ALTERNATIVES TO ZERMELO'S ASSUMPTION*

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1. **The axiom of choice.** The object of this paper is to consider the possibility of setting up a logic in which the axiom of choice is false. The way of approach is through the second ordinal class, in connection with which there appear certain alternatives to the axiom of choice. But these alternatives have consequences not only with regard to the second ordinal class but also with regard to other classes, whose definitions do not involve the second ordinal class, in particular with regard to the continuum. And therefore it is possible to consider these alternatives as, in some sense, postulates of logic. In what follows we proceed, after certain introductory considerations, to state these postulates, to inquire into their character, and to derive as many as possible of their consequences.

The axiom of choice, which is also known as Zermelo's assumption,† and, in a weakened form, as the multiplicative axiom,‡ is a postulate of logic which may be stated in the following way:

Given any set X of classes which does not contain the null class, there exists a one-valued function, F, such that if x is any class of the set X then $F(x)$ is a member of the class x.

An equivalent statement is that there exists an assignment to every class x belonging to the set X of a unique element ρ such that ρ is contained in x.

The important case is that in which the set X contains an infinite number of classes, because the assertion of the postulate is obviously capable of proof when the number of classes is finite. Accordingly a convenient, although not quite precise, characterization of the axiom of choice is obtained by saying that it is a postulate which justifies the employment of an infinite number of acts of arbitrary choice.

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† E. Zermelo. *Beweis dass jede Menge wohlgeordnet werden kann*, Mathematische Annalen, vol. 59 (1904), p. 514. See also *Neuer Beweis für die Möglichkeit einer Wohlordnung*, Mathematische Annalen, vol. 65 (1908), p. 110, and *Untersuchungen über die Grundlagen der Mengenlehre*, Mathematische Annalen, vol. 65 (1908), p. 266, where Zermelo states the weaker form of the axiom of choice which Russell has called the multiplicative axiom.

‡ B. Russell, *On some difficulties in the theory of transfinite numbers and order types*, Proceedings of the London Mathematical Society, (2), vol. 4 (1907), p. 48; Whitehead and Russell, *Principia Mathematica*, vol. I, 1910, p. 561.

Instead of assuming that the function F exists in the case of every set X of classes, it is possible to assume only that the function F exists if the set X contains a denumerable infinity of classes.* Or we may assume that the function F exists if the set X contains either $\aleph_1$ classes or some less number of classes. In this way we obtain a sequence of postulates, each stronger than those which precede it, all of them weakened forms of Zermelo's assumption, which we may call, respectively, the axiom of choice for sets of $\aleph_0$ classes, the axiom of choice for sets of $\aleph_1$ classes, and so on.

For our present purpose we wish to exclude all forms of the axiom of choice from among the postulates of logic, so that in what follows no appeal to the axiom of choice is to be allowed.

2. **The second ordinal class.** As defined by Cantor,† the second ordinal class consists of all those ordinals α such that a well-ordered sequence of ordinal number α has $\aleph_0$ as its cardinal number. Instead of this definition we prefer a definition in terms of order alone such as that given in the next paragraph. This definition probably cannot be proved equivalent to Cantor's except with the aid of the axiom of choice for sets of $\aleph_0$ classes. The relation between the two definitions will appear more clearly in §§ 8, 9, and 10 below.

We shall, therefore, define the second ordinal class by means of the following set of postulates:‡

1. *The second ordinal class is a simply ordered aggregate.*

2. *There is a first ordinal ω in the second ordinal class.*

3. *If α is any ordinal of the second ordinal class, there is a first ordinal, $\alpha+1$, of the set of ordinals of the second ordinal class which follow α.*

4. *If the ordinals $\beta_0, \beta_1, \beta_2, \cdots$ of the second ordinal class are all distinct and form, in their natural order, an ordered sequence ordinally similar to the sequence* 0, 1, 2, 3, $\cdots$ *of positive integers, there is an ordinal β of the second ordinal class, the upper limit of the sequence $\beta_0, \beta_1, \beta_2, \cdots$, which is the first ordinal in the set of ordinals which follow every ordinal β_i of this sequence.*

5. *There is no proper subset of the second ordinal class which contains the ordinal ω and which has the property that if it contains the ordinal α it contains also $\alpha+1$, and if it contains a sequence $\beta_0, \beta_1, \beta_2, \cdots$ of the kind described in Postulate* 4 *it contains also the upper limit β.*

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* Cf. B. Russell, *Introduction to Mathematical Philosophy*, 1919, p. 129.

† G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, zweiter Artikel, Mathematische Annalen, vol. 49 (1897), p. 227.

‡ A closely similar definition of the second ordinal class has been given by O. Veblen, *Definition in terms of order alone in the linear continuum and in well-ordered sets*, these Transactions, vol. 6 (1905), p. 170.

The fifth postulate makes possible the process of transfinite induction.

The positive integers, including 0, are thought of as forming the first ordinal class and as preceding the ordinal ω in their natural order, so that ω is the upper limit of the sequence $0, 1, 2, 3, \cdots$.

The ordinal Ω is the first ordinal which follows all the ordinals of the second ordinal class. It belongs to the third ordinal class, or would belong to this class if we chose to construct it, as could be done by means of a set of postulates analogous to Postulates 1–5.

If an ordinal α precedes an ordinal β in the arrangement of the ordinals just described (which we shall call the natural order of the ordinals), we say that *α is less than β* and *β is greater than α*.

An ordinal β, other than 0, of the first or the second ordinal class is of the first kind or of the second kind according as there is or is not a greatest ordinal less than β.

The *upper limit* α of a sequence s of ordinals in their natural order such that s contains no greatest ordinal is the least ordinal greater than all the ordinals in s. This ordinal α always exists if s contains no greatest ordinal, but α may sometimes belong to a higher ordinal class than any ordinal in s.

A sequence of distinct ordinals of the first and second ordinal classes, in their natural order, $\beta_0, \beta_1, \beta_2, \cdots$, ordinally similar to the sequence 0, 1, 2, 3, $\cdots$ of positive integers, is said to be a *fundamental sequence* of its upper limit β.

A sequence t of ordinals in their natural order is *internally closed* if it contains the upper limits of all its sub-sequences which have an upper limit different from the upper limit of t.

We shall not prove explicitly as consequences of Postulates 1–5 all the theorems about the second ordinal class which we shall need, but we shall make use freely of known theorems whenever these theorems do not depend on the axiom of choice.

The set of all ordinals which are less than a given ordinal α forms, when these ordinals are arranged in their natural order, a well-ordered sequence, which is called the *segment* determined by α, and α is said to be the *ordinal number* of this sequence and of all well-ordered sequences ordinally similar to it. In particular, ω is the ordinal number of the sequence of positive integers in their natural order.

The notions of addition, multiplication, and exponentiation of ordinals, of which we shall need to make some use, either may be defined* in terms of

*G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, zweiter Artikel, Mathematische Annalen, vol. 49 (1897), pp. 207–218, and pp. 231–235.

the notion of the ordinal number of a well-ordered sequence or may be defined more directly from the postulates by means of a process of induction.

3. **Notations for cardinal numbers.** The cardinal number of the segment determined by an ordinal α is called the cardinal number corresponding to α. Those cardinals which correspond to some ordinal greater than or equal to ω are called aleph cardinals. When arranged in order of magnitude they form a well ordered sequence. The first of them is the cardinal number corresponding to ω, which we call* $\aleph_0$. The remainder of them are denoted by the letter $\aleph$ (aleph) with an ordinal as a subscript, this ordinal indicating the position of the number in the well-ordered sequence of aleph cardinals.

The cardinal number corresponding to Ω is different† from $\aleph_0$. The question whether or not it is $\aleph_1$, the first aleph cardinal after $\aleph_0$, is left open for discussion below.

If the cardinal number of an aggregate S is $\aleph_\alpha$, the cardinal number $\beth_\alpha$ of the aggregate of subsets of S is greater than $\aleph_\alpha$. These cardinal numbers we call beth cardinals and denote them by the letter $\beth$ (beth) with the same subscript as the corresponding aleph. Besides these we may define a set of beth cardinals with two subscripts as follows. If the cardinal number of an aggregate S is $\aleph_\alpha$, and $\aleph_\beta$ is an aleph cardinal less than $\aleph_\alpha$, then $\beth_{\alpha,\beta}$ is the cardinal number of the aggregate of those subsets of S which have a cardinal number not greater than‡ $\aleph_\beta$.

The question of the distinctness of these cardinals from the aleph cardinals and from one another must be left open.

The cardinal number $\beth_0$ is, by definition, the cardinal number of the class of all classes of positive integers. It is the cardinal number of the continuum of real numbers.§ It is also the cardinal number of the set of real numbers on a segment of the continuum, the cardinal number of the set of irrational numbers of the continuum, and the cardinal number of the set of irrational

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* G. Cantor *Beiträge zur Begründung der transfiniten Mengenlehre*, erster Artikel, Mathematische Annalen, vol. 46 (1895), p. 492.

† G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, zweiter Artikel, Mathematische Annalen, vol. 49 (1897), p. 227.

‡The cardinal numbers defined in this paragraph are all infinite both in the sense of being non-inductive and in the sense of being reflexive. We shall not be concerned with non-inductive non-reflexive cardinals, although the existence of these cardinals seems to be possible if we deny the axiom of choice. On this class of cardinals see Whitehead and Russell, *Principia Mathematica*, vol. II, 1912, p. 278 and p. 288.

§G. Cantor, loc. cit., erster Artikel, p. 488. The class of subsets of a class of cardinal number b evidently has the cardinal number 2^b as defined by Cantor.

numbers on a segment of the continuum. And by means of the expansion of an irrational number as a continued fraction it can be shown that $\beth_0$ is the cardinal number of the class of sequences of positive integers of ordinal number ω.

