CONCERNING CONTINUA IN THE PLANE*

BY

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In this paper a study will be made of plane continua. Part I deals with continua which constitute the boundary of a connected domain and is concerned in particular with (1) properties of domains which are consequences of certain conditions imposed upon their boundaries, (2) properties of the boundaries of domains which are consequences of conditions imposed upon the domains, and (3) conditions under which the boundary of a domain is accessible from that domain. Part II is concerned with the cut points and end points of continua.

I wish to acknowledge my indebtedness to Professor R. L. Moore, to whom the success of this investigation should be largely attributed. Credit is due him for the suggestion of most of the problems treated in Part I; and it is his stimulating personality, constant encouragement, and many helpful suggestions and criticisms which have attracted my interest to this field of mathematics and have made possible the solution of the problems treated in this paper.

I. Domains and their boundaries

Definitions. A domain $D$ is said to have property $S$† provided it is true that for any positive number $\epsilon$, $D$ may be expressed as the sum of a finite number of connected point sets each of diameter less than $\epsilon$. A point set $K$ will be said to be uniformly connected im kleinen with reference to every one of its bounded subsets provided it is true that if $M$ is any bounded point set whatever and $\epsilon$ is any positive number, then there exists a positive number $\delta$, such that every two points which are common to $M$ and $K$ and whose distance apart is less than $\delta$, lie together in a connected subset of $K$ of diameter less than $\epsilon$. A boundary point $P$ of a domain $D$ is accessible from all sides from $D$‡ provided it is true that if $A$ and $B$ are any two points

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* Various parts of this paper were presented to the Society under different titles, December 29, 1925, February 27, April 2, May 1, and June 12, 1926; it was received by the editors in its present form in June, 1926.


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of the boundary of $D$ and $AXB$ is an arc such that $AXB - (A + B)$ is a subset of $D$ and such that $AXB$ separates $D$ into two domains $D_1$ and $D_2$, then $P$ is accessible from every one of the domains $D_1$, $D_2$ to whose boundary it belongs. Two point sets will be said to be mutually separated if they are mutually exclusive and neither contains a limit point of the other. The point $P$ of a continuum $M$ will be called a cut point of $M$ provided the set $M - P$ is not connected, i.e., is the sum of two mutually separated sets.

**Notation.** In this paper wherever a symbol $X$ is used to denote a point set, the symbol $\bar{X}$ will be used to denote the set $X$ plus all those points which are limit points of $X$. And wherever a symbol $AB$ is used to designate a simple continuous arc, the symbol $(AB)$ will be used to denote the point set $AB - (A + B)$.

**Theorem 1.** In order that a bounded domain $D$ should have property $S$ it is necessary and sufficient that every point of the boundary of $D$ should be accessible from all sides from $D$.

**Proof.** I shall first show that the condition is necessary. Suppose $D$ is a domain having property $S$ and $P$ is a point of its boundary. Let $A$ and $B$ be two points of the boundary of $D$, and let $AXB$ be an arc from $A$ to $B$ such that $(AXB)$ is a subset of $D$, and such that $AXB$ separates $D$ into two domains $D_1$ and $D_2$. In an unpublished paper,* C. M. Cleveland has proved the following theorem. In order that a bounded domain $D$ should have property $S$ it is necessary and sufficient that (1) every maximal connected subset of the boundary of $D$ should be a continuous curve, and (2) for any given positive number $\varepsilon$, there should be not more than a finite number of these continuous curves of diameter greater than $\varepsilon$. Now since $D$ has property $S$, it follows that the boundary of $D$ satisfies conditions (1) and (2) of Cleveland's theorem. And since the boundary of $D$ satisfies these conditions, it can easily be shown by methods almost identical with those used by R. L. Moore to prove Theorem 4 of his paper *Concerning connectedness im kleinen and a related property*† that the boundary of $D_1$, and also that of $D_2$, must satisfy these conditions. Hence, it follows by Cleveland's theorem that each of the domains $D_1$ and $D_2$ must have property $S$. Now let $R$ denote either one of the domains $D_1$, $D_2$ which has the point $P$ in its boundary. It is sufficient, then, to show that $P$ is accessible from $R$.

