

# ON THE BOUND OF THE LEAST NON-RESIDUE OF $n$ TH POWERS\*

BY

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1. In my paper *On the distribution of residues and non-residues of powers* (Journal of the Physico-Mathematical Society of Perm, 1919) I demonstrated that the least quadratic non-residue of a prime  $p$  is less than

$$p^{1/2}e^{1/2}(\log p)^2$$

for all sufficiently great values of  $p$ .

Using the same method one can establish a more general theorem:

**THEOREM I.** *If  $p$  is a prime and  $n$  a divisor of the number  $p-1$  distinct from 1, the least non-residue of  $n$ th powers modulo  $p$  is less than*

$$p^{1/2k}(\log p)^2; \quad k = e^{(n-1)/n}$$

for all sufficiently great values of  $p$ .

This bound may be considerably lowered, by means of very simple changes in our method. For example one can demonstrate the following theorems:

**THEOREM II.** *If  $p$  is a prime and  $n$  a divisor of the number  $p-1$  greater than 20, the least non-residue of  $n$ th powers modulo  $p$  is less than  $p^{1/6}$  for all sufficiently great values of  $p$ .*

**THEOREM III.** *If  $p$  is a prime and  $n$  a divisor of the number  $p-1$  greater than 204, the least non-residue of  $n$ th powers modulo  $p$  is less than  $p^{1/8}$  for all sufficiently great values of  $p$ .*

We prove finally the general theorem:

**THEOREM IV.** *If  $p$  is a prime and  $n$  a divisor of the number  $p-1$  greater than  $m^m$ , where  $m$  is an integer  $\geq 8$ , the least non-residue of  $n$ th powers modulo  $p$  is less than  $p^{1/m}$  for all sufficiently great values of  $p$ .*

2. First we shall demonstrate Theorem I. We use the notations

$$P = p^{1/2}(\log p)^2; \quad T = p^{1/2k}(\log p)^2; \quad k = e^{(n-1)/n},$$

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and assume that there are no non-residues of  $n$ th powers modulo  $p$  less than  $T$ . Then only numbers divisible by integers greater than  $T$  and less than  $P$  can be non-residues of  $n$ th powers less than  $P$ . But evidently, of such numbers, there are not more than

$$\sum_{\substack{q < P \\ q > T}} \left[ \frac{P}{q} \right],$$

where  $q$  runs only over primes. Using the known law of distribution of primes, we may bring this expression to the form

$$\begin{aligned} P \log \frac{\log P}{\log T} + O\left(\frac{P}{\log p}\right) &= P \left( \frac{n-1}{n} + \log \frac{1 + \frac{4 \log \log p}{\log p}}{1 + \frac{4k \log \log p}{\log p}} \right) + O\left(\frac{P}{\log p}\right) \\ &= \left( \frac{n-1}{n} + \frac{(4-4k) \log \log p}{\log p} \right) + O\left(\frac{P}{\log p}\right). \end{aligned}$$

On the other hand, according to my previous work, the number of residues of  $n$ th powers modulo  $p$  in the range

$$1, 2, \dots, [P]$$

may be given as follows:

$$\frac{[P]}{n} + \Delta; \quad |\Delta| < p^{1/2} \log p.$$

Thus the number of non-residues in the same range may be expressed by the formula

$$P \left( \frac{n-1}{n} \right) + \rho; \quad |\rho| < p^{1/2} \log p + 1.$$

Hence

$$P \left( \frac{n-1}{n} \right) + \rho \leq P \left( \frac{n-1}{n} + \frac{(4-4k) \log \log p}{\log p} \right) + O\left(\frac{P}{\log p}\right)$$

which brings us to the inequality

$$(4k-4) \log \log p \leq O(1),$$

which is impossible for sufficiently great  $p$ . This proves Theorem I.