4. **The cardinal number of the class of all well-ordered rearrangements of the positive integers.** By a well-ordered rearrangement of the positive integers is to be understood a well-ordered sequence of positive integers such that in it every positive integer occurs once and but once. The well-ordered sequence may be of any possible ordinal number, but no positive integer may be repeated in the sequence and none may be omitted from it. With this understanding we shall prove the following theorem:

THEOREM 1. *The class of all well-ordered rearrangements of the positive integers has the cardinal number* $\beth_0$ *of the continuum.*

For the class of all ordered pairs of positive integers is of cardinal number* $\aleph_0$. Therefore the set P of all classes of ordered pairs of positive integers is of cardinal number $\beth_0$.

Now to every well-ordered rearrangement W of the positive integers corresponds a class Q of ordered pairs of positive integers such that the ordered pair (a, b) is contained in Q if and only if a precedes b in W. And no such class Q of ordered pairs of positive integers corresponds in this way to more than one well-ordered rearrangement of the positive integers. Therefore the class of all well-ordered rearrangements of the positive integers can be put into one-to-one correspondence with a part of the set P and therefore with a part of the continuum (because P can be put into one-to-one correspondence with the continuum).

But the class of all well-ordered rearrangements of the positive integers contains a part which can be put into one-to-one correspondence with the continuum, namely the class O of those well-ordered rearrangements of ordinal number ω which have the property that in them the set of odd positive integers and the set of even positive integers occur each in its natural order (so that O contains, for example, the sequence 0, 2, 1, 4, 6, 3, 8, 10, 5, 12, 14, 7, $\cdots$). For to the well-ordered rearrangement $a_0, a_1, a_2, \cdots$ contained in O can be correlated the irrational number $(b_0/2)+(b_1/2^2)+(b_2/2^3)+ \cdots$ where b_i is equal to 0 or to 1 according as a_i is even or odd. In this way can be set up a one-to-one correspondence between O and the set of irrational numbers between 0 and 1 and therefore between O and the continuum.

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*G. Cantor, loc. cit., erster Artikel, p. 494.

The theorem to be proved, therefore, follows by an appeal to the theorem of Schröder and Bernstein* which states that if each of two classes can be put into one-to-one correspondence with a part of the other, then the two classes can be put into one-to-one correspondence.‡

COROLLARY 1. *The continuum can be divided into* $\aleph_1$ *mutually exclusive subsets, each of cardinal number*† $\beth_0$.

For the class of well-ordered rearrangements of the positive integers can be so divided by classifying the well-ordered rearrangements of the positive integers according to their ordinal number.

COROLLARY 2. *The class of all well-ordered sequences of positive integers which contain no repetition of any integer is of cardinal number* $\beth_0$.

This corollary and the two following can be proved by the same argument as that used in proving Theorem 1.

COROLLARY 3. *The class of all permutations of the positive integers is of cardinal number*§ $\beth_0$, *where a permutation of the positive integers is restricted to be of ordinal number* ω.

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* E. Schröder, *Über zwei Definitionen der Endlichkeit und G. Cantor'sche Sätze*, Nova Acta Academiae Caesareae Leopoldino-Carolinae Germanicae Naturae Curiosorum, vol. 71 (1898), pp. 336–340; E. Borel, *Leçons sur la Théorie des Fonctions*, pp. 102–107; A. Korselt, *Über einen Beweis des Äquivalenzsatzes*, Mathematische Annalen, vol. 70 (1911), pp. 294–296.

A very neat proof is given by J. König, *Sur la théorie des ensembles*, Comptes Rendus, vol. 143 (1906), pp. 110–112.

†The Schröder-Bernstein theorem can be proved in such a way that an explicit one-to-one correspondence between the classes in question is set up. Therefore we are able here actually to set up an explicit one-to-one correspondence between the continuum and the class of well-ordered rearrangements of the positive integers. And this means, of course, that the sets of Corollary 1 are explicitly defined subsets of the continuum.

When we have made explicit in this way the one-to-one correspondence between the continuum and the class of well-ordered rearrangements of the positive integers, it is not difficult to say in simple cases what well-ordered rearrangement of the positive integers corresponds to a given number a of the continuum. But in certain cases the answer to this question involves the solving of difficult (conceivably unsolvable) problems about the dual fractional expansion of a similar to that proposed by L. E. J. Brouwer, Mathematische Annalen, vol. 83 (1921), pp. 209–210, for the decimal expansion π, but more complicated in character. But the correspondence which we have set up is none the less explicit.

‡ W. Sierpinski, in Bulletin International de l'Académie des Sciences de Cracovie, 1918, p. 110, gives another proof, independent of the axiom of choice, that the continuum can be divided into $\aleph_1$ mutually exclusive subsets. The division of the continuum which he effects is actually a division into subsets each of cardinal number $\beth_0$, although he does not prove this.

§ F. Bernstein, *Untersuchungen aus der Mengenlehre*, Mathematische Annalen, vol. 61 (1905), p. 142.

COROLLARY 4. *The class of all simply ordered sequences of positive integers which contain no repetition of any integer is of cardinal number** $\beth_0$.

The preceding theorem and corollaries are independent of the axiom of choice.

5. **The categorical character of the set of postulates 1-5.** Between any two aggregates J and J' both of which satisfy Postulates 1–5 of §2 it is possible to set up in the following way a one-to-one correspondence which preserves order. Let the first element ω of J correspond to the first element ω' of J'. Then make the requirement that if the element α of J correspond to the element α' of J' then the element $\alpha+1$ of J shall correspond to the element $\alpha'+1$ of J' and to no other element of J', and that if the elements of a fundamental sequence $\beta_0, \beta_1, \beta_2, \cdots$ in J correspond respectively to the elements of a fundamental sequence $\beta_0', \beta_1', \beta_2', \cdots$ in J' then the upper limit β of the first sequence shall correspond to the upper limit β' of the second sequence and to no other element of J'.

Now no element of either aggregate follows next after more than one element of the aggregate. And in either aggregate two fundamental sequences have the same upper limit if and only if it is true that any given ordinal of either sequence precedes some ordinal of the other sequence. From this it follows that if a correspondence between J and J' which satisfies the requirement just stated is one-to-one and preserves order in the case of every element which precedes a given element α of J, then it is one-to-one and preserves order in the case of α also. Therefore we can establish by transfinite induction the existence of a one-to-one correspondence between J and J' which preserves order, the correspondence being constructed in a step by step fashion in accordance with the requirement of the preceding paragraph.

If α is an element of the second kind in J, the corresponding element α' in J' is obtained by choosing a fundamental sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$ for α. Then if $\alpha_0', \alpha_1', \alpha_2', \cdots$ in J' correspond respectively to $\alpha_0, \alpha_1, \alpha_2, \cdots$ in J, α' is the upper limit of the fundamental sequence $\alpha_0', \alpha_1', \alpha_2', \cdots$ in J'. It is true, however, that no matter what fundamental sequence is chosen for α the same corresponding element α' in J' is obtained. The construction of the one-to-one correspondence between J and J' involves, accordingly, no appeal to the axiom of choice.

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* Cf. F. Bernstein, *Untersuchungen aus der Mengenlehre*, Mathematische Annalen, vol. 61 (1905), pp. 140–145, where it is proved that $\beth_0$ is the cardinal number of the class of all order types in which the set of positive integers can be arranged. Two proofs of this theorem are given, but in both of them it is necessary to use the axiom of choice. Nevertheless an obvious modification of the first of these proofs suffices to prove without the aid of the axiom of choice, not the theorem stated by Bernstein, but the related theorem stated in Corollary 4 above.

It will be convenient in this connection to use the following definition of categorical character. A set of postulates is categorical if, given any two systems both of which satisfy all the postulates of the set, there exists a one-to-one correspondence between these two systems which preserves all the relations among their elements which appear as undefined terms in the postulates.

The only such relation which appears in the set of postulates 1–5 under discussion is the relation of order among the ordinals. Therefore we have proved that this set is categorical in the sense just defined.

6. **Nature of a categorical set.** Suppose that a categorical set S of postulates is given, and two contradictory statements K and L in the form of theorems about the system described by S. Then we are not at liberty to suppose that there exist two systems s_1 and s_2, both of which satisfy S, in one of which K is true, and in the other of which L is true, because, if this were the case, we could obtain a contradiction at once by means of the one-to-one correspondence between s_1 and s_2. In a certain sense, therefore, a categorical set is a complete set, because it is impossible to employ simultaneously two distinct systems which satisfy the same categorical set of postulates.

It does not, however, follow that one of the statements K or L must be inconsistent with the set of postulates* S. It is quite conceivable that, although the coexistence of s_1 and s_2 lead to contradiction, nevertheless neither the existence of s_1 alone nor that of s_2 alone should lead to contradiction.

It is clear that the completeness of the set of postulates at the basis of our logic is involved.† We might, not unnaturally, make it one of the requirements for completeness of a set of postulates for logic that, in all such cases as that described above, one of the statements K or L should lead to a contradiction when taken in conjunction with the set S. In the absence,

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* E. B. Wilson, *Logic and the continuum*, Bulletin of the American Mathematical Society, vol. 14 (1908), pp. 432–443. See also E. V. Huntington, *A set of postulates for ordinary complex algebra*, these Transactions, vol. 6 (1905), p. 210 and the second footnote on that page.

†It is possible to demonstrate the completeness of a certain portion of the postulates of logic in the sense that no new independent and consistent postulate can be added to this portion without introducing a new undefined term. The portion in question consists of the postulates given by Whitehead and Russell in part I, section A, of the *Principia Mathematica*. The proof of the completeness of these postulates has been given by E. L. Post, *Introduction to a general theory of elementary propositions*, American Journal of Mathematics, vol. 43 (1921), p. 163–185. But this does not imply the completeness in any sense of the full set of postulates for logic (as at present known), because this full set involves additional undefined terms.

however, of any demonstration that the set of postulates on which our logic is based satisfies this requirement, we must not infer from the fact that a set of postulates S is categorical that there do not exist one or more independent postulates which can be added to the set.

