# AN EXTENSION OF LAGRANGE'S EXPANSION*

BY

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1. **Solution of a partial differential equation by means of a definite integral.** Soon after the publication of Whittaker's solution of Laplace's equation† it was realized that the linear homogeneous partial differential equation

$$f\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)V=0 \tag{1}$$

can be satisfied by a definite integral of type

$$V=\int_0^\theta F[\sigma,\tau]d\tau\ ,$$

where

$$\sigma=x\xi(\tau)+y\eta(\tau)+z\zeta(\tau)-\chi(\tau)$$

by simply choosing functions $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$ such that‡

$$f[\xi(\tau)\ ,\ \eta(\tau)\ ,\ \zeta(\tau)]=0$$

and making the upper limit $\theta$ a constant.

It has been known for some time§ that if the upper limit $\theta$ is not a constant but a function of $x$, $y$, and $z$ defined by the equation

$$x\xi(\theta)+y\eta(\theta)+x\zeta(\theta)=\chi(\theta) \tag{2}$$

the differential equation still may be satisfied. A direct verification of this result for the case when $f(\xi,\eta,\zeta)$ is any homogeneous polynomial of degree $m$ in $\xi$, $\eta$ and $\zeta$ is rather tedious but it may be accomplished by establishing the following rule for the differentiation of $V$.

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* Presented to the Society, San Francisco Section, April 4, 1925; received by the editors in April, 1925.

† Monthly Notices, Royal Astronomical Society, vol. 62 (1902); Mathematische Annalen, 1903, p. 333.

‡ H. Bateman, Proceedings of the London Mathematical Society, ser. 2, vol. 1 (1904), p. 451.

§ A particular case is mentioned in the author's *Electrical and Optical Wave Motion*, p. 12.

Let us write

$$F_n[\sigma,\tau]=\frac{\partial^n}{\partial\sigma^n}F[\sigma,\tau] ,$$

$$M=\chi'(\theta)-x\xi'(\theta)-y\eta'(\theta)-z\zeta'(\theta) ,$$

$$\Phi(\tau)=f[\xi(\tau) , \eta(\tau) , \zeta(\tau)] ,$$

where at present $f$ is any homogeneous polynomial of degree $m$; then

$$\begin{aligned} f\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)V=&\int_0^\theta F_m[\sigma,\tau]\Phi(\tau)d\tau+\frac{1}{M}F_{m-1}[0,\theta]\Phi(\theta)\\ &+\frac{1}{M}\frac{\partial}{\partial\theta}\left\{\frac{1}{M}F_{m-2}[0,\theta]\Phi(\theta)\right\}+\cdots\\ &+\left(\frac{1}{M}\frac{\partial}{\partial\theta}\right)^{m-1}\left\{\frac{1}{M}F[0,\theta]\Phi(\theta)\right\}. \end{aligned} \tag{3}$$

This formula clearly shows that if $\Phi(\tau)\equiv 0$ the expression for $V$ satisfies the differential equation (1).

The formula indicates that if

$$W=\int_0^\omega F[\rho,\tau]d\tau ,$$

where

$$\rho=(x+a)\xi(\tau)+(y+b)\eta(\tau)+(z+c)\zeta(\tau)-\chi(\tau) ,$$

$$(x+a)\xi(\omega)+(y+b)\eta(\omega)+(z+c)\zeta(\omega)=\chi(\omega) ,$$

the Taylor expansion of $W$ in powers of $a$, $b$ and $c$ is

$$\begin{aligned} W=&\int_0^\theta F[\sigma,\tau]d\tau+\int_0^\theta F_1[\sigma,\tau]\phi(\tau)d\tau+\frac{1}{M}F[0,\tau]\phi(\theta)\\ &+\frac{1}{2!}\int_0^\theta F_2[\sigma,\tau][\phi(\tau)]^2d\tau+\frac{1}{2!}[\phi(\theta)]^2\frac{1}{M}F_1[0,\theta]\\ &+\frac{1}{2!}\frac{1}{M}\frac{\partial}{\partial\theta}\left\{\frac{1}{M}[\phi(\theta)]^2F[0,\theta]\right\}\\ &+\frac{1}{3!}\int_0^\theta F_3[\sigma,\tau][\phi(\tau)]^3d\tau+\frac{1}{3!}[\phi(\theta)]^3\frac{1}{M}F_2[0,\theta]\\ &+\frac{1}{3!}\frac{1}{M}\frac{\partial}{\partial\theta}\left\{\frac{1}{M}[\phi(\theta)]^3F_1(0,\theta)\right\}\\ &+\frac{1}{3!}\frac{1}{M}\frac{\partial}{\partial\theta}\left\{\frac{1}{M}\frac{\partial}{\partial\theta}\left(\frac{1}{M}[\phi(\theta)]^3F(0,\theta]\right)\right\}\\ &+\cdots , \end{aligned} \tag{4}$$

