

SOME PROBLEMS IN THE THEORY OF INTERPOLATION BY STURM-LIOUVILLE FUNCTIONS*

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A problem which receives a large share of attention in modern analysis consists in the determination of the properties of linear combinations of orthogonal functions, particularly with reference to the possibility of obtaining, by these combinations, approximate representations of certain classes of functions. Let $\varphi_0(x), \varphi_1(x), \dots, \varphi_p(x)$ denote a sequence of functions orthogonal in the interval $a \leq x \leq b$, and let $f(x)$ represent an arbitrary function. The linear combination

$$(1) \quad \alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + \dots + \alpha_p\varphi_p(x),$$

where the coefficients are defined thus:

$$\alpha_n = \frac{\int_a^b f(x)\varphi_n(x)dx}{\int_a^b \varphi_n^2(x)dx} \quad (n=0, 1, \dots, p),$$

is studied with reference to the question of its convergence toward $f(x)$, as p is allowed to increase without limit. The classical example of such series is the Fourier cosine series.

The value of the coefficient α_n , as defined above, depends upon the behavior of $f(x)$ everywhere throughout the interval $a \leq x \leq b$. Another class of problems arises if the coefficient is defined so that its value shall depend upon the values of $f(x)$ only at discrete points of the interval. In particular, let the interval $a \leq x \leq b$ be subdivided into p equal parts by the points $x_0 = a, x_1, x_2, \dots, x_p = b$, and consider the sum

$$(2) \quad \alpha_{0p}\varphi_0(x) + \alpha_{1p}\varphi_1(x) + \dots + \alpha_{pp}\varphi_p(x),$$

where

$$\alpha_{np} = \frac{\sum_{k=0}^p f(x_k)\varphi_n(x_k)}{\sum_{k=0}^p \varphi_n^2(x_k)} \quad (n=0, 1, \dots, p),$$

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the symbol \sum' being used in the following sense:

$$\sum'_{k=0}^p y_k = \frac{1}{2} y_0 + \sum_{k=1}^{p-1} y_k + \frac{1}{2} y_p.$$

The expression given in (2) will be referred to as the interpolating formula, of order p , for $f(x)$ with respect to the orthogonal system $\varphi_0(x), \varphi_1(x), \dots$, in the sense that it is a formula of approximation determined by the values of $f(x)$ at a finite number of points, not that it necessarily takes on the values of $f(x)$ at these points. The classical example of such a formula is found in the cosine interpolating formula, which, as we shall presently indicate, may be regarded as a special case of the ordinary formula for trigonometric interpolation. Investigations into the properties of the latter, by de la Vallée Poussin,* Faber,† Jackson,‡ and others, have yielded results which are noteworthy because of their close parallelism, both in substance and mode of attainment, to those obtaining in the case of Fourier series. Further problems suggested by a consideration of (2) are quite similar to those studied in connection with (1), but in the case of (2) the solutions have not, in general, been so extensively worked out.

The particular orthogonal function system with which we shall deal in this paper is formed by the characteristic functions of the so-called Sturm-Liouville differential system

$$\begin{aligned} (I) \quad & u''(x) + [\rho^2 - \lambda(x)]u(x) = 0, \\ & u'(0) - hu(0) = 0, \\ & u'(\pi) + Hu(\pi) = 0, \end{aligned}$$

where h and H are real, but unrestricted as to sign, and $\lambda(x)$ is for the present merely defined and continuous in the interval $0 \leq x \leq \pi$. Let the solutions of this system corresponding to the characteristic numbers $\rho_0^2, \rho_1^2, \dots$, arranged in order of magnitude, be denoted by $u_0(x), u_1(x), \dots$, and consider the sum

$$\sum_p [f(x)] = \alpha_{0p} u_0(x) + \alpha_{1p} u_1(x) + \dots + \alpha_{pp} u_p(x),$$

* C. de la Vallée Poussin, *Sur la convergence des formules d'interpolation entre ordonnées équidistantes*, Bulletins de l'Académie Royale de Belgique, Classe des Sciences, 1908, pp. 319-403.

† Faber, *Über stetige Funktionen*, Mathematische Annalen, vol. 69 (1910), pp. 372-441; pp. 417-424.

‡ D. Jackson, *On the accuracy of trigonometric interpolation*, these Transactions, vol. 14 (1913), pp. 453-461; pp. 453-456.

where

$$\alpha_{np} = \frac{\sum_{k=0}^p f(x_k) u_n(x_k)}{\sum_{k=0}^p u_n^2(x_k)} \quad (n=0, 1, \dots, p).$$

The expression $\sum_p [f(x)]$, which will be referred to as the Sturm-Liouville interpolating formula for $f(x)$, constitutes the subject for investigation.

In the discussion, some reference will be made to sums closely allied, in one way or another, with $\sum_p [f(x)]$. These are the following:

(a) the partial sum of the Sturm-Liouville series,

$$\sigma_p [f(x)] = \alpha_0 u_0(x) + \alpha_1 u_1(x) + \dots + \alpha_p u_p(x),$$

where

$$\alpha_n = \frac{\int_0^\pi f(x) u_n(x) dx}{\int_0^\pi u_n^2(x) dx};$$

(b) the cosine interpolation formula, of order p ,

$$T_p [f(x)] = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_p \cos px,$$

where

$$a_{np} = \frac{\sum_{k=0}^p f(x_k) \cos nx_k}{\sum_{k=0}^p \cos^2 nx_k};$$

(c) the partial sum of the Fourier cosine series,

$$t_p [f(x)] = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_p \cos px,$$

where

$$a_n = \frac{\int_0^\pi f(x) \cos nx dx}{\int_0^\pi \cos^2 nx dx}.$$

It should be noticed here that if we specialize the Sturm-Liouville differential system by setting $h = H = 0$, $\lambda(x) \equiv 0$, the resulting characteristic solutions are precisely the cosine functions. Furthermore, it can be readily shown that, if $f(x)$ is defined outside the interval $0 \leq x \leq \pi$ so as to make it

an even periodic function of period 2π , then the ordinary formula for trigonometric interpolation, using an even number ($2p$) of interpolating points* evenly distributed throughout the interval $0 \leq x \leq 2\pi$, reduces precisely to the cosine formula $T_p[f(x)]$. An analogous relation exists between $t_p[f(x)]$ and the partial sum of the ordinary Fourier series.

A brief outline of the topics to be treated is as follows. In the first section are listed a number of facts concerning the nature of the characteristic numbers and solutions of the Sturm-Liouville differential system. In Section 2 there is outlined a proof of the convergence of $\sum_p[f(x)]$ to $f(x)$, provided $f(x)$ satisfies suitable conditions, followed in Section 3 by a detailed proof of the so-called "equivalence" theorem. In the last section there is outlined briefly the method by which we establish another theorem, concerning the rapidity of convergence of $\sum_p[\varphi_p(x)]$ to $\varphi_p(x)$, where $\varphi_p(x)$ is itself a Sturm-Liouville sum. The statement of a corollary, relative to the rapidity of convergence of $\sum_p[f(x)]$ to $f(x)$, when the latter satisfies a Lipschitz condition, concludes the paper. The analysis is rather laborious, especially in the last section, and, to keep the paper from running to inordinate length, the exposition has been much condensed. It is believed, however, that the indications are sufficient to enable the reader to supply the missing details with reasonable directness (except possibly in the case of Theorem IV, which is merely stated without proof, and of which no further use is made). Copies of a more complete version in manuscript are on file in the library of the University of Minnesota and in the library of the Society.