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† Loc. cit.
Let $R$ be expressed as the sum of $n_1$ connected point sets $K_{11}, K_{12}, \ldots, K_{1n_1}$, all of diameter less than $1/3$. Let $G_1$ denote this collection of point sets. Let $S_1$ denote the collection of all those elements of $G_1$ which have $P$ for a limit point, and let $T_1$ denote the sum of all the point sets of the collection $S_1$. There exists a circle $C_1$ having $P$ as center and neither containing nor enclosing any point of $R - T_1$. Let $X_1$ denote a point common to $T_1$ and to the interior of $C_1$. Let $I_1$ denote the sum of all those elements of $S_1$ which can be joined to that element of $S_1$ which contains $X_1$ by a connected subset of $R$ lying wholly within $C_1$. Every point of $I_1$ which is a limit point of $R - I_1$ lies within a circle $c$ such that $c$ plus its interior belongs to $R$ and is of diameter less than $1/9$. Add to $I_1$ the interiors of all such circles $(c)$, and let $R_1$ denote the domain thus obtained. Clearly $R_1$ is of diameter less than 1, and $P$ is a boundary point of $R_1$. Now let $R$ be expressed as the sum of $n_2$ connected point sets $K_{21}, K_{22}, K_{23}, \ldots, K_{2n_2}$, all of diameter less than $1/6$ and also less than the radius of $C_1$. Let $G_2$ denote this collection of point sets. And let $T_2$ and $C_2$ be point sets which, with respect to $G_2$, correspond to $T_1$ and $C_1$ selected previously with respect to $G_1$. Let $X_2$ be a point common to $T_2$, to the interior of $C_2$, and to $I_1$. Let $I_2$ denote the sum of all those elements of $S_2$ which can be joined to the element of $S_2$ which contains $X_2$ by a connected subset of $R$ lying wholly within $C_2$. Clearly $I_2$ is a subset of $I_1$, and hence also of $R_1$. Every point of $I_2$ which is a limit point of $R - I_2$ lies within some circle $c$ such that $c$ plus its interior belongs to $R$ and to $R_1$ and is of diameter less than $1/18$. Add to $I_2$ the interiors of all such circles $(c)$, and let $R_2$ denote the domain thus obtained. Clearly $R_2$ is a subset of $R_1$, is of diameter less than $\frac{1}{4}$, and has the point $P$ in its boundary. This process may be continued indefinitely, and thus we obtain a sequence of subdomains of $R$: $R_1, R_2, R_3, \ldots, R_n$, such that for every positive integer $n$, $R_{n+1}$ has $P$ in its boundary and is a subset of $R_n$, and such that the diameter of $R_n$ approaches zero as a limit as $n$ increases indefinitely.

Now let $Q$ denote any point of $R$. For each positive integer $n$, let $P_n$ denote a point of $R_n$. There exists an arc $QP_1$ lying in $R$, and for each $n$, there exists an arc $P_nP_{n+1}$ lying in $R_n$. It is easy to see* that the point set $P + QP_1 + P_1P_2 + P_2P_3 + \cdots$ is closed and that it contains as a subset an arc $QP$ such that $QP - P$ is a subset of $R$. Hence, $P$ is accessible from $R$, and since $R$ is either one of the domains $D_1, D_2$ which has $P$ in its boundary, it follows that $P$ is accessible from all sides from $D$.