where

$$\phi(\tau)=a\xi(\tau)+b\eta(\tau)+c\zeta(\tau) \ .$$

The form of this expansion reminds one of Lagrange's well known expansion and suggests the following theorem.*

2. **Extension of Lagrange's expansion.** *Let z be defined by the equation*

$$z=a+x\phi(z) \ ;$$

*then under suitable conditions we have the expansion*

$$\begin{aligned}\int_0^z F[a+x\phi(\tau)-\tau,\tau]d\tau &= \int_0^a F[a-\tau,\tau]\,d\tau \\ &+\frac{x}{1!}\left[\int_0^a F_1[a-\tau,\tau]\phi(\tau)d\tau+F[0,a]\phi(a)\right] \\ &+\frac{x^2}{2!}\left[\int_0^a F_2[a-\tau,\tau][\phi(\tau)]^2d\tau+F_1[0,a][\phi(a)]^2\right. \\ &\qquad\left.+\frac{d}{da}\{F[0,a][\phi(a)]^2\}\right] \\ &+\frac{x^3}{3!}\left[\int_0^a F_3[a-\tau,\tau][\phi(\tau)]^3d\tau+F_2[0,a][\phi(a)]^3\right. \\ &\qquad\left.+\frac{d}{da}\{F_1[0,a][\phi(a)]^3\}+\frac{d^2}{da^2}\{F[0,a][\phi(a)]^3\}\right] \\ &+\cdots,\end{aligned} \tag{1}$$

*where as before*

$$F_n[\sigma,\tau]=\frac{\partial^n}{\partial\sigma^n}F[\sigma,\tau] \ .$$

To investigate this expansion we shall endeavor to represent the definite integral on the left hand side by a contour integral

$$I=\frac{1}{2\pi i}\int_C\frac{1-x\phi'(s)}{s-x\phi(s)-a}\,ds\int_0^s F[a+x\phi(\tau)-\tau,\tau]d\tau \ ,$$

where $C$ is a simple contour enclosing the point $s=a$ and also just one root of the equation

$$s-x\phi(s)=a \ . \tag{2}$$

* The method used for this simplified form of the theorem can readily be extended so as to be applicable to the general expansion. In §4 the extension is given in detail for the case in which $F[\sigma,\tau]=1$.

When the contour $C$ is given we may choose $x$ so that for all points on the perimeter of $C$ we have the inequality

$$|x\phi(s)| < |s-a| \; ;$$

then, just as in Hermite's proof of Lagrange's theorem,* it can be shown that the equation (2) has just one root within $C$.

Let us now suppose that the functions $\phi(s)$ and $F[a+x\phi(s)-s,\, s]$ are analytic within $C$ and on its boundary for values of $x$ that enter into consideration; then expanding by Taylor's theorem we have

$$\begin{aligned}F[a+x\phi(\tau)-\tau,\tau] &= F[a-\tau,\tau]+\frac{x}{1!}\phi(\tau)F_1[a-\tau,\tau]\\ &+\frac{x^2}{2!}[\phi(\tau)]^2F_2[a-\tau,\tau]+\cdots+\frac{x^n}{n!}[\phi(\tau)]^nF_n[a-\tau,\tau]\\ &+\frac{1}{n!}\int_a^{a+x\varphi(\tau)}[x\phi(u)+a-u]^nF_{n+1}[a-u,\tau]du\ ,\end{aligned}$$

$$\begin{aligned}\frac{1-x\phi'(s)}{s-x\phi(s)-a} &= \frac{1}{s-a}-x\frac{d}{ds}\,\frac{\phi(s)}{s-a}-\frac{1}{2}x^2\frac{d}{ds}\left[\frac{\phi(s)}{s-a}\right]^2-\cdots\\ &-\frac{1}{n}x^n\frac{d}{ds}\left[\frac{\phi(s)}{s-a}\right]^n+\frac{x^{n+1}}{(s-a)^{n+1}}\left[\frac{\{\phi(s)\}^{n+1}}{s-a-x\phi(s)}\right.\\ &\left.+\phi'(s)\{\phi(s)\}^n\right].\end{aligned}$$