It is not improbable that the set of postulates at the basis of our logic is not complete, even if the axiom of choice is included in the set, because if it were complete it ought to be possible, in the case of every set X of classes which does not contain the null class, to construct a particular function F of the kind whose existence is required by the axiom of choice, a construction the possibility of which is, in many cases, doubtful. If the axiom of choice is excluded the probability of completeness is even more remote.

The question whether the set of postulates at the basis of our logic is or is not complete is evidently equivalent to the question whether or not every mathematical problem can be solved.* On the other hand, since each of these questions is in the form of a theorem about the postulates of logic, into the truth of which theorem it is proposed to inquire, neither has a direct connection with the law of excluded middle,† which is itself a postulate of logic. Suppose, for example, that we have before us a certain consistent set W of postulates for logic among which is the law of excluded middle. There may be, if W is not complete, a postulate p such that either p or *not-p* can be added to the set W without destroying the consistent character of the set. In this case there may be a universe of discourse U_1 in which p and the postulates of W are satisfied and also a universe of discourse U_2 in which *not-p* and the postulates of W are satisfied. Then p would satisfy the law of excluded middle both in U_1 and in U_2, in U_1 by being true, and in U_2 by being false. Accordingly our inability to conclude on the basis of W whether p is true or false does not prevent our concluding on the basis of W that p is either true or false.

7. **Alternatives to Zermelo's assumption.** The foregoing discussion is intended to prepare the way for the suggestion that there may be one or more additional independent postulates which can be added to the set of postulates 1–5 and to forestall the objection that this set is already categorical.

With this possibility in mind we propose to examine the consequences of each of the following postulates when it is taken in conjunction with Postulates 1–5:

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* The latter question is proposed by D. Hilbert in *Mathematische Probleme*, Göttinger Nachrichten, 1900, pp. 261–262, and Archiv der Mathematik und Physik, (3), vol. 1 (1901), p. 52, and *Mathematical Problems*, Bulletin of the American Mathematical Society, vol. 8 (1902), pp. 444–445.

† The opposite view is maintained by L. E. J. Brouwer. See for example, *Intuitionistische Mengenlehre*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 28 (1920), pp. 203–208.

A. *There exists an assignment of a unique fundamental sequence to every ordinal of the second kind in the second ordinal class.*

B. *There exists no assignment of a unique fundamental sequence to every ordinal of the second kind in the second ordinal class; but given any ordinal α of the second ordinal class, there exists an assignment of a unique fundamental sequence to every ordinal of the second kind less than α.*

C. *There is an ordinal ϕ of the second ordinal class such that there exists no assignment of a unique fundamental sequence to every ordinal of the second kind less than ϕ.*

It is an immediate consequence of Postulates 1–5 that there exist fundamental sequences for any particular ordinal of the second kind in the second ordinal class. The preceding postulates A, B, and C are concerned with the possibility of assigning a particular fundamental sequence to every such ordinal in a simultaneous manner.*

Postulate A can be derived as a consequence of the axiom of choice for sets of $\aleph_1$ classes. Postulate B implies a denial of the axiom of choice for sets of $\aleph_1$ classes but seems to be consistent with this axiom for sets of $\aleph_0$ classes. Postulate C implies a denial of the axiom of choice for sets of $\aleph_0$ classes. There is no reason, however, to suppose that Postulate B implies the axiom of choice for sets of $\aleph_0$ classes or that Postulate A implies this axiom for sets of $\aleph_1$ classes.

Postulates A, B, and C are mutually exclusive and it is clear that, together, they exhaust the conceivable alternatives. There are, therefore, three conceivable kinds of second ordinal classes, one corresponding to each of these postulates. If any one of these involve a contradiction it is reasonable to expect that a systematic examination of its properties will ultimately reveal this contradiction. But if a considerable body of theory can be developed on the basis of one of these postulates without obtaining inconsistent results, then this body of theory, when developed, could be used as presumptive evidence that no contradiction existed.

If there be two of these postulates neither of which leads to contradiction, then there are corresponding to them two distinct self-consistent second ordinal classes, just as euclidean geometry and Lobachevskian geometry are distinct self-consistent geometries, with, however, this difference, that the two

*For a discussion of the problem of carrying out such an assignment of fundamental sequences see O. Veblen, *Continuous increasing functions of finite and transfinite ordinals*, these Transactions, vol. 9 (1908), pp. 280–292.

second ordinal classes are incapable of existing together in the same universe of discourse.

It is not unlikely that no one of the three postulates A, B, C leads to any contradiction.

8. **Consequences of Postulate A.** THEOREM A_1. *If α is any ordinal of the second ordinal class, the cardinal number corresponding to α is** $\aleph_0$.

For assign to every ordinal β of the second kind which is less than or equal to α a fundamental sequence u_β (Postulate A).

The ordinals which precede ω form, when arranged in their natural order, a sequence of ordinal number ω. With this as a starting point, we assign to the ordinals which follow ω, one by one in order, an arrangement of all preceding ordinals in a sequence of ordinal number ω, in the following way.

When we have assigned an arrangement in a sequence t of ordinal number ω of all ordinals which are less than an ordinal γ, an arrangement in a sequence of ordinal number ω of all ordinals which are less than $\gamma+1$ is obtained by placing γ before t.

When, to every ordinal ζ which is less than an ordinal β of the second kind, we have assigned an arrangement in a sequence t_ζ of ordinal number ω of all ordinals which are less than ζ, the sequences $t_{\beta_0}, t_{\beta_1}, t_{\beta_2}, \cdots$, where $\beta_0, \beta_1, \beta_2, \cdots$ is the fundamental sequence u_β, may be written one below the other so as to obtain the following array:

$$\begin{array}{cccc} \delta_{00}, & \delta_{01}, & \delta_{02}, & \cdots \\ \delta_{10}, & \delta_{11}, & \delta_{12}, & \cdots \\ \delta_{20}, & \delta_{21}, & \delta_{22}, & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

where $\delta_{i0}, \delta_{i1}, \delta_{i2}, \cdots$ is the sequence t_{β_i} and contains all the ordinals which are less than β_i. Then the ordinals δ_{ij} may be arranged in a sequence of ordinal number ω as follows:

$$\delta_{00}, \ \delta_{01}, \ \delta_{10}, \ \delta_{01}, \ \delta_{11}, \ \delta_{20}, \ \delta_{03}, \ \delta_{12}, \ \delta_{21}, \ \delta_{30}, \ \cdots.$$

By omitting from this sequence all occurrences of any ordinal after the first occurrence, so that a sequence without repetitions results, we obtain an arrangement in a sequence of ordinal number ω of all ordinals less than β.

*See G. Cantor, loc. cit., zweiter Artikel, p. 221. As already explained, Cantor uses the property described in Theorem A_1 in defining the second ordinal class. He then proves, with the aid of the axiom of choice, that the second ordinal class has the properties expressed by the postulates of §2 above.

The way in which the axiom of choice is involved is pointed out by Whitehead and Russell, *Principia Mathematica*, vol. III, 1913, p. 170.

We may prove by induction that this process continues until we obtain an arrangement in a sequence of ordinal number ω of all ordinals less than α. Therefore the cardinal number corresponding to α is $\aleph_0$.

COROLLARY 1. *There exists an assignment to every ordinal β in the second ordinal class of an arrangement in a sequence of ordinal number ω of all ordinals less than β.*

For, under Postulate A, we may assign a fundamental sequence u_β to every ordinal of the second kind in the second ordinal class. The process just described then continues until we have assigned to every ordinal β of the second ordinal class an arrangement in a sequence of ordinal number ω of all ordinals less than β.

COROLLARY 2. *There exists an assignment to every ordinal β in the second ordinal class of a well-ordered rearrangement of the positive integers of ordinal number β.*

Because an arrangement in a sequence of ordinal number ω of all ordinals less than β determines a one-to-one correspondence between the ordinals less than β and the positive integers, and this one-to-one correspondence determines in turn a well-ordered rearrangement of the positive integers of ordinal number β.

THEOREM A_2. *If the class R consists of all well-ordered sequences of positive integers (allowing any number of repetitions of the same integer) whose ordinal numbers belong to the second ordinal class, the cardinal number of R is $\beth_{1,0}$.*

Let O be the class of ordinals less than Ω. Then the cardinal number of O is $\aleph_1$. Let Q be the class of the subsets of O which have a cardinal number not greater than $\aleph_0$. Then the cardinal number of Q is $\beth_{1,0}$.

The class Q can be put into one-to-one correspondence with a part of R as follows. To the subset S of O contained in Q correlate the sequence $a_0, a_1, a_2, \cdots a_\omega, \cdots$ of R, where a_μ is 1 or 0 according as μ is or is not contained in S, and the sequence consists of those a's whose subscripts are less then β, where β is the least ordinal greater than every ordinal in S and not less than ω.

The class R can be put into one-to-one correspondence with a part of Q as follows. To the sequence $b_0, b_1, b_2, \cdots, b_\omega, \cdots$ of R, of ordinal number γ, correlate the subset of O consisting of the following ordinals: $0, 1, 2, \cdots, b_0; \omega, \omega+1, \cdots, \omega+b_1; 2\omega, 2\omega+1, \cdots, 2\omega+b_2; \cdots \gamma\omega$.*

*We adopt Cantor's earlier notation, placing the multiplier before the multiplicand.

Therefore, by the Schröder-Bernstein theorem,* Q and R can be put into one-to-one correspondence.

Therefore the cardinal number of R is $\beth_{1,0}$.

THEOREM A_3. *The cardinal numbers* $\beth_0$ *and* $\beth_{1,0}$ *are identical.*

In accordance with Corollary 1 of Theorem A_1, assign to every ordinal β of the second ordinal class an arrangement in a sequence t_β of ordinal number ω of all ordinals less than β.

