'D'B'X'A'$, whereas the arc $\underbrace{B'Q'A'}$ is exterior to $A'D'B'X'A'$. This is the desired contradiction.

Theorem II. *If M and M' are continuous curves lying in planes S and S' respectively, and if there exists a continuous* (1-1) *correspondence T, such that $T(M)=M'$, and such that sides are preserved under T, then there exists a continuous* (1-1) *correspondence U, such that $U(S)=S'$, and such that if, for any point P of M, $T(P)=P'$, then $U(P)=P'$.*

Proof. Since we have considered in Theorem I the case where M contains no simple closed curve, we shall assume here that M contains at least one simple closed curve. The proof of Theorem II follows, in the main, that of Theorem I. We shall go into detail only when the proof differs from that of Theorem I.

By Schoenflies's definition of a continuous curve,* $S-M$ consists of one unbounded domain and a countable set of bounded domains, the same being true for $S'-M'$. The outer boundary of each bounded domain is a simple closed curve J whose interior contains a countable set of maximal

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* Report, p. 290 and p. 295.

connected subsets of $M-J$ (which we shall call "trees"), each having one and only one limit point on the simple closed curve.

If M is a continuous curve, and N is a closed and connected subset of M, we shall call a maximal connected subset of $M-N$ which has one and only one limit point in N a *tree with respect to* N or a *tree in* $M-N$. The limit point in N is called the *foot* of the tree. The number of trees is countable, and if N is the outer boundary of a domain complementary to M, then not more than a finite number of trees can be of diameter greater than any given positive number. In case M contains no simple closed curve this definition is equivalent to the one given previously.

If α is the boundary of D_1, the unbounded domain in $S-M$, then $T(\alpha)=\alpha'$ is the boundary of D_1', the unbounded domain in $S'-M'$. For suppose it were possible that for some point P of α, $T(P)=P'$ is not in β', the boundary of D_1'. Then P' is separated from any point Q' in D_1' by β', and P' and Q' are therefore separated by some simple closed curve J' which is a subset of β'.* The point Q' is not interior to J'; therefore P' is interior to J'. But P is in α and is therefore not interior to $T^{-1}(J')=J$ or any other simple closed curve in M, and in this case interiors (and, therefore, sides) have not been preserved under T, contrary to hypothesis. A similar contradiction is arrived at if we suppose that for some point P' of β', $T^{-1}(P')=P$ is not a point of α. Therefore $\beta'=\alpha'$.

We shall next show that if α is the boundary of D_2, a bounded domain in $S-M$, then $T(\alpha)=\alpha'$ is the boundary of a bounded domain in $S'-M'$, which we may call D_2'. The boundary α contains a simple closed curve J, and α' contains J'. If N denotes the points of M interior to J, N consists of a countable set of trees. Since N' is identical with the points of M' interior to J', any maximal connected subset of N' is a tree with respect to J', otherwise T is not (1-1) and continuous, and therefore N' also consists of a countable set of trees. Let these be N_1', N_2', N_3', $\cdots$. Let L_i' be N_i' plus its foot on J' plus any points of $S'-M'$ lying in a complementary domain of M' whose boundary is in N_i'. The sets L_1', L_2', L_3', $\cdots$ are closed and connected and have no points in common. Furthermore, only a finite number of these sets can be of diameter greater than any given positive number. Under these conditions, $L_1'+L_2'+L_3'+\cdots$ is not connected.† Therefore not every point of the interior of J' lies in $L_1'+L_2'+\cdots$. Let P' be a point of the interior of J' not in $L_1'+L_2'+\cdots$, and let D_2' be the domain in $S'-M'$

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* R. L. Moore, *Concerning continuous curves in the plane*, loc. cit., Theorem V.