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The condition is also sufficient. Suppose $D$ is a bounded domain such that its boundary, $M$, is accessible from all sides from $D$. Condition (I) will be said to exist if some maximal connected subset of the boundary of $D$ fails to be a continuous curve; Condition (II) will be said to exist if it is true that for some positive number $\varepsilon$, there exist infinitely many maximal connected subsets of $M$ of diameter greater than $\varepsilon$. Suppose Condition (II) exists, and let $G$ denote the set of all those maximal connected subsets of $M$ which are of diameter greater than $\varepsilon$. Since the sum of all the continua of $G$ is bounded and $G$ contains infinitely many elements, it follows* that there exists a continuum $T$ of diameter at least as great as $\varepsilon$ which is the sequential limiting set of some sequence $T_1, T_2, T_3, \ldots$ of elements of $G$. There exist two points $E$ and $F$ of $T$ whose distance apart is $\geq \varepsilon$. Let $C_1$ be a circle with $E$ as center and of radius $\frac{1}{4}\varepsilon$. Let $C_2$ be a circle with $E$ as center and of radius $\frac{3}{4}\varepsilon$. Then since $E$ is within $C_2$ and $F$ is without $C_1$, there exists a positive integer $d$ such that for every $n > d$, $T_n$ contains a point $x_n$ within $C_2$ and a point $y_n$ without $C_1$. It follows from a theorem due to Janiszewski† that for every $n > d$, $T_n$ contains a subcontinuum $t_n$ which contains at least one point on each of the circles $C_1$ and $C_2$ and is a subset of the point set $H$ consisting of the circles $C_1$ and $C_2$ together with all those points of the plane which lie between $C_1$ and $C_2$. For every positive integer $i$, let $M_i$ denote $t_{d+i}$. The continuum $T$ contains a subcontinuum $M_\infty$ which has at least one point on each of the circles $C_1$ and $C_2$, is a subset of $H$, and is the sequential limiting set of the sequence $M_1, M_2, M_3, \ldots$. Now suppose Condition (I) exists. It follows directly from a theorem of R. L. Moore’s‡ that there exist circles $C_1$ and $C_2$, and that $M$ contains a countable infinity of continua $M_1, M_2, M_3, \ldots$, having exactly the same properties as the point sets of the same notation whose existence was shown as a consequence of Condition (II). Hence, we see that the existence of either Condition (I) or Condition (II) leads to exactly the situation described above.

Let $A$ (Fig. 1) denote a point common to $M_\infty$ and $C_2$, and $B$ a point common to $M_\infty$ and $C_1$. Since $M$ is accessible from $D$, it follows that there exists an arc $AB$ such that $(AB)$ is a subset of $D$. It can be shown that there exists a bounded complementary domain $R$ of the point set $AB + M_\infty$ such

that every point of the arc $AB$ belongs to the boundary of $R$. It can be shown that the arc $AB$ separates $D$ into two domains $D_1$ and $D_2$ such that $D_1$ lies wholly within $R$, and $D_2$ lies wholly in $K$, the unbounded complementary domain of the boundary of $R$. Since no member of the sequence of continua $M_1, M_2, M_3, \ldots$ has a point in common with $AB + M_\infty$, it follows that for every positive integer $i$, $M_i$ lies either wholly in $R$ or wholly in $K$. Hence, either $R$ or $K$ must contain infinitely many members of the sequence $M_1, M_2, M_3, \ldots$.

Suppose $R$ contains infinitely many members of this sequence. Then every point of $M_\infty$ is a limit point of a set of points common to $D$ and $R$. And since all such points belong to $D_1$, it follows that every point of $M_\infty$ is a boundary point of $D_1$ and, by hypothesis, is therefore accessible from $D_1$. Let $O$ be a point of $K$ and let $P$ be a point of $M_\infty$ distinct from $A$ and from $B$. Then since the arc $AB$ does not of itself separate the plane, there exists an arc $OP$ which contains no point of the arc $AB$. On $OP$, in the order from $O$ to $P$, let $z$ denote the first point belonging to $M_\infty$. Then the point set $Oz - z$ is a subset of $K$. Now either (a) there exists a point $x$ on the arc $Oz$ such that the arc $zx$ of $Oz$ contains no point of $M$, or (b) $z$ is a limit point of a set of points common to $D$ and $K$. In case (b), since all points common to $K$ and $D$
belong to $D_2$, then $z$ is a boundary point of $D_2$ and is, therefore, accessible from $D_2$. Hence, if $x$ is a point of $D_2$, there exists an arc $xz$ such that $xz - z$ is a subset of $D_2$. Hence, in either case, (a) or (b), there exists an arc $xz$ such that $xz - z$ is a subset of $K$ and contains no point whatever of $M$. It was shown above that $z$ is accessible from $D_1$. Hence, if $y$ is a point of $D_1$, there exists an arc $yz$ such that $yz - z$ is a subset of $D_1$. The two arcs $xz$ and $yz$ can have in common only the point $z$.