Let b be one of the sequences belonging to the class R of the preceding theorem, and let β be the ordinal number of b. Then, corresponding to b, we can determine a sequence c of positive integers of ordinal number ω by the rule that if κ_i is the ith ordinal of t_β then the ith positive integer in c shall be the κ_ith positive integer of b. In this way we can set up a one-to-one correspondence between all the sequences b of R which have a fixed ordinal number β and the class of sequences of positive integers of ordinal number ω. And in exactly the same way we can set up a one-to-one correspondence between those well-ordered rearrangements of the positive integers which have a fixed ordinal number β and the class of permutations of the positive integers, where a permutation of the positive integers is restricted to be of ordinal number ω. But the class of sequences of positive integers of ordinal number ω and the class of permutations of the positive integers can be put into one-to-one correspondence, since each is of cardinal number $\beth_0$. Therefore, choosing a particular such one-to-one correspondence C, we can set up a one-to-one correspondence K_β between the sequences of R which have a fixed ordinal number β and the well-ordered rearrangements of the positive integers which have the ordinal number β. Moreover, since C can be chosen once for all, we have a uniform method of setting up the one-to-one correspondences K_β, and therefore, without appeal to the axiom of choice, we can suppose them all set up, for every ordinal β of the second ordinal class. But as soon as this is done we have a one-to-one correspondence between R and the class of well-ordered rearrangements of the positive integers. And, in Theorems A_2 and 1, we have shown that these two classes have, respectively, the cardinal number $\beth_{1,0}$ and the cardinal number $\beth_0$. Therefore these two cardinal numbers are identical.

COROLLARY. *The class R of Theorem* A_2 *can be put into one-to-one correspondence with the continuum.*

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* Loc. cit.

THEOREM A_4. *The continuum contains a subset of cardinal number** $\aleph_1$.

For the class R, which has just been shown to have the cardinal number of the continuum, contains such a subset, namely the set of all those sequences belonging to R which consist entirely of 2's.

The same theorem can be proved by means of Corollary 2 of Theorem A_1, because, in accordance with that corollary, the class of well-ordered rearrangements of the positive integers contains a subset of cardinal number $\aleph_1$, and, by Theorem 1, this class can be put into one-to-one correspondence with the continuum.

The fact that the set of postulates 1–5 and A, which are all statements about the second ordinal class, has consequences about an entirely different aggregate, namely the continuum, is evidently connected with the fact that these postulates contain more than is necessary to render them categorical.

We have already observed that if we are to think of the three second ordinal classes, that corresponding to Postulate A, that corresponding to B, and that corresponding to C, as all existing we must think of them as each existing in a different universe of discourse. Therefore when we single out one of these second ordinal classes for consideration we thereby restrict the character of the universe of discourse within which we are working. In this way we may think of Postulate A as being indirectly a postulate of logic, although it is in form a statement about the second ordinal class. And the same remark applies to Postulates B and C.

9. **Consequences of Postulate B.** THEOREM B_1. *If α is any ordinal of the second ordinal class, the cardinal number corresponding to α is* $\aleph_0$.

The proof of Theorem A_1 applies without change.

THEOREM B_2. *There exists no assignment to every ordinal β in the second ordinal class of an arrangement in a sequence of ordinal number ω of all ordinals less than β.*

For, given an arrangement in a sequence t, of ordinal number ω, of all ordinals less than β, we could obtain a fundamental sequence for β by omitting from t all the ordinals α which did not have the property that every ordinal which preceded α in t also preceded α in the natural order of the ordinals. Therefore if there existed an assignment to every ordinal β in the second ordinal class of an arrangement in a sequence of ordinal number ω

* A proof of this theorem has been given by G. H. Hardy, *A theorem concerning the infinite cardinal numbers*, Quarterly Journal of Pure and Applied Mathematics, vol. 35 (1903), pp. 87–94. The use of a simultaneous assignment of a fundamental sequence to every ordinal of the second kind in the second ordinal class is essential to Hardy's proof, as it is to the proof given above.

of all ordinals less than β, there would exist an assignment of a fundamental sequence to every ordinal β in the second ordinal class, contrary to Postulate B.

COROLLARY. *There exists no assignment to every ordinal β in the second ordinal class of a well-ordered rearrangement of the positive integers of ordinal number β.*

THEOREM B_3. *The continuum cannot be well-ordered.*

For it follows from Theorem 1 that, if the continuum could be well-ordered, the set of all well-ordered rearrangements of the positive integers could be arranged in a well-ordered sequence u. And it would then be possible to assign to every ordinal β in the second ordinal class a well-ordered rearrangement of the positive integers of ordinal number β, because we could choose, for every β, the first rearrangement of ordinal number β that occurred in u. This, however, is contrary to the corollary of the preceding theorem.

THEOREM B_4. *If the class R consists of all well-ordered sequences of positive integers (allowing any number of repetitions of the same integer) whose ordinal numbers belong to the second ordinal class, the cardinal number of R is $\gimel_{1,0}$.*

The proof of Theorem A_2 applies without change.

DEFINITION. Let t be an internally closed sequence of ordinals of the first and second ordinal classes, all distinct, and arranged in their natural order. Then there is one and only one way in which t can be put into one-to-one correspondence (preserving order) with the sequence of ordinals less than Ω or a segment of this sequence. The ordinals of the first and second ordinal classes which are correlated to themselves in this correspondence form, when arranged in their natural order, an internally closed sequence t', the *first derived sequence** of t. The first derived sequence of t' is the *second derived sequence* of t and so on. If ν is an ordinal of the second kind in the second ordinal class, the *νth derived sequence t^ν* of t consists of those ordinals which are contained in all previous derived sequences. If μ is an ordinal of the first or second ordinal class the *$(\mu+1)$th derived sequence $t^{\mu+1}$* of t is the first derived sequence of t^μ.

If μ is an ordinal of the first or second class, the μth derived sequence t^μ of t is, like t, an internally closed sequence of ordinals of the first and second

*The sequence t determines by its correspondence with the sequence of ordinals less than Ω or a segment thereof a continuous increasing function. The sequence t' consists of the set of values of the first derived function. See O. Veblen, *Continuous increasing functions of finite and transfinite ordinals*, these Transactions, vol. 9 (1908), p. 281.

ordinal classes, all distinct, and arranged in their natural order. And if the ordinal number of t is Ω that of t^μ is also* Ω.

If the first term of t is greater than 0 the sequence formed by taking the first term of each of the derived sequences of t in order is an internally closed sequence of distinct ordinals in their natural order.†

Definition. An internally closed sequence r of ordinals of the second kind of the second ordinal class, all distinct and arranged in their natural order, is a *reduction sequence* if there exists an assignment to every ordinal κ of the second kind in the second ordinal class of a sequence v_κ of distinct ordinals arranged in their natural order, such that the upper limit of v_κ is κ and the ordinal number of v_κ is either ω or one of the ordinals of r.

It follows at once from Postulate B that the ordinal number of a reduction sequence must be Ω.

The sequence of self-residual‡ ordinals of the second ordinal class in their natural order is a reduction sequence. Its first derived sequence, the sequence of ϵ-numbers,‡ is also a reduction sequence.

Theorem B_5. *If the first derived sequence of a reduction sequence r is a reduction sequence, then all the successive derived sequences of r are reduction sequences.*

For let r^ν be the νth derived sequence of r. Assign a fundamental sequence to every ordinal of the second kind less than ν (Postulate B). And to every ordinal of the second kind in the second ordinal class assign a sequence v_κ of distinct ordinals arranged in their natural order such that the upper limit of v_κ is κ and the ordinal number of v_κ is either ω or one of the ordinals of r'. This is possible since, by hypothesis, r' is a reduction sequence.

Let ρ be an ordinal of the second kind which occurs in r' but not in r^ν. Then there is a last ordinal μ, which is less than ν, such that ρ occurs in r^μ. Let α be the ordinal which corresponds to ρ in a one-to-one correspondence (preserving order) between r^μ and the sequence of all ordinals less than Ω.

If μ is of the second kind, let $\mu_0, \mu_1, \mu_2, \cdots$ be the fundamental sequence which we have assigned to μ.

If μ is of the second kind and α is equal to 0, so that ρ stands in the first place in r^μ, the ordinals $\rho_0, \rho_1, \rho_2, \cdots$ which stand in the first place in r^{μ_0}, $r^{\mu_1}, r^{\mu_2}, \cdots$, respectively, constitute a fundamental sequence for ρ.

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*O. Veblen, loc. cit., pp. 283–285.

†O. Veblen, loc. cit., p. 285.

‡For definitions see G. Cantor, loc. cit., zweiter Artikel.

If μ is of the second kind and α is of the first kind, let σ be the ordinal which next precedes ρ in r^{μ}, and let ρ_i be the ordinal which next follows σ in r^{μ_i}. The ordinals $\rho_0, \rho_1, \rho_2, \cdots$ are in their natural order, because if ρ_{i+1} were less than ρ_i it would follow that ρ_{i+1} did not occur in r^{μ_i}, contrary to the definition of derived sequences, which requires that all the terms of $r^{\mu_{i+1}}$ shall be ordinals of r^{μ_i}. And $\rho_0, \rho_1, \rho_2, \cdots$ are all distinct, because if ρ_{i+1} were equal to ρ_i it would follow that both were equal to $\beta+1$, where β was the ordinal which corresponded to σ in a one-to-one correspondence (preserving order) between r^{μ_i} and the sequence of all ordinals less than Ω, contrary to the requirement that all the ordinals of r, and therefore all those of r^{μ_i}, shall be ordinals of the second kind. The upper limit of the sequence $\rho_0, \rho_1, \rho_2, \cdots$, since it is also the upper limit of the sequence $\rho_i, \rho_{i+1}, \rho_{i+2}, \cdots$, necessarily occurs in r^{μ_i}, and, since it occurs in every r^{μ_i}, it occurs also in r^{μ}. This upper limit cannot be greater than ρ, because, if it were, some ρ_i, say ρ_n, would be greater than ρ, and it would follow that ρ did not occur in r^{μ_n}. But the upper limit of $\rho_0, \rho_1, \rho_2, \cdots$ is greater than σ, because each term is greater than σ. Therefore the upper limit is ρ, so that $\rho_0, \rho_1, \rho_2, \cdots$ is a fundamental sequence for ρ.