† See abstract by J. R. Kline, Bulletin of the American Mathematical Society, vol. 31 (1925), p. 300.

containing P'. The outer boundary of D_2' is a simple closed curve K'. If K' were entirely interior to J' or had one point in common with J', then K', D_2', and P' would be points of one of the sets L_i', contrary to our selection of P'. If K' had an arc in common with J', but were not identical with J', K' would contain an arc interior to J' joining two points of J', and such an arc cannot exist in any of the trees N_1', N_2', $\cdots$. Therefore, K' is identical with J'. If we suppose that any point P of α is such that P' is not in the boundary of D_2', or that any point Q' of the boundary of D_2' is such that Q is not in α, we obtain a contradiction by the same argument as that used in the case of the boundary of the unbounded domain. Therefore the boundary of D_2' is α'.

We shall next discuss the definition of "sides are preserved under T" for the case where M is any continuous curve.

Let D be any domain in $S-M$, and A and B two points on its boundary β; then any arc AXB such that $\underline{AXB}$ is in D, divides D into two domains, D_1 and D_2, such that $D=D_1+D_2+\underline{AXB}$. The boundary of D_i $(i=1, 2)$ is composed of $\underline{AXB}$ and β_i, a subset of β. Evidently, $\beta=\beta_1+\beta_2$. Suppose that A and B are two points such that β_1 and β_2 are the same for any choice of the arc $\underline{AXB}$. This will be the case if each of the points A and B is a non-cut point of β, i. e., a point whose removal does not disconnect β. If in $\beta_i(i=1, 2)$ we draw the arc AYB (there is only one such arc), and draw in D an arc $\underline{AXB}$ forming a simple closed curve J with AYB, then the discussion under Theorem I of "sides are preserved under T" holds with slight modifications. Therefore, under the above conditions, we can add to our definition of "sides are preserved under T" that J' contains $A'B'$ but no other points of M', that $J'-A'B'$ lies in D' (where D' is the domain in $S'-M'$ whose boundary is $T(\beta)=\beta'$), and that the interior of J' contains N' and no other points of M'.

As in the proof of Theorem I, we can construct a circle C containing M in its interior I, and a circle C' containing M' in its interior I'. Let α denote the boundary of the unbounded domain in $S-M$. Since α is a non-dense continuous curve separating the plane, α contains a simple closed curve J.* Let A and B be two points of J which are not feet of trees in $\alpha-J$. We can join A to D, any point of C, by an arc in $(I-M)+A+D$, and we can join B to any other point E of C by an arc in $(I-M)+B+E-AD$. If D' and E' are arbitrary points on C' we can draw similar arcs $A'D'$ and $B'E'$. Each of the arcs AB of J forms with AD, BE, and the proper one of the arcs DE of C a simple closed curve, and these two simple closed

* Report, pp. 295–296.

curves C_1 and C_2 contain and enclose all points of α. Similarly in S' are two simple closed curves C_1' and C_2' such that the points contained and enclosed by C_i' $(i=1, 2)$ are the points corresponding to those contained and enclosed by C_i.

Let us select an arbitrary positive number ϵ. Let $D_1, D_2, \cdots, D_n$ be a finite set of domains complementary to M, and $D_1', D_2', \cdots, D_n'$ a set of domains complementary to M', such that (1) if α_i is the boundary of D_i $(i=1, 2, \cdots, n)$, then $T(\alpha_i)=\alpha_i'$ is the boundary of D_i' and (2) all domains complementary to either M or M' and of diameter greater than ϵ occur in one set or the other. In each of the bounded domains, such as D_2, select two points A and B on the outer boundary J_2, which are not feet of trees in α_2-J_2, and join A to B by an arc in D_2 save for its end points. If we join A' to B' by an arc in D_2' save for its end points, this arc divides the interior of J_2' into the interiors of two simple closed curves, such that the points of M' contained and enclosed by either one of them are the points corresponding to those contained and enclosed by the corresponding one of the simple closed curves in S which AB forms with J_2.