1. **Preliminary statements.** (a) The solutions of the system

$$\begin{aligned}u''(x) + [\rho^2 - \lambda(x)]u(x) &= 0, \\ u'(0) - hu(0) &= 0,\end{aligned}$$

cannot be essentially complex, provided that ρ^2 , h , and $\lambda(x)$ are real.

(b) There are infinitely many real values of ρ^2 for which the system (I) is compatible; they have no cluster point in the finite plane, and only a finite number of them can be negative. To each of these values of ρ^2 corresponds a solution $u(x)$ uniquely determined except for a multiplicative constant.

(c) If the index n be chosen such that $\rho_0^2 < \rho_1^2 < \rho_2^2 \cdots$, then the characteristic function $u_n(x)$ corresponding to ρ_n will possess precisely n zeros in the interval $\dagger 0 \leq x \leq \pi$.

* Cf. de la Vallée Poussin, loc. cit., p. 370.

† M. Bôcher, *Leçons sur les Méthodes de Sturm*, p. 69.

(d) When n is sufficiently large so that $\rho_n^2 > 0$, the following asymptotic formula holds:*

$$u_n(x) = \cos \rho_n x + \frac{h \sin \rho_n x}{\rho_n} + \frac{1}{\rho_n} \int_0^x \lambda(t) u_n(t) \sin \rho_n(x-t) dt.$$

(e) If $\rho_n^2 > 0$, then †

$$\rho_n = n + \frac{C_n}{n},$$

where ρ_n denotes the positive square root of ρ_n^2 .

(f) If $\lambda(x)$ is further restricted so as to possess a continuous derivative in the interval $0 \leq x \leq \pi$, the last asymptotic relation is capable of further refinement, as indicated thus:‡

$$\rho_n = n + \frac{C}{n} + \frac{r_n}{n^2},$$

where C is independent of n .

(g) Under the hypothesis just stated, §

$$u_n(x) = \cos nx + \frac{\beta(x) \sin nx}{n} + \frac{\alpha(x, n)}{n^2},$$

where $\beta(x)$ is independent of n , and has a continuous second derivative in $(0, \pi)$.

It will be assumed throughout the remainder of this paper that $\lambda(x)$ does possess a continuous derivative in the interval $0 \leq x \leq \pi$.

2. Convergence of the Sturm-Liouville interpolating formula. The proof of the equivalence theorem, which will occupy our attention in the following section, depends in part upon the uniform convergence to the right values of the Sturm-Liouville interpolating development of an analytic function. The demonstration of this fact will be outlined in the present section, although, instead of limiting ourselves to the consideration of analytic functions, we

* Kneser, *Darstellung willkürlicher Funktionen*, *Mathematische Annalen*, vol. 58 (1904), pp. 81-147; p. 118.

† Cf. Kneser, loc. cit., p. 120. Throughout this paper, any functional symbol involving x and one or more integral parameters, either as arguments or subscripts, shall denote a function of x continuous in the interval $0 \leq x \leq \pi$ and uniformly bounded for all values of the parameters involved, and, likewise, any letter affected with one or more subscripts shall denote a constant with respect to x bounded for all values of the subscripts, with the exception of ρ_n , which is the standard notation for the characteristic numbers.

‡ E. W. Hobson, *On a general convergence theorem, and the theory of the representation of a function by series of normal functions*, *Proceedings of the London Mathematical Society*, vol. 6 (1908), pp. 349-395; p. 378.

§ Cf. Hobson, loc. cit., p. 378.

shall indicate the proof for a wider class of functions, namely, those satisfying Lipschitz conditions. The method to be used parallels to a large extent that employed by Jackson in establishing the order of convergence of the Sturm-Liouville series.* The theorem to be proved may be stated as follows:

THEOREM I. *If $f(x)$ satisfies a Lipschitz condition,*

$$|f(x_2) - f(x_1)| \leq \mu |x_2 - x_1|,$$

throughout the interval $0 \leq x \leq \pi$, then

$$\lim_{p \rightarrow \infty} \sum_p [f(x)] = f(x)$$

uniformly in the interval.

The proof consists mainly in obtaining a suitable dominating expression for the difference $\alpha_{np} u_n(x) - a_{np} \cos nx$. The essential properties of the coefficients α_{np} are brought out in the following lemmas.

LEMMA I. *If $f(x)$ satisfies a Lipschitz condition*

$$|f(x_2) - f(x_1)| \leq \mu |x_2 - x_1|$$

throughout the interval $0 \leq x \leq \pi$, then†

$$\begin{aligned} \frac{1}{p} \left| \sum_{k=0}^p f(x_k) \cos nx_k \right| &= \frac{r_{np}}{n}, \\ \frac{1}{p} \left| \sum_{k=0}^p f(x_k) \sin nx_k \right| &= \frac{r_{np}}{n} \quad (n = 1, 2, \dots, p). \end{aligned}$$

The method by which these results are obtained is set forth in a paper by Jackson,‡ although under somewhat different conditions with regard to the function $f(x)$ and to the length of interval over which the summation is extended. The proof, as adapted to the particular situation under consideration, is similar in character.

* Jackson, *On the degree of convergence of Sturm-Liouville series*, these Transactions, vol. 15 (1914), pp. 439-466; see pp. 453-456.

† When no confusion is likely, the same letter may be used to denote different constants or functions, subject to the conditions laid down in a previous footnote.

‡ D. Jackson, *On the order of magnitude of the coefficients in trigonometric interpolation*, these Transactions, vol. 21 (1920), pp. 321-332; pp. 323, 324.

LEMMA II. For all values of $p > n > 0$,

$$\left[\sum_{k=0}^p u_n^2(x_k) \right]^{-1} = \begin{cases} \frac{2}{p} \left(1 + \frac{r_{np}}{n^2} \right) & (n=1, 2, \dots, s-1), \\ \frac{2}{p} \left(1 + \frac{r_{np}}{p} \right) & (n=s, s+1, \dots, p-1), \end{cases}$$

where s denotes $p/2$ or $(p+1)/2$, according as p is even or odd. When $n = p$, the factor $2/p$ must be replaced by $1/p$.

The expression $\sum' u_n^2(x_k)$ is different from zero in all cases, since $u_n(0) = 1$. Squaring both sides of the equality

$$u_n(x) = \cos nx + \frac{\beta(x) \sin nx}{n} + \frac{\alpha(x, n)}{n^2},$$

we obtain an expression for $u_n^2(x)$ of the form

$$u_n^2(x) = \cos^2 nx + \frac{\beta(x) \sin 2nx}{n} + \frac{\gamma(x, n)}{n^2}.$$

Writing $\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$ and performing the indicated summation, we obtain, for $n = 1, 2, \dots, p-1$,

$$(3) \quad \sum_{k=0}^p u_n^2(x_k) = p \left(\frac{1}{2} + \frac{1}{np} \sum_{k=0}^p \beta(x_k) \sin 2nx_k + \frac{r_{np}}{n^2} \right),$$

since $\sum' \cos 2nx_k = 0$. The modification for the case $n = p$ is apparent, and need not be explicitly mentioned further. Let us consider the sum

$$\frac{1}{p} \sum_{k=0}^p \beta(x_k) \sin 2nx_k.$$

For $n = 1, 2, \dots, s-1$, the fact that $2n < p$, together with the additional fact that $\beta'(x)$ is continuous, permits the application of the preceding lemma to the sum in question, enabling us to write

$$\frac{1}{p} \sum_{k=0}^p \beta(x_k) \sin 2nx_k = \frac{r_{np}}{n} \quad (n=1, 2, \dots, s-1),$$

and, from (3),

$$\sum_{k=0}^p u_n^2(x_k) = p \left(\frac{1}{2} + \frac{r_{np}}{n^2} \right) \quad (n=1, 2, \dots, s-1).$$