The coefficient of $x^n$ in the Taylor expansion of $I$ in powers of $x$ is thus

$$\begin{aligned}\frac{1}{2\pi i}\int_C\Bigg\{ &\frac{ds}{s-a}\int_0^s\frac{1}{n!}[\phi(\tau)]^nF_n[a-\tau,\tau]d\tau\\ &-ds\frac{d}{ds}\left[\frac{\phi(s)}{s-a}\right]\int_0^s\frac{d\tau}{(n-1)!}\,[(\phi\tau)]^{n-1}F^{n-1}[a-\tau,\tau]\\ &-\frac{1}{2}\,ds\frac{d}{ds}\left[\frac{\phi(s)}{s-a}\right]^2\int_0^s\frac{d\tau}{(n-2)!}[\phi(\tau)]^{n-2}F_{n-2}[a-\tau,\tau]-\cdots\\ &-\frac{1}{n}\,ds\frac{d}{ds}\left[\frac{\phi(s)}{s-a}\right]^n\int_0^s d\tau F[a-\tau,\tau]\Bigg\}\ .\end{aligned}$$

The first term in this series may be evaluated at once by Cauchy's theorem while the other terms may be evaluated by Cauchy's theorem after they

* Hedrick-Dunkel, Goursat's *Mathematical Analysis*, vol. 2, Part 1, p. 106.

have been integrated by parts, the terms that are complete differentials disappearing in an integration round a closed contour. The result is

$$\frac{1}{n!}\left[\int_0^a [\phi(\tau)]^n F_n[a-\tau,\tau]d\tau + \binom{n}{1}[\phi(a)]^n F_{n-1}[0,a]\right.$$

$$+\binom{n}{2}\left[\frac{d}{ds}\{[\phi(s)]^n F_{n-2}[a-s,s]\}\right]_{s=a}+\cdots$$

$$\left.+\binom{n}{n}\left[\frac{d^{n-1}}{ds^{n-1}}\{[\phi(s)]^n F[a-s,s]\}\right]_{s=a}\right],$$

where

$$\binom{n}{m}=\frac{n!}{m!(n-m)!}.$$

Let us now write

$$\frac{d}{ds}=D_1+D_2,$$

where $D_1$ operates only on the $s$ in the first argument of a function $F_k(a-s,s)$, while $D_2$ operates on the $s$ in $\phi(s)$ and also on the $s$ in the second argument of $F_k(a-s,s)$. The coefficient of

$$\frac{1}{n!}\frac{d^k}{da^k}\{F_{n-k-1}[0,a][\phi(a)^n]\}$$

in the above expression is then found to be

$$C_n=\binom{n}{k+1}\binom{k}{k}-\binom{n}{k+2}\binom{k+1}{k}+\cdots+(-1)^{n-k-1}\binom{n}{n}\binom{n-1}{k}.$$

Making use of the identity

$$\binom{n+1}{r}=\binom{n}{r}+\binom{n}{r-1}$$

we find that

$$C_{n+1}=C_n+\binom{n}{k}(1-1)^{n-k}$$

$$=C_n, \qquad n>k.$$

But

$$C_{k+1}=1.$$

Therefore

$$C_n=1, \qquad n>k.$$

The term involving $x^n$ is thus

$$
(3)\qquad \begin{aligned} &\frac{x^n}{n!}\left[\int_0^a F_n[a-\tau,\tau][\phi(\tau)]^n d\tau+F_{n-1}[0,a][\phi(a)]^n\right. \\ &\left.+\frac{d}{da}\{F_{n-2}[0,a][\phi(a)]^n\}+\cdots+\frac{d^{n-1}}{da^{n-1}}\{F[0,a][\phi(a)[^n\}\right], \end{aligned}
$$

and so the expansion has the form given above.

It is easy to obtain a formula for the remainder after the term (3) but the formula is complicated and hardly worth writing down. An expression for the remainder may be obtained also by remarking that the integral

$$
\int_0^z F[a+x\phi(\tau)-\tau,\tau]d\tau
$$

is a function of $z$ and, when this function is analytic, Lagrange's theorem may be applied to it in the usual way and a formula for the remainder written down.