Let $I$ denote the point set consisting of $M_\infty$ plus all of its bounded complementary domains. Let $L$ denote the closed point set $I + M_1 + M_2 + M_3 + \cdots$. It can easily be shown that neither of the points $x$ and $y$ belongs to $I$. Now $I$ does not separate the plane, and it is a maximal connected subset of the closed set $L$. By a theorem of R. L. Moore's* it follows that there exists a simple closed curve $J$ such that $J$ encircles $I$ and contains no point of $L$, and every point on or within $J$ is at a distance from some point of $I$ less than the minimum distance from $x$ to $I$ and also less than the minimum distance from $y$ to $I$. Hence both $x$ and $y$ are without $J$. On the arc $xz$, in the order from $z$ to $x$, and on $zy$, in the order from $z$ to $y$, let $X$ and $Y$ respectively denote the first points belonging to $J$. Denote the two arcs of $J$ from $X$ to $Y$ by $XTY$ and $XSY$ respectively, and let $R_1$ and $R_2$ denote the interiors of the closed curves $XzYTX$ and $XzYSX$ respectively. On the arc $XzY$ there exist points $E, U, H, \text{ and } G$ in the order $X, E, U, z, H, G, Y$ such that within some circle which has $z$ as center and which neither contains nor encloses any point of the arc $AB$ there exist arcs $EFG$ and $UCH$ such that $(EFG)$ and $(UCH)$ are subsets of $R_1$ and $R_2$ respectively. Since $E$ and $U$ lie in $K$, and $H$ and $G$ lie in $R$, it follows that both $(EFG)$ and $(UCH)$ must contain a point in common with $M_\infty$. But $(EFG)$ belongs to $R_1$, and $(UCH)$ belongs to $R_2$. Hence $R_1$ contains a point $u$ of $M_\infty$, and $R_2$ contains a point $v$ of $M_\infty$. Let $C_u$ and $C_v$ be circles having $u$ and $v$ respectively as centers and such that $C_u$ belongs to $R_1$ and $C_v$ belongs to $R_2$. Now since $J$ encloses $M_\infty$ and contains no point of $L$, there exists a positive number $d_1$ such that for every integer $n > d_1$, $M_n$ lies wholly within $J$. There exists a positive number $d_2$ such that for every integer $n > d_2$, $M_n$ has a point within $C_u$ and also a point within $C_v$. Let $i$ be an integer which is greater than each of these two numbers $d_1$ and $d_2$. Then $M_i$ lies wholly within $J$ and contains at least one point in each of the domains $R_1$ and $R_2$. Hence, since it is connected, $M_i$ must contain a point $p$ of the arc $XzY$. Since $M_i$ has no point in common with $M_\infty$, therefore $p \neq z$. Hence $p$ must belong either to $(zX)$ or to $(zY)$. But $p$

belongs to \( M \), and neither \((aX)\) nor \((aY)\) contains any point whatever of \( M \). Thus in case \( R \) contains infinitely many members of the sequence \( M_1, M_2, M_3, \ldots \), the supposition that either Condition (I) or Condition (II) exists leads to a contradiction. In an entirely similar manner the same supposition leads to a contradiction in case \( K \) contains infinitely many members of this sequence.

Since neither Condition (I) nor Condition (II) can exist, then (1) every maximal connected subset of the boundary of \( D \) is a continuous curve, and (2) for every positive number \( \epsilon \), there are not more than a finite number of these maximal connected subsets of diameter greater than \( \epsilon \). Since \( D \) is bounded, it follows from Cleveland’s theorem quoted above that \( D \) has property \( S \).

**Theorem 2.** If the domain \( D \) is uniformly connected im kleinen with reference to every one of its bounded subsets, then every point of the boundary of \( D \) is accessible from \( D \).