If μ is of the first kind (so that $\mu=\lambda+1$) and α is equal to 0, let ρ_0 be the first ordinal of r^{λ}, ρ_1 the ρ_0th ordinal of r^{λ}, ρ_2 the ρ_1th ordinal of r^{λ}, and so on. Then the sequence $\rho_0, \rho_1, \rho_2, \cdots$ is an increasing sequence, because $\rho_{i+1}<\rho_i$ is impossible on account of the increasing character of r^{λ}, and $\rho_{i+1}=\rho_i$ would imply $\rho_1=\rho_0$, a situation which is impossible because ρ_0 cannot be equal to 0 and ρ_1 is the ρ_0th ordinal of r^{λ}. And the upper limit of $\rho_0, \rho_1, \rho_2, \cdots$ is ρ.

If μ is of the first kind (so that $\mu=\lambda+1$) and α is also of the first kind, let σ be the ordinal which next precedes ρ in r^{μ}. Let ρ_0 be the $(\sigma+1)$th ordinal of r^{λ}, ρ_1 the ρ_0th ordinal of r^{λ}, ρ_2 the ρ_1th ordinal of r^{λ}, and so on. Then the sequence $\rho_0, \rho_1, \rho_2, \cdots$ is an increasing sequence, because $\rho_{i+1}<\rho_i$ is impossible on account of the increasing character of r^{λ}, and $\rho_{i+1}=\rho_i$ would imply $\rho_0=\sigma+1$, whereas ρ_0 must be an ordinal of the second kind. And the upper limit of $\rho_0, \rho_1, \rho_2, \cdots$ is ρ.

The case $\mu=1$ is taken account of in each of the two preceding paragraphs, for in that case $\lambda=0$ and r^{λ} is the sequence r itself.

If α is of the second kind, we have assigned to α a sequence v_{α} whose upper limit is α and whose ordinal number is an ordinal τ of r'. If γ is any ordinal less than τ, let α_{γ} be the γth term of v_{α}, and let ρ_{γ} be the α_{γ}th term of r^{μ}. Then the sequence $\rho_2, \rho_3, \rho_4, \cdots, \rho_{\omega}, \rho_{\omega+1}, \cdots$ is a sequence of ordinal number τ whose upper limit is ρ. And α, and therefore τ, is less than ρ. The problem of finding an increasing sequence whose upper limit is ρ and whose ordinal number is either ω or an ordinal of r^{ν} is, therefore, reduced to the corresponding problem for the smaller ordinal τ, and this reduction continues until we obtain such a sequence or until one of the cases already considered arises.

We can now conclude that it is possible to assign to every ordinal ρ in r' an increasing sequence of ordinals whose upper limit is ρ and whose ordinal number is either ω or an ordinal of r^ν, because we have just described a systematic method of making such an assignment. But we have assigned to every ordinal κ of the second kind in the second ordinal class an increasing sequence v_κ of ordinals whose upper limit is κ and whose ordinal number is either ω or an ordinal of r'. Therefore we can assign to every ordinal κ of the second kind in the second ordinal class an increasing sequence of ordinals whose upper limit is κ and whose ordinal number is either ω or an ordinal of r^ν. Therefore r^ν is a reduction sequence.

COROLLARY 1. *The sequence $\bar{r}$ of those ordinals which occur in the first place in the sequences r^θ, where θ takes on all values which make it an ordinal of the second kind, is a reduction sequence.*

For, by the method just given, we can assign to every ordinal ρ of r' an increasing sequence of ordinals whose upper limit is ρ and whose ordinal number is an ordinal of $\bar{r}$.

If ρ occurs in one of the sequences r^θ in such a way that it has an immediate predecessor σ in r^θ, let ρ_α be the ordinal which next follows σ in r^α, where α is any ordinal less than θ. Then the ordinals ρ_α form an increasing sequence of ordinal number θ whose upper limit is ρ. From this, by means of the sequence v_θ, we obtain an increasing sequence whose upper limit is ρ and whose ordinal number is an ordinal ρ' of r' less than ρ. The problem of finding an increasing sequence of ordinals whose upper limit is ρ and whose ordinal number is an ordinal of $\bar{r}$ therefore reduces to the corresponding problem for the smaller ordinal ρ'. And this reduction continues until we obtain such a sequence or one of the other possible cases arises.

In all other cases we proceed exactly as we did in proving Theorem B_5.

COROLLARY 2. *The sequence $\tilde{r}$ of those ordinals which occur in the first place in the successive derived sequences of r is a reduction sequence.*

For $\tilde{r}$ contains $\bar{r}$.

COROLLARY 3. *The first derived sequence $\bar{r}'$ of $\bar{r}$, and therefore all the derived sequences of $\bar{r}$, are reduction sequences.*

Since $\bar{r}$ is a reduction sequence, we can assign to every ordinal κ of the second kind in the second ordinal class an increasing sequence v_κ of ordinals whose upper limit is κ and whose ordinal number α is either ω or an ordinal of $\tilde{r}$. If α is not an ordinal of $\bar{r}'$, it occurs in the κ'th place in $\bar{r}$, where κ' is an ordinal of the second kind less than κ. The problem of finding an increasing sequence of ordinals whose upper limit is κ and whose ordinal number is either ω or an ordinal of $\bar{r}'$ therefore reduces to the corresponding problem for the smaller ordinal κ', and this reduction continues until an ordinal $\kappa^{(k)}$ is obtained which is either ω or an ordinal of $\bar{r}'$.

COROLLARY 4. *The first derived sequence $\tilde{r}'$ of $\tilde{r}$, and therefore all the derived sequences of $\tilde{r}$, are reduction sequences.*

For $\tilde{r}$ contains $\bar{r}'$, and therefore $\tilde{r}'$ contains $\bar{r}''$.

It is possible that it can be proved that every internally closed sequence of ordinal number Ω and consisting of ordinals of the second kind is a reduction sequence. At any rate, we are not able to show the contrary.

In particular, the question naturally arises in this connection whether there exists a reduction sequence whose first derived sequence is not a reduction sequence. It is clear, in view of Theorem B_5, that if such a reduction sequence existed, it could not be the first derived sequence of any sequence. But it does not follow from this alone that such reduction sequences do not exist. In fact it is possible to find internally closed sequences of ordinal number Ω in the second ordinal class which are not first derived sequences of other sequences. In constructing an example we have only to choose 2ω as the first ordinal of the sequence, because, in any order preserving one-to-one correspondence between the set of ordinals less than 2ω and a subset of them, the ordinal ω necessarily corresponds to itself. As an example of as equence which not only is not the first derived sequence of any sequence but retains that property no matter how many ordinals are omitted from the beginning of it, we may take the sequence s of the ordinals $(2\omega)^\alpha$ arranged in order of magnitude, where α takes on all values less than Ω except the value 0. The sequence s and its first derived sequence (namely the sequence of ϵ-numbers) are, however, both reduction sequences.

10. **Consequences of Postulate** C. DEFINITION. Under Postulate C there is an ordinal ϕ of the second ordinal class such that there exists no assignment of a unique fundamental sequence to every ordinal of the second kind less than ϕ. The ordinal υ_1 is the least such ordinal ϕ in the second ordinal class.

THEOREM C_1. *The ordinal υ_1 is an ordinal of the second kind.*

For if υ_1 were an ordinal of the first kind, there would be an ordinal α which next preceded υ_1, and there would exist an assignment of a unique fundamental sequence to every ordinal of the second kind less than α. Then, in order to obtain an assignment of a unique fundamental sequence to every ordinal of the second kind less than υ_1, we would have to make at most one arbitrary choice, namely a choice of a fundamental sequence for α if α were of the second kind. Since a single arbitrary choice is always permissible, this would lead to a contradiction with the definition of υ_1. Therefore υ_1 is an ordinal of the second kind.

The way in which Postulate C involves a denial of the axiom of choice for sets of $\aleph_0$ classes can be made clear in this connection by proposing the following argument which purports to show that Postulate C leads to a contradiction. Let $\alpha_0, \alpha_1, \alpha_2, \cdots$ be a fundamental sequence for υ_1. Since α_0 is less than υ_1 it follows at once from the definition of υ_1 that there exists an assignment A_0 of a unique fundamental sequence to every ordinal of the second kind less than α_0, and similarly that there exists an assignment A_1 of a unique fundamental sequence to every ordinal of the second kind less than α_1, an assignment A_2 of a unique fundamental sequence to every ordinal of the second kind less than α_2, and so on. Then, in order to obtain an assignment of a unique fundamental sequence to every ordinal of the second kind less than υ_1, we may use A_0 for ordinals less than α_0, A_1 for ordinals less than α_1 and not less than α_0, A_2 for ordinals less than α_2 and not less than α_1, and so on.

This argument fails to obtain a contradiction from Postulate C because the assignments A_0, A_1, $A_2, \cdots$ are each of them only one of the many existing assignments of the required character, so that there is an element of arbitrary choice involved in fixing upon the particular assignment A_0, another in fixing upon A_1, and so on. The use of all the assignments, $A_0, A_1, A_2, \cdots$, simultaneously, therefore, involves an appeal to Zermelo's assumption, and this is, of course, inadmissible in this connection.

THEOREM C_2. *If α is an ordinal of the second ordinal class less than υ_1, the cardinal number corresponding to α is $\aleph_0$.*

Since α is less than υ_1 it is possible to assign to every ordinal β of the second kind less than or equal to α a fundamental sequence u_β. The proof of Theorem A_1 can, therefore, be used here without change.