3. To prove Theorem II, let

$$P = p^{1/2}(\log p)^2 ; T = p^{1/6},$$

and assume that there are no non-residues of  $n$ th powers modulo  $p$  less than  $T$ . Then only numbers divisible by primes greater than  $T$  and less than  $P$  can be non-residues less than  $P$ . The number of such numbers is evidently equal to

$$(1) \quad \sum_{q>T}^{q<P} \left[ \frac{P}{p} \right] - \sum_{q>T}^{q<P^{1/2}} \sum_{q_1>q}^{q_1<P/q} \left[ \frac{P}{qq_1} \right] + \sum_{q>T}^{q<P^{1/3}} \sum_{q_1>q}^{q_1<(P/q)^{1/2}} \sum_{q_2>q_1}^{q_2<P/q_1} \left[ \frac{P}{qq_1q_2} \right],$$

where  $q, q_1, q_2$  run over primes.

But, according to the law of the distribution of primes, the first sum may be written

$$P \log \frac{\log P}{\log T} + O\left(\frac{P}{\log p}\right) = P \log 3 + O\left(\frac{P \log \log p}{\log p}\right),$$

which for sufficiently great  $p$  is less than

$$P \cdot 1.0987.$$

The second double sum may be put into the form

$$P \sum_{q>T}^{q<P^{1/2}} \frac{1}{q} \log \frac{\log(P/q)}{\log p} + O\left(\frac{P}{\log p}\right) = P \sum_{q>T^{1/6}}^{q<P^{1/4}} \frac{1}{q} \log \frac{\log p^{1/2}}{\log q} + O\left(\frac{P \log \log p}{\log p}\right).$$

But applying the law of distribution of primes we have

$$P \int_{p^{1/6}}^{p^{1/4}} \log \frac{\log(p^{1/2}/z)}{\log z} \cdot \frac{dz}{z \log z} + O\left(\frac{P \log \log p}{\log p}\right) = P \int_{1/3}^{1/2} \log \frac{1-u}{u} \cdot \frac{du}{u} + O\left(\frac{P \log \log p}{\log p}\right),$$

which, for  $p$  sufficiently great, is greater than

$$P \cdot 0.147.$$

The last triple sum evidently is a quantity of the order

$$P \frac{\log \log p}{\log p},$$

so that the expression (1) for sufficiently great  $p$  is less than

$$P(1.0988 - 0.147) = P \cdot 0.9518.$$

On the other hand, the number of non-residues of  $n$ th powers modulo  $p$  in the series

$$1, 2, \dots, [P],$$

as seen in § 2, is equal to

$$P \left( 1 - \frac{1}{n} \right) + O \left( \frac{P}{\log p} \right).$$

So, for  $p$  sufficiently great, we have the inequality

$$P \left( 1 - \frac{1}{n} \right) < P \cdot 0.952.$$

The impossibility of this inequality for  $n > 20$  proves Theorem II.

4. To prove Theorem III we let

$$P = p^{1/2}(\log p)^2; T = p^{1/8},$$

and assume that there are no non-residues of  $n$ th powers, modulo  $p$ , less than  $T$ . It is easy to show that the number of such numbers is less than

$$(2) \quad \sum_{q>T}^{q<P} \left[ \frac{P}{q} \right] - \sum_{q>T}^{q<P^{1/2}} \sum_{q_1>q}^{q_1<P/q} \left[ \frac{P}{qq_1} \right] + \sum_{q>T}^{q<P^{1/8}} \sum_{q_1>q}^{q_1<(P/q)^{1/2}} \sum_{q_2>q_1}^{q_2<P/q_1} \left[ \frac{P}{qq_1q_2} \right],$$

where  $q, q_1, q_2$  run over primes only.