For $n = s, s + 1, \dots, p - 1$, the factor $1/n$ appearing in (3) is at most equal to $2/p$, consequently

$$\sum_{k=0}^p u_n^2(x_k) = p \left(\frac{1}{2} + \frac{r_{np}}{p} \right) \quad (n = s, s + 1, \dots, p - 1).$$

We are now prepared to consider the reciprocal of the sum. Choose N sufficiently large, so that

$$\left| \frac{r_{np}}{n^2} \right| < \frac{1}{4}, \quad \left| \frac{r_{np}}{p} \right| < \frac{1}{4},$$

for all values of n and p subject to the inequality $p > n \geq N$. It is then legitimate, for these values of the indices, to write the reciprocals in the form*

$$\left[\sum_{k=0}^p u_n^2(x_k) \right]^{-1} = \begin{cases} \frac{2}{p} \left(1 + \frac{r_{np}}{n^2} \right) & (n = N, N + 1, \dots, s - 1) \\ \frac{2}{p} \left(1 + \frac{r_{np}}{p} \right) & (n = s, s + 1, \dots, p - 1). \end{cases}$$

For the remaining values of n , ranging from 1 to $N - 1$, inclusive, we can choose r_{np} , bounded for all values of $p > n$, so that the above expression for $\left[\sum u_n^2(x_k) \right]^{-1}$ still remains valid. The proof of this assertion is essentially contained in the facts that $u_n(x)$ is continuous, and that $\sum u_n^2(x_k) \geq 1$, since $u_n(x)$ is real and $u_n(0) = 1$.

LEMMA III. For $0 \leq x \leq \pi$,

$$\left| \alpha_{np} u_n(x) - a_{np} \cos nx \right| < \frac{C}{n^2} \quad (n = 1, 2, \dots, p),$$

where

$$\alpha_{np} = \frac{\sum_{k=0}^p f(x_k) u_n(x_k)}{\sum_{k=0}^p u_n^2(x_k)},$$

$$a_{np} = \frac{2}{p} \sum_{k=0}^p f(x_k) \cos nx_k,$$

C being independent of p and n .

* For present purposes it will suffice to use a less refined form of this equality, namely,

$$\left[\sum_{k=0}^p u_n^2(x_k) \right]^{-1} = \frac{2}{p} \left(1 + \frac{r_{np}}{n} \right) \quad (n = 1, 2, \dots, p - 1),$$

but the proof of the equivalence theorem, in the next section, demands the more elaborate form.

With the aid of the asymptotic formula for $u_n(x)$ and the preceding lemma we may write, for $n = 1, 2, \dots, p - 1$,

$$\begin{aligned} \alpha_{np} = & 2 \left(1 + \frac{r_{np}}{n} \right) \left[\frac{1}{p} \sum_{k=0}^p f(x_k) \cos nx_k \right. \\ & + \frac{1}{np} \sum_{k=0}^p f(x_k) \beta(x_k) \sin nx_k \\ & \left. + \frac{1}{n^2 p} \sum_{k=0}^p f(x_k) \alpha(x_k, n) \right]. \end{aligned}$$

By Lemma I we obtain, in the product of the bracketed terms, a number of quantities of order $1/n^2$, the sum of which we denote by $(r_{np})/n^2$. Hence

$$\alpha_{np} = \frac{2}{p} \sum_{k=0}^p f(x_k) \cos nx_k + \frac{r_{np}}{n^2}.$$

Multiplying through by $u_n(x)$, expressed in its asymptotic form, and again collecting the terms of order $1/n^2$, we obtain finally

$$\begin{aligned} \alpha_{np} u_n(x) &= \frac{2}{p} \cos nx \sum_{k=0}^p f(x_k) \cos nx_k + \frac{r_{np}(x)}{n^2} \\ &= a_{np} \cos nx + \frac{r_{np}(x)}{n^2} \quad (n=1, 2, \dots, p-1). \end{aligned}$$

The lemma follows directly from the last equality. The proof for $n = p$ is obtained by replacing $2/p$ by $1/p$ at the appropriate stages in the discussion.

The subsequent procedure consists in expressing $f(x) - \sum_p[f(x)]$ as the sum of certain differences which can be made arbitrarily small by a proper choice of p and a subsidiary index N . These differences will involve, besides terms of $\sum_p[f(x)]$, also terms from the sums $\sigma_p[f(x)]$, $T_p[f(x)]$, and $t_p[f(x)]$, and, in order to simplify the notation, we let

$$\sum_r^s, \quad \sigma_r^s, \quad T_r^s, \quad t_r^s$$

denote the sums of the terms of orders r to s inclusive, of the respective formulas. The following inequality will be employed:

$$\begin{aligned} \left| f(x) - \sum_p[f(x)] \right| &\leq \left| \sigma_0^N - \sum_0^N \right| + \left| T_{N+1}^p - \sum_{N+1}^p \right| + \left| T_0^N - t_0^N \right| \\ &+ \left| t_0^N - f(x) \right| + \left| f(x) - T_0^p \right| + \left| f(x) - \sigma_0^N \right|, \end{aligned}$$

where N is an integer presently to be determined. Let the six terms of the right-hand member be denoted by D_1, D_2, \dots, D_6 . Select any $\epsilon > 0$; then choose N sufficiently large so that

$$D_2 < \frac{\epsilon}{6}, \quad D_4 < \frac{\epsilon}{6}, \quad D_6 < \frac{\epsilon}{6},$$

for all values of $p \geq N + 1$. Holding N fast, choose P so large that

$$D_1 < \frac{\epsilon}{6}, \quad D_3 < \frac{\epsilon}{6}, \quad D_5 < \frac{\epsilon}{6},$$

for all values of $p \geq P$. Adding these inequalities, we arrive at the conclusion that, corresponding to any $\epsilon > 0$, there exists an integer P , such that

$$|f(x) - \sum_p [f(x)]| < \epsilon, \quad p \geq P.$$

It remains to justify these inequalities. Applying Lemma III to D_2 , we find without difficulty that

$$\left| T_p - \sum_{N+1}^p \right| < \frac{C}{N}.$$

The inequalities governing D_4, D_5 , and D_6 depend on the uniform convergence to $f(x)$ of $t[f(x)]$,* $T_p[f(x)]$,† and $\sigma[f(x)]$ ‡ respectively. In regard to D_1 and D_3 , we are dealing essentially with the difference between the integral of a continuous function and the finite sum which tends toward the integral as a limit. Since only a finite number N of terms are involved, the conclusion is valid.

3. The equivalence theorem. Let $v_0(x), v_1(x), \dots$, and $\bar{v}_0(x), \bar{v}_1(x), \dots$, represent two function systems, each orthogonal in the interval $a \leq x \leq b$. The statement that the two series§

$$f(x) \sim \sum_{n=0}^{\infty} b_n v_n(x), \quad f(x) \sim \sum_{n=0}^{\infty} \bar{b}_n \bar{v}_n(x)$$

* As the theorem on the convergence of Sturm-Liouville series is needed in any event (cf. footnote †), it is perhaps simplest in this connection merely to point out once more that the cosine series is a special case of the Sturm-Liouville series.