The following alternative method of obtaining the coefficients in the expansion has been developed from a suggestion made by an editor of this journal.

Differentiating the expression

$$
I=\int_0^z F[a+x\phi(\tau)-\tau,\tau]d\tau
$$

$n$ times with respect to $x$ we obtain

$$
\begin{aligned} \frac{d^nI}{dx^n}=\frac{d^{n-1}}{dx^{n-1}}\left[\frac{dz}{dx}F(0,z)\right]&+\frac{d^{n-2}}{dx^{n-2}}\left[\frac{dz}{dx}\,\phi(z)F_1(0,z)\right] \\ +\frac{d^{n-3}}{dx^{n-3}}&\left[\frac{dz}{dx}\{\phi(z)\}^2F_2(0,z)\right]+\cdots \\ &+\int_0^z F_n[a+x\phi(\tau)-\tau,\tau]\{\phi(\tau)\}^n d\tau\,. \end{aligned}
$$

Putting $x=0$ to calculate the coefficients in the Maclaurin expansion in ascending powers of $x$, we may derive the value of

$$
\left\{\frac{d^{n-s}}{dx^{n-s}}\left[\frac{dz}{dx}\{\phi(z)\}^{s-1}F_{s-1}(0,z)\right]\right\}_{x=0}
$$

from the Lagrangian expansion of

$$
\frac{dz}{dx}\{\phi(z)\}^{s-1}F_{s-1}(0,z)
$$

in ascending powers of $x$. The coefficients of $x^n/n!$ is thus seen to have the form

$$\int_0^a F_n[a-\tau,\tau][\phi(\tau)]^n d\tau + F_{n-1}[0,a][\phi(a)]^n + \frac{d}{da}\{F_{n-2}[0,a][\phi(a)]^n\}$$
$$+ \cdot \cdot \cdot + \frac{d^{n-1}}{da^{n-1}}\{F[0,a][\phi(a)]^n\} .$$

An editor has kindly remarked that our expansion problem can be regarded as a particular case of the following more general expansion problem.

Let $w=f(x, z)$ be a function which is holomorphic in the neighborhood of $x=0$ and $z=a$, let $\phi(z)$ be holomorphic in the neighborhood of $z=a$ and let $z=z(x)$ be that solution of the equation $z=a+x\phi(z)$ which is holomorphic in the neighborhood of $x=0$ and tends to the value $a$ as $x\rightarrow 0$. The problem is to expand $w$ in a power series in $x$ convergent for sufficiently small values of $|x|$.

This problem, in its turn, is merely a special case of a more general problem treated by Cauchy.*

Given a function $g(z, x)$ holomorphic for $|z|<r$, $|x|<\rho$, for which $z=0$ is an $m$-fold 0 of $g(z, 0)$. Let $r_1<r$ and $\rho_1<\rho$ be so chosen that for $|x|\leqq\rho_1$, the function $g(z, x)$ is different from zero on the circle $|z|=r_1$ and admits of $m$ zeros $z_1$, $z_2$, $z_m$ inside of this circle, which zeros are continuous functions of $x$, vanishing with $x$. Finally let $f(x, z)$ be an analytic function of $x$ and $z$ which is holomorphic for $|x|<\rho_1$, $|z|<r_1$. Then the sum

$$F(x)=f(x,z_1)+f(x,z_2)+ \cdot \cdot \cdot +f(x,z_m)$$

is a holomorphic function of $x$ in $|x|<\rho_1$ and expressions can be found for the coefficients in the Maclaurin series for $F(x)$ in powers of $x$.

3. **Special cases of the expansion.** When $a=0$, $\phi(\tau)=1$, $F=F(x-\tau, \tau)$ we have the expansion

$$\int_0^x F[x-\tau,\tau]\,d\tau = \frac{x}{1!}F[0,0]+\frac{x^2}{2!}\Big[F_{10}[0,0]+F_{01}[0,0]\Big]$$
$$+\frac{x^3}{3!}\Big[F_{20}[0,0]+F_{11}[0,0]+F_{02}[0,0]\Big]+ \cdot \cdot \cdot ,$$

where now

$$F_{mn}[\sigma,\tau]=\frac{\partial^{m+n}}{\partial\sigma^m\partial\tau^n}\ F[\sigma,\tau] .$$

* E. Lindelöf, *Calcul des Résidus*, 1905, p. 24.