Applying the known laws of distribution of primes, we can put this expression into the form

$$\sum_{q>p^{1/8}}^{q<p^{1/2}} \frac{P}{q} - \sum_{q>p^{1/8}}^{q<p^{1/2}} \sum_{q_1>q}^{q_1<p^{1/2}/q} \frac{P}{qq_1} + \sum_{q>p^{1/8}}^{q<p^{1/8}} \sum_{q_1>q}^{q_1<p^{1/4}/q^{1/2}} \sum_{q_2>q_1}^{q_2<p^{1/2}/q_1} \frac{P}{qq_1q_2} + O \left( \frac{P \log \log p}{\log p} \right).$$

The first sum may be put into the form

$$P \log 4 + O \left( \frac{P}{\log p} \right)$$

which for sufficiently great  $p$  is less than

$$P \cdot 1.3863.$$

Then as in the proof of Theorem II the second double sum may be given in the form

$$P \int_{1/4}^{1/2} \log \frac{1-u}{u} \frac{du}{u} + O\left(\frac{P}{\log p}\right),$$

which for sufficiently great  $p$  is less than

$$P \cdot 0.40609.$$

It remains to estimate the third triple sum. We have

$$\sum_{\substack{q_1 < p^{1/2}/q_2 \\ q_2 > q_1}} \frac{P}{qq_1q_2} = \frac{P}{qq_1} \log \frac{\frac{1}{2} \log p - \log q - \log q_1}{\log q_1} + O\left(\frac{P}{qq_1 \log p}\right).$$

Noting this, it is easy to obtain

$$\sum_{\substack{q_1 < p^{1/4}q^{1/2} \\ q_1 > q}} \sum_{\substack{q_2 < p^{1/2}/q_1 \\ q_2 > q_1}} \frac{P}{qq_1q_2} = \frac{P}{q} \int_q^{p^{1/4}/q^{1/2}} \frac{dy}{y \log y} \cdot \log \frac{\frac{1}{2} \log p - \log q - \log y}{\log y} + O\left(\frac{P}{q \log^2 p}\right) = \frac{P}{q} \int_v^{1/4-v/2} \frac{dz}{z} \log \frac{\frac{1}{2} - v - z}{z} + O\left(\frac{P}{q \log p}\right); v = \frac{\log q}{\log p}.$$

The third triple sum may be given in the form

$$\begin{aligned} & P \int_{1/8}^{1/6} \frac{dv}{v} \int_v^{1/4-v/2} \frac{dz}{z} \left( \log\left(\frac{1}{2} - v\right) - \log z - \frac{z}{\frac{1}{2} - v} - \frac{z^2}{2\left(\frac{1}{2} - v\right)^2} \right. \\ & \quad \left. - \frac{z^3}{3\left(\frac{1}{2} - v\right)^3} - \dots \right) + O\left(\frac{P}{\log p}\right) \\ & = P \int_{1/8}^{1/6} \log \frac{\frac{1}{2}\left(\frac{1}{2} - v\right)}{v} \log \left(\frac{2\left(\frac{1}{2} - v\right)}{v}\right)^{1/2} \frac{dv}{v} \\ & \quad - P \int_{1/8}^{1/6} \left(\frac{1}{2} + \frac{1}{4 \cdot 4} + \frac{1}{8 \cdot 9} + \frac{1}{16 \cdot 16} + \dots\right) \frac{dv}{v} \\ & \quad + P \int_{1/8}^{1/6} \left(\frac{v}{\frac{1}{2} - v} + \frac{1}{4} \left(\frac{v}{\frac{1}{2} - v}\right)^2 + \frac{1}{9} \left(\frac{v}{\frac{1}{2} - v}\right)^3 + \dots\right) \frac{dv}{v}. \end{aligned}$$

Introducing in the first integral the substitution

$$\frac{\frac{1}{2} - v}{v} = u,$$

and in the third the substitution

$$\frac{v}{\frac{1}{2} - v} = u,$$

we easily obtain

$$P \int_2^3 \log \frac{u}{2} \log 2u^{1/2} \frac{du}{1+u} - P \left( \frac{1}{2} + \frac{1}{4 \cdot 4} + \frac{1}{8 \cdot 9} + \dots \right) \log \frac{4}{3} \\ + P \int_{1/3}^{1/2} \left( 1 + \frac{1}{4}u + \frac{1}{9}u^2 + \dots \right) \frac{du}{1+u} + O \left( \frac{P}{\log p} \right).$$