† Cf., e.g., Jackson, these Transactions, vol. 14, loc. cit., see pp. 455,456. The passage cited deals, to be sure, with the case in which an interval of length 2π is divided into an odd number of equal parts, but the same method of treatment applies to the problem involved here, which, it will be remembered, is essentially that of representing an even function of period 2π , with subdivision of a period interval into an even number of equal parts.

‡ Jackson, these Transactions, vol. 15, loc. cit., see p. 453.

§ The symbol \sim signifies that the series represents the formal expansion of $f(x)$.

possess "essentially the same convergence properties," according to Walsh,* means that the series

$$\sum_{n=0}^{\infty} [b_n v_n(x) - \bar{b}_n \bar{v}_n(x)]$$

converges absolutely and uniformly to zero throughout the interval $a \leq x \leq b$. In his papers† on the subject he adduces two cases where the expansions of $f(x)$ based on two function systems are equivalent‡; in one case the systems consist respectively of the sine functions

$$\sqrt{2} \sin k\pi x \quad (k=1, 2, \dots),$$

and the normalized solutions of the differential system§

$$\begin{aligned} u''(x) + [\rho^2 - g(x)] u(x) &= 0, \\ u(0) &= 0, \\ u(1) &= 0, \end{aligned}$$

but as yet the equivalence theorem has not been extended to the case where the function systems consist respectively of the cosine functions and the general Sturm-Liouville functions.

If, however, we widen the significance of the term "equivalence" by dropping the restrictions that $\sum [b_n v_n(x) - \bar{b}_n \bar{v}_n(x)]$ shall converge *absolutely*, we then possess an equivalence theorem, due to Haar,|| for the expansions of $f(x)$ in terms of the cosine and Sturm-Liouville functions, respectively. It is our purpose here to establish an analogous theorem relative to the expansions of $f(x)$ by means of the corresponding interpolating formulas. This theorem is quite directly deducible from another more general conclusion, which may properly be introduced as a separate theorem. The statement of the latter is as follows:

THEOREM II. *If $f(x)$ is defined and bounded in the interval $0 \leq x \leq \pi$, there exists a constant C' , independent of p and $f(x)$, such that, for all values of p ,*

$$|\sum_p [f(x)] - T_p [f(x)]| < C'M,$$

where $M = \max |f(x)|$ in the interval.

* J. L. Walsh, *A generalization of the Fourier cosine series*, these Transactions, vol. 22 (1921), pp. 230-239; see p. 236.

† Cf. Walsh, loc. cit.; also, *On the convergence of the Sturm-Liouville series*, Annals of Mathematics, (2), vol. 24 (1922), pp. 109-120; pp. 117-120.

‡ In the sense of "possessing the same convergence properties."

§ This is essentially a limiting case of the differential system (I) for $h=H=\infty$.

|| Haar, *Über orthogonale Funktionensysteme*, Mathematische Annalen, vol. 69 (1910), pp. 331-371; p. 335.

We have

$$(4) \quad \sum_p [f(x)] - T_p[f(x)] = \sum_{n=0}^p [\alpha_{np} u_n(x) - a_{np} \cos nx],$$

where, for $n = 1, 2, \dots, p-1$,

$$\begin{aligned} & \alpha_{np} u_n(x) - a_{np} \cos nx \\ &= \frac{\sum_{k=0}^p f(x_k) u_n(x_k)}{\sum_{k=0}^p u_n^2(x_k)} u_n(x) - \frac{2}{p} \sum_{k=0}^p f(x_k) \cos nx_k \cos nx, \end{aligned}$$

and corresponding expressions hold when $n = 0$ and $n = p$, except that the factor $2/p$ must be replaced by $1/p$. Let us substitute this expression for $\alpha_{np} u_n(x) - a_{np} \cos nx$ in (4), reverse the order of the resulting double summation with respect to n and k , then divide the factor of $f(x_k)$ through by $2/p$ and represent the resulting denominator $(2/p) \sum u_n^2(x_k)$ by the symbol S_{np} . Thus we obtain

$$(5) \quad \begin{aligned} \sum_p [f(x)] - T_p[f(x)] &= \frac{2}{p} \sum_{k=0}^p f(x_k) \left[\left\{ \frac{u_0(x_k) u_0(x)}{S_{0p}} - \frac{1}{2} \right\} \right. \\ &+ \sum_{n=1}^{p-1} \left\{ \frac{u_n(x_k) u_n(x)}{S_{np}} - \cos nx_k \cos nx \right\} \\ &\left. + \left\{ \frac{u_p(x_k) u_p(x)}{S_{pp}} - \frac{1}{2} \cos px_k \cos px \right\} \right]. \end{aligned}$$

Denoting the terms in braces by $v_n(x, k, p)$ and their sum with respect to n by $F(x, k, p)$, we may write (5) in the form

$$(6) \quad \begin{aligned} & \sum_p [f(x)] - T_p[f(x)] \\ &= \frac{2}{p} \sum_{k=0}^p f(x_k) \left[v_0(x, k, p) + \sum_{n=1}^{p-1} v_n(x, k, p) + v_p(x, k, p) \right] \\ &= \frac{2}{p} \sum_{k=0}^p f(x_k) F(x, k, p). \end{aligned}$$

This quantity $F(x, k, p)$, which is independent of $f(x)$, possesses an important property which leads directly to the theorem. This property is established in the following lemma.

LEMMA IV. *There exists a constant Q , independent of k and p , such that*

$$|F(x, k, p)| < Q,$$

for $0 \leq x \leq \pi$, and for all values of p and of $k \leq p$.

From (6), we have

$$F(x, k, p) = v_0(x, k, p) + \sum_{n=1}^{p-1} v_n(x, k, p) + v_p(x, k, p).$$

We need consider only the sum, for the single terms $v_0(x, k, p)$ and $v_p(x, k, p)$ are readily seen to be bounded, when it is recalled that $u_0(0) = 1$ and $u_0(x)$ is continuous, and (Lemma II) that $\lim_{p \rightarrow \infty} S_{pp} = 2$. The product $u_n(x_k)u_n(x)$, which is involved in $v_n(x, k, p)$, can be expanded by means of the asymptotic formula for $u_n(x)$ into the form

$$\begin{aligned} \cos nx_k \cos nx + \frac{\beta_1(x, k)}{n} \sin n(x_k + x) \\ + \frac{\beta_2(x, k)}{n} \sin n(x_k - x) + \frac{\delta(x, k, n)}{n^2}, \end{aligned}$$

where

$$\beta_1(x, k) = \frac{1}{2} [\beta(x_k) + \beta(x)],$$

$$\beta_2(x, k) = \frac{1}{2} [\beta(x_k) - \beta(x)].$$

Recalling the definition of $v_n(x, k, p)$ given by (6), we apply Lemma II to S_{np} and utilize the expression just worked out for $u_n(x_k)u_n(x)$, thereby obtaining

$$\begin{aligned} v_n(x, k, p) &= \left[1 + \left\{ \frac{r_{np}}{n^2} \text{ or } \frac{r_{np}}{p} \right\} \right] u_n(x_k)u_n(x) - \cos nx_k \cos nx \\ &= \frac{\beta_1(x, k)}{n} \sin n(x_k + x) + \frac{\beta_2(x, k)}{n} \sin n(x_k - x) \\ &\quad + \left\{ \frac{r_{np}(x, k)}{n^2} \text{ or } \frac{r_{np}(x, k)}{p} \right\}, \end{aligned}$$

the first or second terms in the braces being used according as $n < s$ or $n \geq s$. For the sum we may therefore write

$$\begin{aligned} \sum_{n=1}^{p-1} v_n(x, k, p) &= \beta_1(x, k) \sum_{n=1}^{p-1} \frac{\sin n(x_k + x)}{n} + \beta_2(x, k) \sum_{n=1}^{p-1} \frac{\sin n(x_k - x)}{n} \\ &\quad + \sum_{n=1}^{s-1} \frac{r_{np}(x, k)}{n^2} + \frac{1}{p} \sum_{n=s}^{p-1} r_{np}(x, k). \end{aligned}$$