In particular, if $F=f(x-\tau)g(\tau)$ where $f(x)$ and $g(x)$ are analytic in the neighborhood of $x=0$, we have the result that for sufficiently small values of $x$, the definite integral

$$h(x)=\int_0^x f(x-t)g(t)dt$$

can be expanded in the form

$$h(x)=\frac{x}{1!}f(0)g(0)+\frac{x^2}{2!}[f'(0)g(0)+f(0)g'(0)]$$
$$+\frac{x^3}{3!}[f''(0)g(0)+f'(0)g'(0)+f(0)g''(0)]+\cdots,$$

where primes denote differentiations with respect to the argument. Thit result occurs in a slightly different form in Borel's theory of divergens series.* Writing

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\cdots,$$

$$g(x)=g(0)+\frac{x}{1!}g'(0)+\frac{x^2}{2!}g''(0)+\cdots,$$

$$F(y)=\int_0^\infty e^{-xy}f(x)dx,\quad G(y)=\int_0^\infty e^{-xy}g(x)dx,\quad F(y)G(y)=\int_0^\infty e^{-xy}h(x)dx,$$
$$F(y)=f(0)y^{-1}+f'(0)y^{-2}+f''(0)y^{-3}+\cdots,$$
$$G(y)=g(0)y^{-1}+g'(0)y^{-2}+g''(0)y^{-3}+\cdots,$$
$$F(y)G(y)=f(0)g(0)y^{-2}+[f'(0)g(0)+f(0)g'(0)]y^{-3}+\cdots,$$

the form of the series for $h(x)$ is at once suggested.

Since integral equations of type

$$h(x)=\int_0^x f(x-\tau)\ g(\tau)d\tau$$

are of some importance in analysis† it should be worth while to consider the more general class of integral equations of type

* *Leçons sur les Séries Divergentes*, Paris, 1901. This expansion and the previous one may be obtained by writing $\tau=sx$ in the integral and expanding in powers of $x$ by Maclaurin's theorem. The most direct method, however, is that given at the end of §2.

† A method of solving equations of this type which depends on a determination of the functions $F(y)$ and $G(y)$ was suggested by Vilfredo Pareto, Journal für Mathematik, vol. 110 (1892), p. 290. The method was suggested again in a more general form by the present author *Report on integral equations*, British Association Report, 1910, and was actually used in the solution of

$$h(x) = \int_a^z F[a + x\phi(\tau) - \tau]g(\tau)d\tau \ ,$$

where

$$z = a + x\phi(z) \ ,$$

and $g(\tau)$ is the function to be determined.

When the definite integral can be expanded in a convergent power series

$$h(x) = \frac{x}{1!}F(0)g(a)\phi(a) + \frac{x^2}{2!}\Big[ F'(0)\{g(a)\}\{\phi(a)\}^2$$
$$+ F(0)\frac{d}{da}\{g(a)[\phi(a)]^2\}\Big] + \frac{x^3}{3!}\Big[F''(0)g(a)\{\phi(a)\}^3$$
$$+ F'(0)\frac{d}{da}\{g(a)[\phi(a)]^3\} + F(0)\frac{d^2}{da^2}\{g(a)[\phi(a)]^3\}\Big] + \cdots \ ,$$

the quantities $g(a)$, $g'(a)$, $g''(a)$, $\cdots$ can be determined uniquely from the coefficients in the known power series for $h(x)$. The function $g(\tau)$ can then be calculated for other values of $\tau$ by Taylor's theorem. This solution is, of course, of a purely formal nature and must be supplemented by convergence theorems.

4. **An application of Lagrange's expansion.** To prove that the quantity $\theta$ of § 1 is a solution of

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\theta = 0 \ ,$$

it is convenient to establish the formula

$$(1) \qquad g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\theta = \left(\frac{1}{M}\frac{\partial}{\partial \theta}\right)^{m-1}\left\{\frac{g[\xi(\theta)\ ,\ \eta(\theta)\ ,\ \zeta(\theta)]}{M}\right\} ,$$

where $g(\xi, \eta, \zeta)$ is any homogeneous polynomial of degree $m$ in $\xi, \eta, \zeta$. Consider the equation

$$(x+a)\xi(\omega) + (y+b)\eta(\omega) + (z+c)\zeta(\omega) = \chi(\omega) \ .$$

Writing

$$\sigma = \chi(\omega) - x\xi(\omega) - y\eta(\omega) - x\zeta(\omega) \ ,$$

$$\omega = \phi(\sigma) \ , \qquad \theta = \phi(0) \ ,$$

$$\sigma = a\xi[\phi(\sigma)] + b\eta[\phi(\sigma)] + c\zeta[\phi(\sigma)] \ ,$$

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a problem in Messenger of Mathematics, vol. 49 (1920), p. 1. The method has been used recently by J. R. Carson in a number of articles appearing in the Bell System Technical Journal.