But this expression for sufficiently great  $p$  is less than

$$P \cdot 0.01489.$$

Comparing this result with those obtained for simple and double sums we find that the expression (2) for sufficiently great  $p$  is less than

$$P(1.38631 - 0.40609 + 0.01489) < P \left( 1 - \frac{1}{205} \right),$$

whence, reasoning as in Theorem II, we prove Theorem III.

5. Passing to the demonstration of Theorem IV let us prove first the following lemma:

**LEMMA.** *If  $k$  be a positive number increasing indefinitely, and  $s$  an integer  $\geq 2$ , then the number  $T$  of numbers less than  $t_s$  and not divisible by primes greater than  $k$ , where  $t_s$  is any number satisfying the condition*

$$k^s < t_s \leq k^{s+1/(s+2)},$$

*is greater than*

$$\frac{t_s}{s!(s+2)^s}$$

*for all sufficiently great values of  $k$ .*

**Demonstration.** Let

$$\epsilon = \frac{1}{s+2}.$$

(i) Taking any number  $t_1$  such that

$$k < t_1 < k^{2-2\epsilon},$$

we find a lower bound of the number  $T_1$  of numbers which are  $\leq t_1$  and divisible at least by one prime greater than  $k^{1-\epsilon}$  and  $\leq k$ . Evidently

$$T_1 = \sum_{\substack{q \leq k \\ q > k^{1-\epsilon}}} \left[ \frac{t_1}{q} \right],$$

where  $q$  runs over primes only. Considering certain laws of distribution of primes, this number may be written in the form

$$t_1 \log \frac{\log t_1}{(1-\epsilon) \log k} + O\left(\frac{t_1}{\log k}\right).$$

But this last expression is greater than

$$t_1 \log \frac{1}{1-\epsilon} + O\left(\frac{t_1}{\log k}\right)$$

which for sufficiently great  $k$  is greater than  $\epsilon t_1$ .

So for sufficiently great  $k$  we have

$$T_1 > \epsilon t_1.$$

(ii) Taking any number  $t_2$ ,

$$k^2 < t_2 \leq k^{3-3\epsilon},$$

we find a lower bound of the number  $T_2$  of numbers which are  $\leq t_2$  and divisible by the product of any two primes, greater than  $k^{1-\epsilon}$  and  $\leq k$ . Products differing in the order of divisors, we shall consider as different.

Let  $q$  be a prime greater than  $k^{1-\epsilon}$  and  $\leq k$ . The numbers not surpassing  $t_2$  and divisible by  $q$  are

$$q, 2q, \dots, \left[ \frac{t_2}{q} \right] q.$$

Consequently, we must find how many numbers of the series

$$1, 2, \dots, \left[ \frac{t_2}{q} \right]$$

are still divisible by primes greater than  $k^{1-\epsilon}$  and  $\leq k$ . Since

$$k = k^{2-1} < \frac{t_2}{q} < k^{3-3\epsilon-(1-\epsilon)} = k^{2-2\epsilon},$$

then, according to (i), we find that this number for sufficiently great  $k$  is greater than

$$\epsilon \frac{t_2}{q}.$$

Hence, as in (i), we find that

$$T_2 > \epsilon^2 t_2$$

for all sufficiently great values of  $k$ .

(iii) Arguing thus, we finally find that, if  $t_s$  is any number satisfying the condition

$$k^s < t_s \leq k^{s+1-(s+1)\epsilon},$$

and  $T_s$  denotes the number of numbers  $\leq t_s$  and divisible by the product of  $s$  primes greater than  $k^{1-\epsilon}$  and  $\leq k$  (considering as different the products with different order of divisors), then for sufficiently great  $k$

$$T_s > \epsilon^s t_s = \frac{t_s}{(s+2)^s}.$$

Noting that

$$T > \frac{T_s}{s!},$$

we prove the lemma.