Each of the sine sums, expressed in terms of a variable $y = x_k \pm x$, is the partial sum of a well known convergent Fourier expansion, and is bounded* for all values of p and of the arguments $x_k \pm x$. The remaining sums are obviously bounded likewise. Hence $F(x, k, p)$ must be dominated in absolute value by some constant Q , for all values of p , and all values of x and x_k in the interval.

The theorem follows directly, for we may write

$$\left| \sum_p [f(x)] - T_p [f(x)] \right| \leq \frac{2}{p} \sum'_{k=0}^p \left| f(x_k) F(x, k, p) \right| < 2MQ = C'M.$$

This theorem shows, then, that as far as mere boundedness is concerned, $T_p[f(x)]$ and $\sum_p[f(x)]$ behave in a similar manner, provided only that $f(x)$ is defined and bounded. If the additional restriction of continuity is imposed upon $f(x)$, we obtain very easily our equivalence theorem, which may be stated thus:

THEOREM III. *If $f(x)$ is continuous, $0 \leq x \leq \pi$, then*

$$\lim_{p \rightarrow \infty} \left\{ \sum_p [f(x)] - T_p [f(x)] \right\} = 0$$

uniformly in the interval.

Let $f_1(x), f_2(x), \dots$ represent a sequence of analytic functions, such that

$$\lim_{\nu \rightarrow \infty} [f(x) - f_\nu(x)] = 0$$

uniformly for $0 \leq x \leq \pi$. Let $\delta_\nu(x) = f(x) - f_\nu(x)$. Then

$$(7) \quad \begin{aligned} & \left| \sum_p [f(x)] - T_p [f(x)] \right| \\ & \leq \left| \sum_p [f_\nu(x)] - T_p [f_\nu(x)] \right| \\ & \quad + \left| \sum_p [\delta_\nu(x)] - T_p [\delta_\nu(x)] \right|. \end{aligned}$$

Choose $\epsilon > 0$. Then there exists a value of ν , say N , such that

$$|\delta_N(x)| \leq \frac{\epsilon}{2C'}.$$

By the preceding theorem, therefore,

$$\left| \sum_p [\delta_N(x)] - T_p [\delta_N(x)] \right| < C' \frac{\epsilon}{2C'} = \frac{\epsilon}{2}.$$

* Cf., e. g., D. Jackson, *Über eine trigonometrische Summe*, Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 257-262; see p. 257.

With N fixed, we can choose an integer P , such that

$$|\sum_p [f_N(x)] - T_p[f_N(x)]| < \frac{\epsilon}{2},$$

for all values of $p \geq P$. This choice of P is rendered possible because $f_N(x)$, being analytic, is capable of representation by both the trigonometric and Sturm-Liouville interpolating formulas with errors arbitrarily small. With the application of the last two inequalities to (7), the theorem follows directly:

$$|\sum_p [f(x)] - T_p[f(x)]| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad p \geq P.$$

An immediate corollary is that if $f(x)$ is any continuous function for which $T_p[f(x)]$ converges uniformly to $f(x)$, then $\sum_p [f(x)]$ will do the same. In particular, a sufficient condition for convergence is that $f(x)$ satisfy the familiar Lipschitz-Dini condition.*

It may be added, in passing, that we have in this section the materials with which to demonstrate the existence of a Sturm-Liouville interpolating formula which converges for all continuous functions. It is formed by analogy with the Fejér mean,† but is not identical with the arithmetical mean of the sums $\sum_0 [f(x)]$, $\sum_1 [f(x)]$, \dots , $\sum_p [f(x)]$. We shall merely state the facts, without proof, in the following theorem:

THEOREM IV. *If $f(x)$ is continuous, $0 \leq x \leq \pi$, then the interpolating formula*

$$\overline{\sum}_p [f(x)] = \bar{\alpha}_{0,p} u_0(x) + \bar{\alpha}_{1,p} u_1(x) + \dots + \bar{\alpha}_{p-1,p} u_{p-1}(x),$$

where

$$\bar{\alpha}_{n,p} = \frac{p-n}{p} \alpha_{n,p} \quad (n=0, 1, \dots, p-1),$$

converges uniformly to $f(x)$ throughout the interval $0 \leq x \leq \pi$.

4. Degree of convergence of the Sturm-Liouville interpolating formula.
In the course of the discussion concerning the degree of convergence of the

* Cf. Faber, loc. cit., p. 422; D. Jackson, these Transactions, vol. 14, loc. cit., p. 456.

† Cf. D. Jackson, *A formula of trigonometric interpolation*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 371-375; p. 372.

trigonometric interpolating formula,* we find reference made to the fact that the trigonometric interpolating expansion of a finite trigonometric sum is identically that sum, a consequence of the well known fact that the trigonometric functions are, if we may use the term in this connection, orthogonal with respect to summation over the interval $(0, 2\pi)$. This fact, in the case of the cosine formula, may be expressed in the form

$$\sum_{k=0}^p \cos mx_k \cos nx_k = 0, \quad m \neq n \quad (m \leq p, n \leq p).$$

In the case of the Sturm-Liouville functions, however, $\sum' u_m(x_k)u_n(x_k)$ is not generally equal to zero, but only tends toward zero as p increases, hence the Sturm-Liouville interpolating formula for a finite sum is not identically that sum, but only an approximation to it. The degree of this approximation can, however, be determined; the conditions and solution of the problem thus suggested find precise formulation in the following

THEOREM V. *Given an infinite sequence of functions $\varphi_p(x)$ of the type*

$$\varphi_p(x) = c_{0p}u_0(x) + c_{1p}u_1(x) + \cdots + c_{pp}u_p(x) \quad (p = 1, 2, \cdots),$$

and a constant K , independent of p , such that

$$|\varphi_p(x)| < K, \quad 0 \leq x \leq \pi \quad (p = 1, 2, \cdots),$$

and another constant A , independent of n and p , such that

$$|c_{0p}| < A, \quad |c_{np}| < \frac{A}{n}, \quad 1 \leq n \leq p \quad (p = 1, 2, \cdots);$$

then there exists a constant C , independent of p , such that

$$\left| \sum_p [\varphi_p(x)] - \varphi_p(x) \right| < \frac{C}{p}, \quad 0 \leq x \leq \pi \quad (p = 1, 2, \cdots).$$

The complete proof of this theorem is quite involved and tedious; it seems best, therefore, to present in this paper the mere outline of the proof, containing some of the more important subordinate results, and other details sufficient to indicate the methods employed.