**Proof.** Let \( P \) denote a point of the boundary of \( D \). Let \( C_1 \) be a circle having \( P \) as center and of diameter less than 1. For every point \( x \) common to \( D \) and the interior of \( C_1 \), let \( K_x \) denote the greatest connected point set which contains \( x \) and is common to \( D \) and the interior of \( C_1 \). Let \( G_1 \) denote the collection of all such sets \( (K_x) \). Since \( D \) is uniformly connected im kleinen with respect to every one of its bounded subsets, it follows that if \( C \) is any circle concentric with and within \( C_1 \), then there are not more than a finite number of elements of \( G_1 \) which have points on or within \( C \) and have a limit point on \( C_1 \). Hence there exists a circle \( k_1 \), concentric with and within \( C_1 \), such that \( k_1 \) neither contains nor encloses any point of any element of \( G_1 \) which does not have \( P \) for a limit point. Let \( S_1 \) denote the finite collection of all those elements of \( G_1 \) which have points on or within \( k_1 \), and let \( T_1 \) denote the sum of all the point sets of this collection. Let \( X_1 \) be a point common to \( T_1 \) and the interior of \( k_1 \). Let \( D_1 \) denote the element of \( S_1 \) which contains \( X_1 \). Clearly \( D_1 \) is a domain which (1) is a subset of \( D \) and of the interior of \( C_1 \), (2) has \( P \) in its boundary, and (3) contains every point common to \( D \) and the interior of \( C_1 \) which can be joined to \( X_1 \) by an arc which is also a subset of \( D \) and of the interior of \( C_1 \). Now let \( C_2 \) denote a circle which is concentric with \( C_1 \) and of diameter less than \( \frac{1}{2} \) and also less than the radius of \( k_1 \). Let \( G_2, k_2, S_2, T_2, X_2, \) and \( D_2 \) be collections and sets which with respect to \( C_2 \) correspond to \( G_1, k_1, S_1, T_1, X_1, \) and \( D_1 \) selected above with respect to \( C_1 \), with the additional condition that \( X_2 \) shall belong to \( D_1 \). Then \( D_2 \) is a domain which (1) is a subset of \( D \), of \( D_1 \), and of the interior of \( C_2 \), (2) has \( P \) in its boundary, and (3) contains every point common to \( D \) and the interior
of $C_2$ which can be joined to $X_2$ by an arc which is also a subset of $D$ and of the interior of $C_2$. This process may be continued indefinitely, and thus we obtain a sequence of subdomains of $D$: $D_1, D_2, D_3, \cdots$, such that for every positive integer $n$, $D_{n+1}$ has $P$ in its boundary and is a subset of $D_n$, and such that the diameter of $D_n$ approaches zero as a limit as $n$ increases indefinitely. By an argument which is identical with the third paragraph of the proof of Theorem 1, it follows that if $A$ is any point of $D$, then there exists an arc $AP$ such that $AP - P$ is a subset of $D$. Hence, every point of the boundary of $D$ is accessible from $D$.

**Theorem 3.** In order that a domain $D$ should be uniformly connected im kleinen with reference to every one of its bounded subsets it is necessary and sufficient that (1) every maximal connected subset of the boundary of $D$ should be either a point, a simple closed curve, or an open curve, and (2) if $\varepsilon$ is any positive number and $J$ is any simple closed curve, there should be not more than a finite number of maximal connected subsets of the boundary of $D$ which have points within $J$ and are of diameter greater than $\varepsilon$.

Proof. I shall show that the conditions are necessary. This may be done with the use of methods only slightly different from those used by R. L. Moore in his paper *A characterization of Jordan regions by properties having no reference to their boundaries* to prove the proposition that every bounded, simply connected, and uniformly connected im kleinen domain is bounded by a simple closed curve. I shall simply indicate the modifications necessary in his argument to establish Theorem 3.