THEOREM C_3. *The cardinal number corresponding to υ_1 is not $\aleph_0$.*

For suppose that the set of ordinals less than υ_1 could be arranged in a sequence t of ordinal number ω. Then for any ordinal κ less than υ_1 we could obtain a fundamental sequence by omitting from t, first all ordinals which were not less than κ, then all ordinals which did not have the property of being greater than every ordinal less than κ which preceded them in t. This would enable us to assign a fundamental sequence to every ordinal κ of the second kind less than υ_1, contrary to the definition of υ_1.

COROLLARY. *The cardinal number corresponding to* υ_1 *is* $\aleph_1$.

We have pointed out in §2 above that the definition of the second ordinal class which we are using differs from Cantor's definition. Under the latter the second ordinal class consists of all those ordinals to which the corresponding cardinal number is $\aleph_0$. In connection with Postulate C, these would be the ordinals less than υ_1 and not less than ω, whereas, under the definition which we are using, the second ordinal class contains ordinals greater than υ_1.

The convenience of a definition in terms of order, such as the one which we are using, lies in the fact that it enables us to use unchanged many known theorems about the second ordinal class which have been proved by means of order properties and therefore apply to the more extensive class of ordinals rather than to the ordinals between ω and υ_1. An example is the theorem that if f is a continuous increasing function* defined for the set of ordinals less than Ω and its value is always an ordinal less than Ω then the first derived function* of f exists.† The truth of this theorem is not affected by our choice among the postulates A, B, C, if we adhere to the definition of Ω which we have given. But, as we shall prove in Theorem C_8 below, Postulate C implies the falsehood of the theorem just stated if we take Ω to be the ordinal which we have called υ_1, as we should have to if we used Cantor's definition of the second ordinal class. Another example is the theorem, to which we shall refer below, that every ordinal of the second ordinal class can be expressed in Cantor's normal form. This theorem is true not only of ordinals of the second ordinal class less than υ_1 but of all the ordinals of the second ordinal class as we have defined it, because Cantor's proof is equally applicable to ordinals less than υ_1 and to ordinals of the second ordinal class greater than υ_1.

Nevertheless it is true that, in many ways, the ordinal υ_1 plays, in connection with Postulate C, a rôle similar to that played by Ω in connection with Postulates A and B.

THEOREM C_4. *There exists a denumerable set of subsets* $S_0, S_1, S_2, \cdots$ *of the continuum such that there is no choice of one element out of each of the sets* $S_0, S_1, S_2, \cdots$.

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*For definitions, see below.

†O. Veblen, loc. cit., p. 283.

By Theorem 1, there exists a one-to-one correspondence between the continuum and the set of well-ordered rearrangements of the positive integers. Let K be such a one-to-one correspondence. Let $\alpha_0, \alpha_1, \alpha_2, \cdots$ be a fundamental sequence for ν_1. And let S_i be the set of those numbers of the continuum which correspond under K to well-ordered rearrangements of the positive integers of ordinal number α_i. Then there is no choice of one element out of each of the sets S_i, because, if there were, this would lead to a choice, for each ordinal α_i, of a well-ordered rearrangement of the positive integers of ordinal number α_i, and this would lead in turn to a choice, for each ordinal α_i, of an arrangement of the ordinals less than α_i in a sequence of ordinal number ω, and then, by the method of Theorem A_1, we could obtain an arrangement of the ordinals less than ν_1 in a sequence of ordinal number ω, contrary to Theorem C_3.

COROLLARY. *The continuum cannot be well-ordered.*

THEOREM C_5. *There exists a set T of points of the continuum and a set I of intervals which covers T such that no denumerable subset of I covers* T.*

On account of the possibility of setting up a one-to-one correspondence between the continuum and the segment $(0, 1/2)$ of the continuum, the sets S_i of the preceding theorem can be so chosen that all their elements lie between 0 and 1/2. Let the sets S_i be chosen in this way, and let T consist of the points $0, 1, 2, 3, \cdots$. Let I_i consist of all intervals $(i-s_i, i+s_i)$, where s_i is an element of S_i. Then there is a one-to-one correspondence between I_i and S_i. Let I be the sum of all the sets I_i, so that I contains every interval which occurs in any one of the sets I_i.

It is clear that I covers T. Suppose that some denumerable subset J of I covered T. Then J would contain at least one interval in common with

*The contrary of this theorem has been proved by W. H. Young with the aid of the axiom of choice. See Proceedings of the London Mathematical Society, vol. 35 (1902), pp. 384–388, and also W. H. Young and G. C. Young, *The Theory of Sets of Points*, 1906, pp. 38–40.

See also E. Lindelöf, *Sur quelques points de la théorie des ensembles*, Comptes Rendus, vol. 137 (1903), pp. 697–700, and *Remarques sur un théorème fondamental de la théorie des ensembles*, Acta Mathematica, vol. 29 (1905), pp. 187–189. Lindelöf's theorem is stronger than that given by Young in that it applies to space of any finite number of dimensions and weaker in that there must be a one-to-one correspondence between the points of I and the intervals (or n-spheres) of T, each point being at the center of the corresponding interval (or n-sphere). The axiom of choice is necessary to the proof.

The falsity of Lindelöf's theorem for one dimension can be proved as a consequence of Postulate C by means of the following modification of the proof of Theorem C_5. Let T consist of all points $i+(1/2)s_i$, $i=0, 1, 2, 3, \cdots$, and to the point $i+(1/2)s_i$ let correspond the interval $(i, i+s_i)$, I_i consisting of all intervals $(i, i+s_i)$ for a fixed value of i, and I being the sum of the sets I_i.

each of the sets I_i. And J could be arranged in a sequence of ordinal number ω, and for each set I_i could be chosen the first interval of J which belonged also to I_i. And this would effect a choice of one interval out of each of the sets I_i, corresponding to which there would be a choice of one element out of each of the sets S_i, contrary to the preceding theorem.

COROLLARY. *No subset of I which can be well-ordered covers T.*

DEFINITION. An internally closed sequence r of ordinals of the second kind less than υ_1 all distinct and arranged in their natural order is a *reduction sequence* if there exists an assignment to every ordinal κ of the second kind less than υ_1 of a sequence v_κ of distinct ordinals arranged in their natural order, such that the upper limit of v_κ is κ and the ordinal number v_κ is either ω or one of the ordinals of r.

We shall use the same definition of a derived sequence as that given in the preceding section.

DEFINITION. A function f defined for all ordinals less than a certain ordinal Ξ, the value of f being always an ordinal, is a *continuous increasing function,** if, for every pair of ordinals ξ_1 and ξ_2, both less than Ξ, such that ξ_1 is less than ξ_2, it is true that $f(\xi_1)$ is less than $f(\xi_2)$, and the set t of all ordinals $f(\xi)$, where ξ takes on all values less than Ξ, forms an internally closed sequence when the ordinals are arranged in their natural order. The αth *derived function** of f is the continuous increasing function of ξ, $f(\xi, \alpha)$, determined by the one-to-one correspondence between the αth derived sequence t^α of t and the whole or a segment of the sequence of all ordinals less than Ξ.

THEOREM C_6. *The ordinal υ_1 is an ϵ-number and occupies the υ_1th place in the sequence of ϵ-numbers arranged in their natural order.*

Since ω^ξ is a continuous increasing function of ξ, we know that ω^{υ_1} is greater than or equal to υ_1.

Suppose that ω^{υ_1} is greater than υ_1. Then, since the sequence of ordinals ω^ξ in their natural order is internally closed, there is a greatest ordinal α such that ω^α is not greater than υ_1. And α is less than υ_1. Assign to every ordinal β of the second kind less than or equal to α a fundamental sequence $\beta_0, \beta_1, \beta_2, \cdots$. Since the ordinals ω^ξ are the self-residual ordinals, every ordinal κ of the second kind less than υ_1 can be written in a unique way in the form $\gamma+\omega^\beta$, where γ has its least possible value, which may be 0, and β is not greater than α. If β is of the first kind it has an immediate predecessor

* O. Veblen, loc. cit.

ζ, and a fundamental sequence for κ is $\gamma+\omega^\zeta$, $\gamma+2\omega^\zeta$, $\gamma+3\omega^\zeta$, $\cdots$ (if ζ is equal to 0, ω^ζ is to be taken equal to 1). If β is of the second kind, the sequence $\gamma+\omega^{\beta_0}$, $\gamma+\omega^{\beta_1}$, $\gamma+\omega^{\beta_2}$, $\cdots$ is a fundamental sequence for κ. In this way we are able to assign a fundamental sequence to every ordinal κ of the second kind less than υ_1, contrary to the definition of υ_1.

Therefore ω^{υ_1} is equal to υ_1. Therefore υ_1 is an ϵ-number.

In the sequence of ϵ-numbers in their natural order let the αth place be the place occupied by υ_1. Then α is less than or equal to υ_1.

Suppose that α is less than υ_1. To every ordinal θ of the second kind less than α assign a fundamental sequence θ_0, θ_1, θ_2, $\cdots$. Every ordinal κ of the second kind less than υ_1 can be written in a unique way in the form $\gamma+\omega^{\kappa'}$, where γ has its least possible value, which may be 0, and κ' is not greater than κ.

If κ' is an ϵ-number, so that $\kappa'=\omega^{\kappa'}=\epsilon_\theta$, then θ is less than α. If θ is of the second kind, a fundamental sequence for κ is $\gamma+\epsilon_{\theta 0}$, $\gamma+\epsilon_{\theta 1}$, $\gamma+\epsilon_{\theta 2}$, $\cdots$. If θ is of the first kind it has an immediate predecessor δ, and a fundamental sequence for κ is*

$$\gamma+\epsilon_\delta+1,\ \gamma+\omega^{\epsilon_\delta+1},\ \gamma+\omega^{\omega^{\epsilon_\delta+1}},\ \cdots.$$

If κ' is of the first kind it has an immediate predecessor ζ, and a fundamental sequence for κ is $\gamma+\omega^\zeta$, $\gamma+2\omega^\zeta$, $\gamma+3\omega^\zeta$, $\cdots$.