If we introduce the notation

$$\sum_{n=0}^p {}^m y_n = \left[\sum_{n=0}^p y_n \right] - y_m$$

* D. Jackson, these Transactions, vol. 14, loc. cit., p. 455.

we can express the difference $\sum_p [\varphi_p(x)] - \varphi_p(x)$ in the compact form

$$\sum_p [\varphi_p(x)] - \varphi_p(x) = \frac{\sum_{m=0}^p u_m(x) \sum_{n=0}^m c_{np} \sum_{k=0}^p u_m(x_k) u_n(x_k)}{\sum_{k=0}^p u_m^2(x_k)} .$$

It will be found convenient to introduce the additional notation

$$S(m, n, p) = \frac{\pi}{p} \sum_{k=0}^p u_m(x_k) u_n(x_k),$$

and to separate out the terms with indices $m = 0$ and $n = 0$, since these terms cannot be represented by the asymptotic formula. We then have

$$(8) \quad \begin{aligned} \sum_p [\varphi_p(x)] - \varphi_p(x) &= u_0(x) \frac{\sum_{n=1}^p c_{np} S(0, n, p)}{S(0, 0, p)} \\ &+ \sum_{m=1}^p u_m(x) \frac{c_{0p} S(m, 0, p)}{S(m, m, p)} + \sum_{m=1}^p u_m(x) \frac{\sum_{n=1}^p c_{np} S(m, n, p)}{S(m, m, p)} . \end{aligned}$$

The problem presented is essentially that of determining the order of magnitude of the quantities $S(m, n, p)$.

To do this, we break up $S(m, n, p)$ into parts corresponding to the several terms of the product $u_m(x)u_n(x)$, when the characteristic functions have been replaced by their asymptotic representations. The coefficients of the sine terms in the product, which appear below, will be separated into linear functions and functions vanishing at the end points 0 and π :

$$\frac{1}{2} \beta(x) = (a + bx) + \eta(x),$$

$$\beta(x)\alpha(x, n) = (a_n + b_n x) + \eta_n(x),$$

where $\eta(0) = \eta_n(0) = \eta(\pi) = \eta_n(\pi) = 0$. Making these substitutions, and effecting certain trigonometric reductions, we obtain for $u_m(x)u_n(x)$ the following expression of eleven terms:

$$\begin{aligned}
u_m(x) u_n(x) &= \frac{1}{2} [\cos(m+n)x + \cos(m-n)x] \\
&+ (a+bx) \left[\frac{\sin(m+n)x + \sin(m-n)x}{m} + \frac{\sin(m+n)x - \sin(m-n)x}{n} \right] \\
&+ \eta(x) \left[\frac{\sin(m+n)x + \sin(m-n)x}{m} + \frac{\sin(m+n)x - \sin(m-n)x}{n} \right] \\
&+ \frac{\beta^2(x)}{2mn} [\cos(m-n)x - \cos(m+n)x] + \frac{\alpha(x, n)}{n^2} \cos mx + \frac{\alpha(x, m)}{m^2} \cos nx \\
&+ \frac{(a_n + b_n x) \sin mx}{n^2 m} + \frac{(a_m + b_m x) \sin nx}{m^2 n} \\
&+ \frac{\eta_n(x) \sin mx}{n^2 m} + \frac{\eta_m(x) \sin nx}{m^2 n} + \frac{\alpha(x, n)\alpha(x, m)}{n^2 m^2}.
\end{aligned}$$

Let these eleven terms be denoted by $g_r(x, m, n)$, $r = 1, 2, \dots, 11$, respectively, so that

$$u_m(x)u_n(x) = \sum_{r=1}^{11} g_r(x, m, n).$$

Since

$$\int_0^\pi u_m(x)u_n(x)dx = 0,$$

it is clear that

$$\begin{aligned}
\frac{\pi}{p} \sum_{k=0}^p u_m(x_k)u_n(x_k) &= \frac{\pi}{p} \sum_{k=0}^p u_m(x_k)u_n(x_k) - \int_0^\pi u_m(x)u_n(x)dx \\
&= \sum_{r=1}^{11} \left[\frac{\pi}{p} \sum_{k=0}^p g_r(x_k, m, n) - \int_0^\pi g_r(x, m, n)dx \right].
\end{aligned}$$

Denoting the bracketed quantity by $S_r(m, n, p)$, and recalling the notation used for the left-hand member, we can write

$$S(m, n, p) = \sum_{r=1}^{11} S_r(m, n, p).$$

The determination of the orders of magnitude of the quantities $S_r(m, n, p)$ involves the use of a number of more or less well known formulas and theorems, which may be summarized thus:

$$(a) \quad \sum_{k=0}^p \cos \nu x_k = \begin{cases} 0, & \nu \neq 2lp, \\ p, & \nu = 2lp, l=0, 1, 2, \dots; \end{cases}$$

$$(b) \quad \sum_{k=0}^p \sin \nu x_k = \begin{cases} 0, & \nu \text{ even}, \\ \cot \frac{\nu\pi}{2p}, & \nu \text{ odd}; \end{cases}$$

$$(c) \quad \sum_{k=0}^p x_k \sin \nu x_k = \begin{cases} 0, & \nu = 2lp, \\ \frac{-\pi(-1)^p}{2} \cot \frac{\nu\pi}{2p}, & \nu \neq 2lp; \end{cases}$$

$$(d) \quad \int_0^\pi \cos \nu x \, dx = \begin{cases} 0, & \nu \neq 0, \\ \pi, & \nu = 0; \end{cases}$$

$$(e) \quad \int_0^\pi \sin \nu x \, dx = \begin{cases} 0, & \nu \text{ even}, \\ \frac{2}{\nu}, & \nu \text{ odd}; \end{cases}$$

$$(f) \quad \int_0^\pi x \sin \nu x \, dx = \begin{cases} 0, & \nu = 0, \\ -(-1)^p \frac{\pi}{\nu}, & \nu \neq 0; \end{cases}$$

(g) if $f''(x)$ is continuous in the interval $0 \leq x \leq \pi$, then $f(x)$ can be expanded in a Fourier series of cosines with coefficients of order* $1/\nu^2$;

(h) if $f(x)$ satisfies the above hypothesis and the additional condition that $f(0) = f(\pi) = 0$, then it can be expanded into a Fourier sine series with coefficients of order $1/\nu^2$.

For $S_1(m, n, p)$ and $S_2(m, n, p)$, we obtain explicit expressions. From (a) and (d) it follows that $S_1(m, n, p) = 0$.

From (b), (c), (e), and (f), we obtain

$$S_2(m, n, p) = \frac{\pi}{p} H_{mn} \frac{(1/m) \sin(m\pi/p) - (1/n) \sin(n\pi/p)}{\sin[(m+n)\pi/(2p)] \sin[(m-n)\pi/(2p)]},$$

where $H_{mn} = -b\pi/2$ or $a + (b\pi/2)$ according as $m + n$ is even or odd.

* Cf. Picard, *Traité d'Analyse*, vol. 1, pp. 255, 256, in edition 2, or p. 334 in edition 3.