Suppose the domain $D$ is uniformly connected im kleinen with reference to every one of its bounded subsets. Then by an argument almost identical with that used by Moore to show that the boundary of his domain in the above mentioned proposition is a continuous curve, it follows that every maximal connected subset of the boundary of $D$ is a continuous curve, and that if $J$ is any simple closed curve and $\varepsilon$ is any positive number, then there are not more than a finite number of these maximal connected subsets of the boundary of $D$ which have points within $J$ and are of diameter greater than $\varepsilon$. Now let $M$ denote any definite maximal connected subset of the boundary of $D$ which consists of more than one point. I shall show that $M$ must be either a simple closed curve or an open curve. Let the points $P_1, P_2, P_3, \cdots$, the arcs $A_1B_1, A_2B_2, A_3B_3, \cdots$, and the point set $N^*$ be selected and defined with respect to $M$ exactly as was done by Moore in the paragraph beginning

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near the bottom of page 366 of his paper. I shall now show that $M$ is neither a simple continuous arc nor a ray of an open curve. Suppose the contrary is true. Then if $M$ is an arc, let $A$ and $B$ denote its end points, and if $M$ is a ray, let $A$ denote its end point. Let $X$ be a point of $M$ which is distinct from $A$ and from $B$, and let $C$ be a circle with $X$ as center and not enclosing or containing either $A$ or $B$. Within $C$ and on $M$ there exist points $E$, $U$, $W$, and $G$ in the order $A$, $E$, $U$, $X$, $W$, $G$, and within $C$ there exist arcs $EFG$ and $UVW$ having only their end points in common with $M$ and such that if $R_1$ and $R_2$ denote the interiors of the closed curves $EFGWXUE$ and $UVWXU$ respectively, then $R_1$ and $R_2$ are mutually exclusive domains each of which lies wholly within $C$. Since under this supposition, $M$ can contain no simple closed curve, it readily follows that $X$ must be a limit point of a set of points $K_x$ common to $D$ and $R_1$ and also of a set $K_t$ common to $D$ and $R_2$. But clearly this is impossible, since $D$ is uniformly connected im kleinen with reference to every one of its bounded subsets. It follows, then, that $M$ is neither an arc nor a ray of an open curve.

Now suppose $M$ is bounded. In this case, since $M$ cannot be an arc, it follows by exactly the same argument given by Moore in the first paragraph of page 369 of his paper that $M$ is a simple closed curve. Suppose $M$ is unbounded. Since $M$ cannot be a ray of an open curve, it readily follows that both of the sequences of points $A_1$, $A_2$, $A_3$, $\ldots$ and $B_1$, $B_2$, $B_3$, $\ldots$ must be infinite and that neither of these sequences can have a limit point. It follows that $N^*$ is a closed point set which is identical with $M$ and which, evidently, must be an open curve. Hence the conditions are necessary.

C. M. Cleveland† has proved that the conditions are sufficient.

**Theorem 4.** If $K$ denotes the set of all the cut points of the boundary $M$ of a complementary domain $D$ of a continuous curve, then $D + K$ is uniformly connected im kleinen.

Proof. By a theorem due to Miss Torhorst,‡ $M$ is a continuous curve. Suppose $D + K$ is not uniformly connected im kleinen. Then, for some positive number $\epsilon$, $D$ contains two infinite sequences of points $X_1$, $X_2$, $X_3$, $\ldots$ and $Y_1$, $Y_2$, $Y_3$, $\ldots$, such that (1) for every positive integer $n$, the distance from $X_n$ to $Y_n$ is less than $1/n$, (2) for no integer $n$ is it true that $X_n$ and $Y_n$ lie together in some connected subset of $D + K$ of diameter less than $\epsilon$, and (3) there exists in $M$ a point $P$ which is the sequential limiting