and introducing a parameter $t$ which is afterwards put equal to unity we may write the last equation in the form

$$\sigma=t\{a\xi[\phi(\sigma)]+b\eta[\phi(\sigma)]+c\zeta[\phi(\sigma)]\} .$$

We may now expand the function $\omega=\phi(\sigma)$ in ascending powers of $t$ by Lagrange's theorem* if $a$, $b$ and $c$ are sufficiently small. The expansion is

$$\omega=\phi(\sigma)=\phi(0)+\sum_{n=1}^{\infty}\frac{t^n}{n!}\left\{\frac{d^{n-1}}{d\sigma^{n-1}}\left[\phi'(\sigma)\{a\xi[\phi(\sigma)]+b\eta[\phi(\sigma)]+c\zeta[\phi(\sigma)]\}\right]\right\}_{\sigma=0}.$$

Putting $t=1$ and transforming back to the variable $\theta$ we obtain the expansion

$$\omega=\theta+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{1}{M}\frac{\partial}{\partial\theta}\right)^{n-1}\left[\frac{\{a\xi(\theta)+b\eta(\theta)+c\zeta(\theta)\}^n}{M}\right]. \tag{2}$$

Regarding this as the Taylor expansion of $\omega$ in powers of $a$, $b$ and $c$, we may calculate the different partial derivatives of order $m$ with respect to the three independent variables $x$, $y$ and $z$ by finding the coefficients of the different products of powers of $a$, $b$ and $c$. The general result can be expressed in the form (1). The result of § 1 may be derived from that of § 2 with the aid of the above device of introducing an auxiliary variable $t$.

This device is a familiar one in the theory of Taylor series in several variables and the theorem used in obtaining (2) is really one generalization of Lagrange's theorem for the case of several independent variables. An entirely different generalization giving a power series in more than one variable has been obtained by Darboux† for the case of two variables and by Stieltjes‡ for the case of $n$ variables. Stieltjes remarks that the method by which Heine derived Lagrange's expansion by the calculus of variations§ can be generalized so as to give more general expansions.|| Many other generalizations of Lagrange's expansion are known. Besides

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* J. L. Lagrange, Berlin Memoirs, 1768.

† G. Darboux, Comptes Rendus, vol. 68, p. 324; C. Hermite, *Cours d'Analyse*, 4th edition, p. 182; H. Poincaré, Acta Mathematica, vol. 9.

‡ T. J. Stieltjes, Annales de l'Ecole Normale, ser. 3, vol. 2 (1885), p. 93. Collected works, vol. 1, p. 445.

§ E. Heine, Journal für die reine und Angewandte Mathematik, vol. 54 (1857).

|| An application of the general expansion of Stieltjes is mentioned in a paper by the author, Bulletin of the American Mathematical Society, vol. 22 (1916), p. 329. There is, however, a mistake in sign in equations (10). These equations should read $\sigma_p=\mu_p-\xi_p\theta$, $\tau_p=\nu_p-\eta_p/\theta$.

those due to Bürmann,* Teixeira† and Rouché‡ that are given in Whittaker and Watson's *Modern Analysis*, there is a recent generalization due to Kössler.§ References to some further developments are given below‖.

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* Bürmann, Mémoires de l'Institut, vol. 2 (1799), p. 13. See also A. C. Dixon Proceedings of the London Mathematical Society, ser. 1, vol. 34 (1902) p. 151.

† F. G. Teixeira, Resal Journal, ser. 3, vol. 7 (1881), p. 277. Académie Royale de Belgique, Mémoires in 4 to, ser. 2, vol. 1. Journal für die reine und angewandte Mathematik, vol. 122 (1900), p. 97.

‡ E. Rouché, Paris Memoirs, vol. 18 (1868), p. 422.

§ M. Kössler, Proceedings of the London Mathematical Society, ser. 2, vol. 20 (1922), p. 365.

‖ G. Zolotareff, Nouvelles Annales, ser. 2, vol. 15 (1876), p. 422.
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