If κ' is of the second kind but is not an ϵ-number, then the problem of finding a fundamental sequence for κ reduces to that of finding a fundamental sequence for the ordinal κ', less than κ, and this reduction continues in the same way until an ordinal $\kappa^{(k)}$ is obtained which either is of the first kind or is an ϵ-number.

In this way we are able to assign a fundamental sequence to every ordinal κ of the second kind less than υ_1, contrary to the definition of υ_1.

Therefore α is equal to υ_1.

COROLLARY 1. *The ordinal υ_1 is a self-residual ordinal.*

COROLLARY 2. *The sequence of self-residual ordinals less than υ_1 arranged in their natural order, and its first derived sequence, the sequence of ϵ-numbers less than υ_1 arranged in their natural order, are reduction sequences.*

THEOREM C_7. *If the first derived sequence of a reduction sequence r is a reduction sequence, then the ordinal number of r is υ_1, and the first υ_1 derived sequences of r exist and are reduction sequences of ordinal number υ_1.*

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*G. Cantor, loc. cit., zweiter Artikel, p. 243.

If ν is an ordinal less than υ_1 we can prove by the same argument as that used in proving Theorem B_6 that r^ν is a reduction sequence, and this argument applies even if we suppose r^ν empty. And then, as soon as we have proved that r^ν is a reduction sequence, it follows from Postulate C that r^ν is not empty. Therefore the first υ_1 derived sequences of r exist and are reduction sequences.

It remains to prove that r and its first υ_1 derived sequences are all of ordinal number υ_1.

The ordinal number α of r^ν cannot be greater than υ_1 because r^ν consists entirely of ordinals less than υ_1. Suppose that α is less than υ_1. Then $r^{\nu+1}$ contains no ordinal greater than or equal to α. Therefore we can assign a fundamental sequence $\rho_0, \rho_1, \rho_2, \cdots$ to every ordinal ρ in $r^{\nu+1}$. But to every ordinal κ of the second kind less than υ_1 we can assign an increasing sequence κ_0, κ_1, $\kappa_2, \cdots \kappa_\omega, \cdots$ of ordinals whose upper limit is κ and whose ordinal number is either ω or an ordinal ρ of $r^{\nu+1}$. If the ordinal number of this increasing sequence is not ω, a fundamental sequence for κ is $\kappa_{\rho_0}, \kappa_{\rho_1}, \kappa_{\rho_2}, \cdots$. Therefore we can assign a fundamental sequence to every ordinal κ of the second kind less than υ_1, contrary to the definition of υ_1.

Therefore the ordinal number α of r^ν is υ_1. And in the same way we can prove that the ordinal number of r is υ_1.

COROLLARY 1. *Let* $f(\xi)=\omega^\xi$. *Then, if* α *is less than* υ_1, $f(\upsilon_1, \alpha)=\upsilon_1$.

COROLLARY 2. *If* $f(\xi)=\omega^\xi$, *then* $f(0, \upsilon_1)=\upsilon_1$.

COROLLARY 3. *The sequence* $\bar{r}$ *of those ordinals which occur in the first place in the successive derived sequences of* r *is a reduction sequence of ordinal number* υ_1, *and the first* υ_1 *derived sequences of* $\bar{r}$ *are reduction sequences of ordinal number* υ_1.

The proof of this is the same as that of the corollaries to Theorem B_6.

COROLLARY 4. *Let* $f(\xi)=\omega^\xi$ *and* $\phi(\xi)=f(0, \xi)$. *Then if* α *is less than* υ_1, $\phi(\upsilon_1, \alpha)=\upsilon_1$, *and* $\phi(0, \upsilon_1)=\upsilon_1$.

THEOREM C_8. *There exists a continuous increasing function defined for the set of ordinals less than* υ_1, *the value of which is always an ordinal less than* υ_1, *and the first derived function of which does not exist.*

Since υ_1 is of the second kind, there exists a fundamental sequence α_0, $\alpha_1, \alpha_2, \cdots$ for υ_1. Let $f(\xi)=\omega^\xi$, and let $\beta_i=f(0, \alpha_i+1)$ for $i=0, 1, 2, \cdots$. Then the upper limit of the sequence $\beta_0, \beta_1, \beta_2, \cdots$ is υ_1. And β_i is the least value of ξ such that $f(\xi, \alpha_i)=\xi$.

Let a function F be defined as follows. If $0 \leqq \xi \leqq \beta_0$, $F(\xi)=f(\xi, \alpha_1)$, and if $\beta_i < \xi \leqq \beta_{i+1}$, $F(\xi)=f(\xi, \alpha_{i+2})$. Then F is a continuous increasing function defined for the set of ordinals less than υ_1, and its value is always an ordinal less than υ_1, but there is no ordinal ξ less than υ_1 such that $F(\xi)=\xi$, and therefore the first derived function of F does not exist.

11. **Properties of υ-numbers.** Definition. The υ-numbers are those ordinals κ of the second ordinal class, greater than ω, which have the property that the cardinal number corresponding to κ is greater than that corresponding to any ordinal less than κ.

If Postulate C is denied, it follows that υ-numbers do not exist (Theorems A_1 and B_1).

If Postulate C is accepted, we are assured of the existence of at least one υ-number, namely υ_1. Postponing the question how many υ-numbers the second ordinal class contains, we are able to say that, in any case, the υ-numbers arranged in order of magnitude form a well-ordered sequence, so that the αth ordinal of this sequence, counting from υ_1 as the first ordinal of the sequence, may be indicated by the symbol υ_α.

The following theorems are consequences of Postulates 1–5 alone, but since they become vacuous if Postulate A or Postulate B is accepted, we think of them as belonging with Postulate C.

Theorem C_9. *The υ-numbers are ordinals of the second kind.*

For let $\alpha+1$ be any ordinal of the first kind in the second ordinal class. Then the set of ordinals of the second ordinal class less than $\alpha+1$ can be arranged in a sequence of ordinal number α by placing α first and letting the remaining ordinals follow in their natural order, thus, $\alpha, 0, 1, 2, 3, \cdots, \omega, \omega+1, \cdots$. But if $\alpha+1$ were a υ-number, the set of ordinals of the second ordinal class less than $\alpha+1$ could not be arranged in a sequence of ordinal number less than $\alpha+1$. Therefore $\alpha+1$ is not a υ-number.

Theorem C_{10}. *Given any increasing sequence s of υ-numbers, $\upsilon_{\alpha_0}, \upsilon_{\alpha_1}, \upsilon_{\alpha_2}, \cdots$, of ordinal number ω, the upper limit υ_α of s is a υ-number.*

The cardinal number corresponding to υ_α cannot be equal to that corresponding to any ordinal υ_{α_i} of s, because, if it were, it would be less than that corresponding to $\upsilon_{\alpha_{i+1}}$. Therefore the cardinal number corresponding to υ_α is greater than that corresponding to any ordinal of s. Therefore the cardinal number corresponding to υ_α is greater than that corresponding to any less ordinal. Therefore υ_α is a υ-number.

Theorem C_{11}. *The υ-numbers are ϵ-numbers.*

We have already shown that υ_1 is an ϵ-number. We shall show by transfinite induction that the remaining υ-numbers (if any other υ-numbers exist) are also ϵ-numbers.

If υ_α is a υ-number which is also an ϵ-number, then the next following υ-number $\upsilon_{\alpha+1}$ (if it exist) is also an ϵ-number. For suppose $\upsilon_{\alpha+1}$ is not an ϵ-number. Then it can be written in Cantor's normal form*:

$$a_0\omega^{\nu_0} + a_1\omega^{\nu_1} + \cdots + a_n\omega^{\nu_n}$$

where $\upsilon_{\alpha+1} > \nu_0 > \nu_1 > \cdots > \nu_n$, the coefficients a_i are finite ordinals, and the sum contains a finite number of terms in all. Since υ_α is an ϵ-number, $\omega^{\upsilon_\alpha} = \upsilon_\alpha$. Therefore ν_0 is greater than υ_α. Therefore ν_0+1 is greater than υ_α. And, since ν_0 is less than $\upsilon_{\alpha+1}$, ν_0+1 is also less than $\upsilon_{\alpha+1}$, by Theorem C_9. Consequently, since $\upsilon_{\alpha+1}$ is the next υ-number after υ_α, the set of ordinals less than ν_0+1 can be put into one-to-one correspondence with the set of ordinals less than υ_α. Let such a one-to-one correspondence be set up, and if κ is any ordinal of the set of ordinals less than ν_0+1, let κ' be the corresponding ordinal of the set of ordinals less than υ_α. Now every ordinal less than than $\upsilon_{\alpha+1}$ can be written in Cantor's normal form, $\sum_i b_i\omega^{\mu_i}$, where $\nu_0+1 > \mu_0 > \mu_1 > \cdots$, the coefficients b_i are finite ordinals, and the sum contains a finite number of terms in all. The understanding is that one of the exponents μ_i may have the value 0, and that ω^0 is to be taken equal to 1. To the ordinal $\sum_i b_i\omega^{\mu_i}$, less than $\upsilon_{\alpha+1}$, let correspond the ordinal $\sum_i b_i\omega^{\mu_i'}$, where the terms of the sum are to be arranged in order of magnitude, the greatest first. Then $\sum_i b_i\omega^{\mu_i'}$ is less than υ_α, because $\upsilon_\alpha = \omega^{\upsilon_\alpha}$, and all the exponents μ_i' are less than υ_α. We have, accordingly, set up in this way a one-to-one correspondence between the set of all ordinals less than $\upsilon_{\alpha+1}$ and a certain set of ordinals less than υ_α. But this is impossible, because the cardinal number corresponding to $\upsilon_{\alpha+1}$ is greater than that corresponding to υ_α. Therefore the supposition that $\upsilon_{\alpha+1}$ was not an ϵ-number was incorrect.