For each of the remaining quantities $S_r(m, n, p)$, we obtain a dominating expression which indicates its order of magnitude. The manner in which this is obtained may be briefly outlined. Leaving $g_7(x, m, n)$, $g_8(x, m, n)$, and $g_{11}(x, m, n)$ out of consideration for the present, we notice that each term $g_r(x, m, n)$ involves either a sine or cosine term as one of its factors. We proceed to expand the other factor into a Fourier series; in the latter case, into a cosine series, and in the former, into a sine series; and then we change the resulting products into the sums and differences of cosines. This, of course, is equivalent to expanding the function $g_r(x, m, n)$ itself into a cosine series. In the case of $g_{11}(x, m, n)$, we expand the product $\alpha(x, n)\alpha(x, m)$ into a cosine series directly. Recalling that

$$S_r(m, n, p) = \frac{\pi}{p} \sum_{k=0}^p g_r(x_k, m, n) - \int_0^\pi g_r(x, m, n) dx,$$

we apply (a) and (d) to the sum and integral, respectively, whereupon all except one out of every $2p$ terms in the expansion of $g_r(x, m, n)$ disappear, leaving $S_r(m, n, p)$ in the form of an infinite series whose sum approaches zero as p increases indefinitely. By the theorems enunciated in (g) and (h), we know the orders of magnitude of the terms of this series, hence we can determine that of $S_r(m, n, p)$.

It should be mentioned here that the function $\alpha(x, n)/n^2$, which is involved indirectly in S_9 and S_{10} , and directly in S_5 , S_6 and S_{11} , possesses a continuous second derivative, uniformly bounded for all values of n . In dealing with S_5 and S_6 , we must know in some detail how the derivatives of $\alpha(x, n)/n^2$ depend upon n . The nature of this dependence is indicated by the relations

$$\frac{d}{dx} \left[\frac{\alpha(x, n)}{n^2} \right] = \frac{P(x, n)}{n},$$

$$\frac{d^2}{dx^2} \left[\frac{\alpha(x, n)}{n^2} \right] = Q_1(x, n) \cos nx + Q_2(x, n) \sin nx + \frac{Q_3(x, n)}{n},$$

where $Q_1'(x, n)$ and $Q_2'(x, n)$, as well as Q_1 , Q_2 , and Q_3 , are uniformly bounded and continuous.

The remaining terms, $g_7(x, m, n)$ and $g_8(x, m, n)$, we treat the same as g_2 obtaining thereby explicit expressions for $S_7(m, n, p)$ and $S_8(m, n, p)$, for which suitable dominating quantities are easily found.

The results obtained through the processes thus briefly sketched appear in the following inequalities:*

* Since $m \neq n$, $m+n$ can never equal $2p$.

$$\begin{aligned}
|S_2(m, n, p)| &< \frac{4LV_1}{p^2} \left[\frac{1}{m} + \frac{1}{n} \right] + \frac{V_1}{(2p - m - n)^2} \left[\frac{1}{m} + \frac{1}{n} \right], \\
|S_4(m, n, p)| &< \frac{1}{mn} \left[\frac{2LV_2}{p^2} + \frac{V_2}{2(2p - m - n)^2} \right], \\
|S_5(m, n, p)| &< \frac{4LV_3}{np^2} + \frac{V_3}{p^2(2p - m - n)}, \\
|S_7(m, n, p)| &< \frac{V_4}{n^2 p^2}, \\
|S_9(m, n, p)| &< \frac{2LV_5}{mp^2}, \\
|S_{11}(m, n, p)| &< \frac{LV_6}{p^2} \left[\frac{1}{m} + \frac{1}{n} \right].
\end{aligned}$$

There is no need for writing down separate inequalities for S_6 , S_8 , and S_{10} , since they are, in form, entirely analogous to those for S_5 , S_7 , and S_9 , respectively. The letter L denotes $\sum(1/n^2)$, $n = 1, 2, \dots$, and the V 's denote constants whose values are related to the maximum values of the derivatives of the functions which were expanded in the Fourier cosine or sine series.

Referring back to (8) and recalling that $u_n(x)$ remains bounded for all values of n , that $S(m, m, p)$ has a positive lower bound independent of m and p , and that the coefficients c_{np} are of order $1/n$, we find that our next problem is essentially that of determining suitable upper bounds for the double summations

$$\sum_{m=1}^p \sum_{n=1}^m \frac{1}{n} \left| S_r(m, n, p) \right| \quad (r = 3, 4, \dots, 11).$$

We notice that the expressions which dominate $|S_r(m, n, p)|$ are simple functions of the discrete variables m and n , so that the double sums can be replaced by double integrals which are easily evaluated, yielding thereby the desired upper bounds. By this method we are enabled to deduce the existence of a constant, say W , such that, for all values of $p \geq 1$,

$$(9) \quad \sum_{m=1}^p \sum_{n=1}^m \frac{1}{n} \left| S_r(m, n, p) \right| < \frac{W}{p} \quad (r = 3, 4, \dots, 11).$$

Thus far we have made use of only one of the properties of the coefficients c_{rp} , namely, that $|c_{np}| < A/n$, but, in dealing with $S_2(m, n, p)$, we must utilize the other property, namely, that they are such that the sequence of

Sturm-Liouville sums $\varphi_p(x)$ remains bounded for all values of p . Hence we must work with the sum

$$\sum_{m=1}^p \left| \sum_{n=1}^p c_{np} S_2(m, n, p) \right|.$$

Recalling the explicit expression previously obtained for $S_2(m, n, p)$, we shall find it expedient, by means of a change of variable, to reduce $S_2(m, n, p)$ to the form of a function of two discrete variables ranging between the limits 0 and π . Since $x_k = k\pi/p$, we can write

$$S_2(m, n, p) = \frac{\pi^2}{p^2} H_{mn} \frac{\frac{\sin x_m}{x_m} - \frac{\sin x_n}{x_n}}{\sin \frac{x_m + x_n}{2} \sin \frac{x_m - x_n}{2}},$$

where $0 < x_m \leq \pi$, $0 < x_n \leq \pi$, $x_m \neq x_n$. Let $L(x_m, x_n)$ denote the complex fraction, so that

$$(10) \quad S_2(m, n, p) = \frac{\pi^2}{p^2} H_{mn} L(x_m, x_n).$$

Denoting by $L(y, z)$ the corresponding function of two continuous variables y, z , and expanding the numerator of $L(y, z)$ into a power series, we find that we can write

$$L(y, z) = \frac{\frac{y+z}{2}}{\sin \frac{y+z}{2}} \cdot \frac{\frac{y-z}{2}}{\sin \frac{y-z}{2}} \cdot \xi(y, z),$$

where $\xi(y, z)$ is analytic in the region $0 \leq y \leq \pi$, $0 \leq z \leq \pi$. The properties of $L(y, z)$ are therefore essentially those of the reciprocals of functions of the familiar type $(\sin x)/x$. Since these properties pertain to certain regions of the y, z domain, denoted by R , we divide the latter into two subregions R_1 and R_2 , where R_2 is defined by the inequalities $s \leq m \leq p$, $s \leq n \leq p$, and $R_1 = R - R_2$.