**Demonstration of Theorem IV.** We have seen that, if  $n$  is a divisor of  $p-1$  differing from 1, the number  $R$  of residues of  $n$ th powers modulo  $p$  less than  $p^{1/2}(\log p)^2$  can be written in the form

$$(3) \quad R = \frac{p^{1/2}(\log p)^2}{n} + O(p^{1/2} \log p).$$

Taking any integer  $m \geq 8$ , and letting  $k = p^{1/m}$ ;  $s = m/2$  for  $m$  even;  $s = (m+1)/2$  for  $m$  odd, according to the lemma the number of numbers less than  $p^{1/2}(\log p)^2$ , divisible only by primes less than  $p^{1/m}$ , is for  $p$  sufficiently great, greater than

$$\frac{p^{1/2}(\log p)^2}{s!(s+2)^s}.$$

Assuming that among the numbers less than  $p^{1/m}$  there are no non-residues of  $n$ th powers modulo  $p$ , we have

$$R > \frac{p^{1/2}(\log p)^2}{s!(s+2)^s}.$$

Comparing this inequality with equation (3) we have

$(1/n) + O(1/\log p) > 1/(s!(s+2)^s)$  whence  $n < s!(s+2)^s + \delta$ , where  $\delta$  goes to 0 with increasing  $p$ . But applying the formula of Stirling, we have  $s!(s+2)^s < m^m$ , from which it follows that, for sufficiently great values of  $p$ ,  $n < m^m$ , which is impossible for  $n > m^m$ . This proves the Theorem IV.

**Remark.** Evidently the bound  $n > m^m$  is very rough. Thus, with  $m=8$ , we get here the inequality  $n > 16777216$  instead of the inequality  $n > 204$  found above.

6. We know that to find a primitive root of a prime  $p$  it is enough, having found different primitive divisors  $2, q_1, q_2, \dots, q_r$  of the number  $p-1$ , to find one further non-residue  $\nu_0, \nu_1, \dots, \nu_r$  of each of the powers  $2, q_1, \dots, q_r$ . By means of the numbers  $\nu_0, \nu_1, \dots, \nu_r$  it is quite easy to find the primitive root. Applying the established theorems it is easy to prove that

(i) If  $p$  is sufficiently great, all the numbers  $\nu_0, \nu_1, \dots, \nu_r$  are found in the range

$$(4) \quad 1, 2, \dots, [p^{1/2e^{1/n}(\log p)^2}].$$

(ii) If  $p$  is not of the form  $8N+1$ , and the numbers  $q_1, q_2, \dots, q_r$  are sufficiently large, then instead of the range (4) we can take shorter ranges, depending on the lowest bound  $Q$  of the numbers  $q$ . For example, if  $Q > 20$ , we take the range

$$(5) \quad -1, 1, 2, \dots, [p^{1/6}];$$

if  $Q > 204$ , then

$$(6) \quad -1, 1, 2, \dots, [p^{1/8}],$$

and finally if  $Q > m^m$ , when  $m$  is an integer  $\geq 8$ ,

$$(7) \quad -1, 1, 2, \dots, [p^{1/m}].$$

These results can be formulated in a different manner.

(i) If  $p$  is a sufficiently great prime, then a complete system of residues modulo  $p$  can be got by multiplying the powers of the numbers of the range (4).

(ii) If  $p$  is not of the form  $8N+1$ , and all the numbers  $q_1, q_2, \dots, q_r$  are not less than  $Q$ , then instead of the range (4) we can take the range (5) for  $Q > 20$ , the range (6) for  $Q > 204$ , and finally the range (7) for  $Q = m^m; m \geq 8$ .