If every ordinal of the increasing sequence of υ-numbers $\upsilon_{\beta_0}, \upsilon_{\beta_1}, \upsilon_{\beta_2}, \cdots$ is an ϵ-number, the upper limit υ_β of the sequence is an ϵ-number, because the ϵ-numbers form in order of magnitude an internally closed sequence.

Therefore, by transfinite induction, every υ-number is an ϵ-number.

COROLLARY. *If α is any ordinal of the second ordinal class, the cardinal number corresponding to α^ω is the same as that corresponding to α. And, therefore, if β is any ordinal greater than α and less than α^ω, the cardinal number corresponding to β is the same as that corresponding to α.*

*G. Cantor, loc. cit., zweiter Artikel, p. 237.

For the least ϵ-number ϵ_β greater than α is the upper limit of the sequence* $\alpha+1, \omega^{\alpha+1}, \omega^{\omega^{\alpha+1}}, \cdots$ and is, therefore, greater than $\omega^{\omega^{\alpha+1}}$, or $(\omega^{\omega^\alpha})^\omega$. And ω^{ω^α} is greater than or equal to α. Therefore ϵ_β is greater than α^ω. But the least υ-number greater than α is greater than or equal to ϵ_β and therefore greater than α^ω. Therefore the cardinal number corresponding to α^ω is the same as that corresponding to α.

12. **Postulates F and G.** In connection with Postulate C there appear two possibilities, which we shall state as Postulates F and G, inconsistent with each other, but each apparently consistent with Postulates 1–5 and C. These possibilities are the following:

F. *If ψ is any ordinal of the second ordinal class, there is some ordinal α of the second ordinal class, such that there exists no assignment to every ordinal κ of the second kind less than α of an increasing sequence v_κ of ordinals such that the upper limit of v_κ is κ and the ordinal number of v_κ is less than ψ.*

G. *There is an ordinal ψ of the second ordinal class such that, given any ordinal α of the second ordinal class, there exists an assignment to every ordinal κ of the second kind less than α of an increasing sequence v_κ of ordinals such that the upper limit of v_κ is κ and the ordinal number of v_κ is less than ψ.*

Postulate F is stated in such a way that it implies Postulate C, but Postulate G does not.

We shall examine briefly the consequences of each of the postulates just stated when taken in conjunction with Postulates 1–5 and C, taking the same experimental attitude as that which we took in the case of Postulates A, B, and C.

13. **Consequences of Postulate F.** THEOREM F_1. *If υ_η is any υ-number, there exists a υ-number greater than υ_η.*

For suppose the contrary. Then there exists a greatest υ-number, υ_β. Let ψ be the ordinal $\upsilon_\beta+1$ and let α be any ordinal of the second ordinal class. Then the set of all ordinals less than α can be arranged in a sequence t_α of ordinal number less than ψ. Then there exists an assignment to every ordinal κ of the second kind less than α of an increasing sequence v_κ of ordinals such that the upper limit of v_κ is κ and the ordinal number of v_κ is less than ψ. For we could choose v_κ to be the sequence obtained by omitting from t_α, first all ordinals not less than κ, and than all ordinals which do not have the property of being greater than every ordinal less than κ which precedes them in t_α.

This, however, is contrary to Postulate F. Therefore if υ_η is any υ-number, there exists a υ-number greater than υ_η.

*G. Cantor, loc. cit., zweiter Artikel, p. 243.

THEOREM F_2. *The sequence of the v-numbers of the second ordinal class arranged in order of magnitude is an internally closed sequence of ordinal number* Ω.

This follows at once from the preceding theorem and Theorem C_{10}.

COROLLARY. *The cardinal number corresponding to* Ω *is* $\aleph_\Omega$.

14. **Consequences of Postulate G.** Turning now to the consequences of Postulates C and G taken together, we recall that, in accordance with Postulate G, there exist ordinals ψ in the second ordinal class such that, given any ordinal α of the second ordinal class, there exists an assignment to every ordinal κ of the second kind less than α of an increasing sequence v_κ of ordinals such that the upper limit of v_κ is κ and the ordinal number of v_κ is less than ψ. Let Υ be the least such ordinal ψ. Then

THEOREM CG_1. *The ordinal* Υ *is an ordinal of the second kind.*

For suppose that Υ is an ordinal of the first kind. Then there exists an ordinal β such that Υ is equal to $\beta+1$.

In accordance with the definition of Υ, given any ordinal α of the second ordinal class, there exists an assignment to every ordinal κ of the second kind less than α of an increasing sequence v_κ of ordinals such that the upper limit of v_κ is κ and the ordinal number of v_κ is less than Υ and therefore less than or equal to β.

If β is an ordinal of the first kind the ordinal number v_κ cannot be equal to β, because only those sequences in which there is no greatest ordinal have an upper limit. Therefore in this case the ordinal number of v_κ is always less than β, contrary to the definition of Υ.

If β is an ordinal of the second kind it has a fundamental sequence β_0, $\beta_1, \beta_2, \cdots$. Those sequences v_κ which are of ordinal number β can then be replaced by sequences v_κ' of ordinal number ω obtained by omitting from v_κ all ordinals except those in the positions $\beta_0, \beta_1, \beta_2, \cdots$. And in this way we obtain again a contradiction of the definition of Υ.

Therefore Υ is an ordinal of the second kind.

THEOREM CG_2. *The ordinal* Υ *is a v-number.*

Suppose that to some ordinal β less than Υ corresponds the same cardinal number as to Υ. Then the set of ordinals less than Υ can be rearranged in a sequence of ordinal number β. Choosing a particular such rearrangement t of the set of ordinals less than Υ, we have a uniform method of rearranging any given well-ordered sequence s of ordinal number greater than β but not greater than Υ in a well-ordered sequence of ordinal number less than or equal to β, because there is a one-to-one correspondence between s and the whole or a segment of the sequence u of ordinals less than Υ arranged in order of magnitude, so that the rearrangment t of u determines the desired rearrangement of s.

Now in accordance with the definition of Υ, given any ordinal α of the second ordinal class, there exists an assignment to every ordinal κ of the second kind less than α of an increasing sequence v_κ of ordinals such that the upper limit of v_κ is κ and the ordinal number of v_κ is less than Υ. In accordance with the preceding paragraph we can rearrange v_κ as a well-ordered sequence of ordinal number less than or equal to β, and by omitting from the rearranged sequence all ordinals which do not have the property of being greater than every ordinal which precedes them in the sequence we obtain an increasing sequence w_κ of ordinals such that the upper linit of w_κ is κ and the ordinal number of w_κ is less than or equal to β and therefore less than $\beta+1$.

Since, by hypothesis, β is less than Υ so that Υ cannot be less than $\beta+1$, it follows, in view of the definition of Υ, that Υ is equal to $\beta+1$. This, however, is contrary to Theorem CG_1.

Therefore there is no ordinal β less than Υ such that the same cardinal number corresponds to β as to Υ.

But it follows at once from Postulate C that Υ is greater than ω. Therefore Υ is a υ-number.

THEOREM CG_3. *The ordinal Υ is the greatest υ-number.*

Let α be an ordinal of the second ordinal class, greater than Υ. Assign to every ordinal κ of the second kind which is less than or equal to α an increasing sequence v_κ of ordinals such that the upper limit of v_κ is κ and the ordinal number of v_κ is less than Υ.

The set of ordinals which precede Υ form, when arranged in their natural order, a sequence of ordinal number Υ. With this as a starting point assign to the ordinals which follow Υ, one by one in order, an arrangement of all preceding ordinals in a sequence of ordinal number Υ, in the following way.

When we have assigned an arrangement in a sequence t of ordinal number Υ of all ordinals which are less than an ordinal γ, an arrangement in a sequence of ordinal number Υ of all ordinals which are less than $\gamma+1$ is obtained by placing γ before t.

When we have assigned to every ordinal ζ which is less than an ordinal β of the second kind an arrangement in a sequence t_ζ of ordinal number Υ of all ordinals which are less than ζ, the sequences t_{β_0}, t_{β_1}, $t_{\beta_2}, \cdots, t_{\beta_\omega}, \cdots$, where $\beta_0, \beta_1, \beta_2, \cdots, \beta_\omega, \cdots$ is the sequence v_β, may be written one after the other so as to obtain a sequence u_β of ordinal number not greater than

Υ^2. In accordance with the corollary of Theorem C_{11} the set of ordinals less than Υ^2 can be rearranged in a sequence of ordinal number Υ. Choosing a particular such rearrangement s of the set of ordinals less than Υ^2, we have a uniform method of rearranging any sequence u_β in a well-ordered sequence w_β of ordinal number Υ, because there is a one-to-one correspondence between u_β and the whole or a segment of the sequence v of ordinals less than Υ^2 arranged in their natural order, so that the rearrangement s of v determines the desired rearrangement w_β of u_β (the ordinal number of w_β cannot be less than Υ on account of the fact that Υ is a υ-number). By omitting from w_β all occurrences of any ordinal after the first occurrence, so that a sequence without repetition results, we obtain an arrangement in a sequence of ordinal number Υ of all ordinals less than β.

We may prove by induction that this process continues until we obtain an arrangement in a sequence of ordinal number Υ of all ordinals less than α. Therefore the cardinal number corresponding to α is the same as that corresponding to Υ. But α was any ordinal of the second ordinal class greater than Υ. Therefore Υ is the greatest υ-number.

COROLLARY. *The sequence of the υ-numbers of the second ordinal class arranged in order of magnitude is an internally closed sequence whose ordinal number is an ordinal of the first kind less than Ω.*

It should be noted that there is nothing in the preceding to preclude the possibility that υ_1 and Υ are the same ordinal, in which case υ_1 would be the only υ-number.

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