In R_1 the following inequalities hold:

$$(11) \quad |L(y, z)| < L_1,$$

$$(12) \quad \left| \frac{\partial}{\partial z} L(y, z) \right| < G_1,$$

where L_1 and G_1 denote constants, independent of m, n , and p . Let

$$\sum_n^m c_{np} S_2(m, n, p)$$

denote the sum, with respect to n , of $c_{np} S_2(m, n, p)$ taken over the values of n for which (x_m, x_n) belongs to R_1 . By using the expression previously obtained for $S_2(m, n, p)$ and an obvious identity, we may write

$$\begin{aligned} \frac{p^2}{\pi^2} \sum_n^m c_{np} S_2(m, n, p) &= \sum_n^m c_{np} H_{mn} L(x_m, x_n) \\ (13) \qquad \qquad \qquad &= L(x_m, 0) \sum_n^m c_{np} H_{mn} \\ &\quad + \sum_n^m c_{np} H_{mn} [L(x_m, x_n) - L(x_m, 0)]. \end{aligned}$$

By the law of the mean,

$$L(x_m, x_n) - L(x_m, 0) = \frac{n\pi}{p} \frac{\partial}{\partial z} L(x_m, h_{mn} x_n),$$

where $0 < h_{mn} < 1$, and, by (12),

$$|L(x_m, x_n) - L(x_m, 0)| < \frac{n\pi G_1}{p}.$$

Since $|c_{np}| < A/n$, we finally obtain

$$(14) \quad \left| \sum_n^m c_{np} H_{mn} [L(x_m, x_n) - L(x_m, 0)] \right| < H \sum_n^m \frac{A}{n} \frac{n\pi G_1}{p} = G_2,$$

where $H > |H_{mn}|$. Upon replacing, in the first sum on the right-hand side of (13), the quantities H_{mn} by their values

$$H_{mn} = \begin{cases} -\frac{b\pi}{2}, & m+n \text{ even,} \\ a + \frac{b\pi}{2}, & m+n \text{ odd,} \end{cases}$$

and condensing the resultant expression, we obtain

$$\begin{aligned} (15) \quad \sum_n^m c_{np} H_{mn} &= (-1)^{m+1} \frac{1}{2} (b\pi + a) \sum_n^m (-1)^n c_{np} \\ &\quad + \frac{1}{2} a \sum_n^m c_{np}. \end{aligned}$$

Since $\varphi_p(x) = \sum c_{np}u_n(x)$ is uniformly bounded for all values of p and for all values of x in the interval $0 \leq x \leq \pi$, it is clear that the sequences of constants $|\varphi_p(0)|$ and $|\varphi_p(\pi)|$ are bounded. But, from the asymptotic formula for $u_n(x)$, we find that

$$u_n(0) = 1,$$

$$u_n(\pi) = (-1)^n + \frac{r_n}{n}, \quad n > 0.$$

Therefore

$$\sum_{n=0}^p c_{np} = \sum_{n=0}^p c_{np} u_n(0) = \varphi_p(0),$$

$$\sum_{n=0}^p (-1)^n c_{np} = \sum_{n=0}^p c_{np} u_n(\pi) - c_{0p}r_0 - \sum_{n=1}^p c_{np} \frac{r_n}{n}$$

$$= \varphi_p(\pi) - c_{0p}r_0 - \sum_{n=1}^p \frac{r_{np}}{n^2},$$

whence it is apparent that $\sum c_{np}$ and $\sum c_{np}(-1)^n$ remain respectively bounded for all values of p . It is easily shown that the same is true of the partial sums

$$\sum_n^m c_{np}, \quad \sum_n^m (-1)^n c_{np},$$

which appear in (15). Hence, combining (13), (14), and (15), we finally deduce the existence of a constant W_1 , such that

$$\left| \sum_n^m c_{np} S_2(m, n, p) \right| < \frac{W_1}{p^2} \quad (p = 1, 2, \dots).$$

Summing this with respect to m over the range $1 \leq m \leq p$, we obtain the desired result, namely

$$(16) \quad \sum_m \left| \sum_n^m c_{np} S_2(m, n, p) \right| < \frac{W_1}{p}.$$

Our next problem is to obtain a similar inequality for this summation extended over the region R_2 , which, as we may recall, constitutes that part of the (x_m, x_n) domain in which m and n are subject to the inequalities $s \leq m \leq p, s \leq n \leq p, m \neq n$. In this region we find that

$$\left| L(x_m, x_n) \right| < \frac{\text{const.}}{2\pi - x_m - x_n}.$$

Replacing x_m and x_n by their respective values, $m\pi/p$ and $n\pi/p$, and recalling (10), we can write*

$$\left| S_2(m, n, p) \right| < \frac{K}{p(2p - m - n)},$$

where K is a constant, independent of p , m , and n . In working with the double sum

$$\sum_m \left| \sum_n \sum_{R_1}^m c_{np} S_2(m, n, p) \right|,$$

we shall have no further occasion to utilize that property of the coefficients c_{np} whereby $\sum c_{np} u_n(x)$ remains bounded for all values of p ; all we require is the fact that $c_{np} < A/n$. Hence it appears that our problem is essentially that of determining the order of magnitude of

$$\frac{1}{p} \sum_{m=s}^p \sum_{n=s}^m \frac{1}{n(2p - m - n)}.$$

This problem can be further simplified by noting that $1/n \leq 2/p$, whereby we may write

$$\sum_m \left| \sum_n \sum_{R_1}^m c_{np} S_2(m, n, p) \right| < \frac{2AK}{p^2} \sum_{m=s}^p \sum_{n=s}^m \frac{1}{2p - m - n}.$$

Replacing the double sum by the corresponding double integral and evaluating the latter, we arrive at the result

$$\sum_{m=s}^p \sum_{n=s}^m \frac{1}{2p - m - n} < pK'.$$

Combining the last two inequalities, we obtain

$$\sum_m \left| \sum_n \sum_{R_1}^m c_{np} S_2(m, n, p) \right| < \frac{W_2}{p},$$

which may be combined with (16) to yield the desired conclusion, namely

$$(17) \quad \sum_{m=1}^p \left| \sum_{n=1}^m c_{np} S_2(m, n, p) \right| < \frac{W_3}{p}.$$

We are now in a position to determine the order of magnitude of the quantity appearing on the right-hand side of (8), by which $|\sum_p[\varphi_p(x)] - \varphi_p(x)|$ is dominated. The inequalities (17) and (9) show that the double

* Since $m \neq n$, $m+n$ never reaches $2p$.

summation in (8) never exceeds some fixed multiple of $1/p$. The same is seen to be true of the single summations in the right-hand member of (8), as a result of reasoning analogous in principle to that which precedes, but materially simpler in execution. Hence the difference $\sum_p [\varphi_p(x)] - \varphi_p(x)$ must, as the theorem states, be dominated in absolute value by some constant multiple of $1/p$.

From this theorem, considered in conjunction with Theorem II, one may derive a conclusion as to the degree of convergence of the Sturm-Liouville formula for a function satisfying a Lipschitz condition, essentially similar in proof to the corresponding theorem in trigonometric interpolation. This conclusion may be stated thus:

THEOREM VI. *If $f(x)$ satisfies a Lipschitz condition,*

$$|f(x_2) - f(x_1)| < \mu(x_2 - x_1), \quad 0 \leq x_1 < x_2 \leq \pi,$$

then there exists a constant G , independent of p , such that

$$\left| \sum_p [f(x)] - f(x) \right| < \frac{G \log p}{p}, \quad p \geq 2,$$

for all values of x in the interval $0 \leq x \leq \pi$; and, for the points of interpolation $x_k, k = 0, 1, \dots, p$,

$$\left| \sum_p [f(x_k)] - f(x_k) \right| < \frac{G}{p}, \quad p \geq 1.$$

We shall not carry out the demonstration of this theorem, but merely point out that, by analogy with the trigonometric case, the proof requires the existence of a Sturm-Liouville formula $\varphi_p(x)$ which shall represent $f(x)$ with an error less in absolute value than a fixed multiple of $1/p$, and whose coefficients c_{np} shall satisfy the hypotheses of the preceding theorem. It is known, however, that such a formula does exist,* hence the proof offers no further difficulty.

* D. Jackson, these Transactions, vol. 15, loc. cit., p. 466.