Let us now consider any one of the above simple closed curves K formed by an arc AXB in M and $\underline{AYB}$ in $S-M$, and its corresponding simple closed curve $K'=A'X'B'Y'A'$ in the plane S'. Let the points of M contained and enclosed by K be denoted by N. Then $T(N)=N'$ is the set contained and enclosed by K'. Suppose either $N-AXB$ or $N'-A'X'B'$ contains a tree of diameter greater than $\frac{1}{9}\epsilon$, and let this tree and its corresponding one be denoted by T and T'. For definiteness let us suppose that the diameter of T is greater than $\frac{1}{9}\epsilon$. Let the foot of T be F. Let G_1 denote some point of T which belongs to α, the boundary of the complementary domain of M in which $\underline{AYB}$ was drawn, and such that an arc FG_1 of α is of diameter greater than $\frac{1}{18}\epsilon$. If $\alpha-G_1$ is connected, we shall denote G_1 by G. If $\alpha-G_1$ is not connected, it consists of a countable set of trees. Let W denote the one which contains F. The set $\alpha-W$ is closed and connected. It has therefore a non-cut point G distinct from G_1. The sets $(\alpha-W-G)$ and $(W+G_1)$ are connected and have G_1 in common. Therefore their sum, $\alpha-G$, is connected, and G is a non-cut point of α. Moreover, any arc in α from F to G passes through G_1 and therefore α contains an arc FG of diameter greater than $\frac{1}{18}\epsilon$.

If H is any point of $\underline{AYB}$, H can be joined to G by an arc interior to K, save for H, and having only G in common with M. In S', a similar arc $H'G'$ can be drawn. The points of M on and interior to $GFAHG$ will have their corresponding points on and interior to $G'F'A'H'G'$. Continuing this process a finite number of times we eventually arrive at the point where none of the simple closed curves K and K' contain a tree of diameter greater than $\frac{1}{9}\epsilon$.

The choice of sets $K_1, K_2, \cdots, K'_{m-1}$, and the simple closed curves $C_1, C_2, \cdots, C'_{m-1}$ is made as in Theorem I without any modifications. The correspondence U is defined as in Theorem I for all points of $S-M$ and for all points which are boundary points of domains complementary to M. If there are any other points in M, we shall define U as being identical with T.

The correspondence U defined in this way is evidently (1-1), and for points of M, U and T are identical. It remains to be shown that U is continuous, i. e., if $P_1, P_2, \cdots$ is a set of points of S approaching P as a sequential limit, then $P'_1, P'_2, \cdots$ approach $T(P)=P'$ as a sequential limit. In case P is a point of $S-M$, all except a finite number of the points of $P_1+P_2+\cdots$ lie in the same domain as does P, and we have defined U for the complementary domains in such a way that $P'_1, P'_2, \cdots$ will approach P'. In case P is a point of M and an infinite number of points of $P_1+P_2+\cdots$ lie in M, then P' is a limit point of $P'_1+P'_2+\cdots$ because U is identical with T for points of M, and T is continuous. In case P is a point of M, and an infinite number of points of $P_1+P_2+\cdots$ lie in a single domain of $S-M$, then P is a boundary point of that domain, and again U has been defined in such a way that $P'_1, P'_2, \cdots$ have P' as a limit point. In case, finally, P is a point of M, and only a finite number of points of $P_1+P_2+\cdots$ lie in M or in any single domain of $S-M$, then we can pick out an infinite subsequence $Q_1, Q_2, \cdots$, such that each Q_i is a point of $S-M$, and no two points Q_i, Q_j lie in the same domain. Let us associate with each Q_i a point R_i of the boundary of the domain in which Q_i lies. For any given ϵ only a finite number of the points R_i can be selected in such a way that the distance between Q_i and R_i is greater than ϵ. Therefore the set of points $R_1, R_2, \cdots$ also approach P as a sequential limit. Similarly in the plane S', the two sequences $Q'_1, Q'_2, \cdots$ and $R'_1, R'_2, \cdots$ approach the same limit. Since $R'_1, R'_2 \cdots$ are points of M', their limit is P'. Therefore in each case, U is continuous.

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