† In unpublished work.
‡ Ueber den Rand der einfach zusammenhängenden ebenen Gebiete, Mathematische Zeitschrift, vol. 9 (1921), p. 64 (73).
set of each of these two sequences of points. Let \( C \) be a circle having \( P \) as center and of diameter \( \epsilon/2 \). Let us first suppose that \( D \) is bounded. It follows by a theorem of R. L. Moore's* that \( D \) has property \( S \). Hence, \( D \) is expressible as the sum of a finite number of connected point sets \( K_1, K_2, K_3, \ldots, K_n \), all of diameter less than \( \epsilon/5 \). Let \( K_{m_1}, K_{m_2}, K_{m_3}, \ldots, K_{m_m} \) denote those sets of this sequence which have \( P \) as a limit point. Clearly \( K_{m_1} + K_{m_2} + \cdots + K_{m_m} \) is a subset of the interior of \( C \). Since \( P \) is not a limit point of \( D - (K_{m_1} + K_{m_2} + \cdots + K_{m_m}) \), there exists a positive integer \( i \) such that both \( X_i \) and \( Y_i \) belong to \( K_{m_1} + K_{m_2} + \cdots + K_{m_m} \). Let \( N_x \) and \( N_y \) denote sets of this sequence which contain \( X_i \) and \( Y_i \) respectively. Let \( R_x \) and \( R_y \) denote the maximal connected subsets of \( D \) which contain \( N_x \) and \( N_y \) respectively and lie within \( C \). Clearly, the domains \( R_x \) and \( R_y \) can have no points in common. The point \( P \) belongs to the boundary of each of these domains, and by the method used in the proof of Theorem 1, it can be shown that \( P \) is accessible from each of them. Hence, there exist arcs \( X_iP \) and \( Y_iP \) such that \( X_iP - P \) and \( Y_iP - P \) are subsets of \( R_x \) and \( R_y \) respectively. There exists an arc \( t \) from \( X_i \) to \( Y_i \) which is a subset of \( D \). The point set \( t + X_iP + Y_iP \) contains a simple closed curve \( J \) which contains \( P \) and lies, except for the point \( P \), wholly in \( D \). Let \( I \) and \( E \) denote the interior and exterior respectively of \( J \). If either \( I \) or \( E \), say \( I \), contains no point of \( M \), then since \( D \) contains points of \( I \), it follows that \( I \) is a subset of \( D \), and clearly in this case \( X_i \) and \( Y_i \) can be joined by a connected subset of \( D \) of diameter less than \( \epsilon \), contrary to supposition. And if both \( I \) and \( E \) contain points of \( M \), then clearly \( P \) is a cut point of \( M \) and belongs to \( K \). And in this case \( R_x + R_y + P \) is a connected subset of \( D + K \) which contains both \( X_i \) and \( Y_i \) and is of diameter less than \( \epsilon \), contrary to supposition. Thus, in any case, the supposition that \( D + K \) is not uniformly connected im kleinen leads to a contradiction. The case where \( D \) is unbounded may be treated in a slightly modified manner.

**Theorem 5.** In order that the simply connected bounded domain \( D \) should become uniformly connected im kleinen upon the addition of a single point \( O \) of its boundary \( B \) it is necessary and sufficient that (1) if \( K \) be any maximal connected subset of \( B - O \), then \( K + O \) is a simple closed curve, and (2) there be not more than a finite number of these curves of \( B \) of diameter greater than any preassigned positive number.

Proof. The conditions are necessary. Suppose \( D \) is a bounded domain with connected boundary \( B \), and \( O \) is a point of \( B \) such that \( D + O \) is uniformly

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* Concerning connectedness im kleinen and a related property, loc. cit., Theorem 4.
connected im kleinen. Then $B$ is a continuous curve. For suppose it is not. Then $B$ contains a point $P$ which is distinct from $O$ and at which $B$ is not connected im kleinen. Then by an argument identical with that used by R. L. Moore in his paper *A characterization of Jordan regions by properties having no reference to their boundaries*,† in the paragraph beginning at the bottom of page 365, with the additional condition that the circle $K$ used in his argument be taken of radius less than $\frac{1}{2}$ the distance between $O$ and $P$, it can be shown that this supposition leads to a contradiction. Hence, $B$ is a continuous curve.

Let $K$ denote a maximal connected subset of $B - O$. Then since $B - K$ is closed, it follows that $K$ is connected im kleinen. Now let an inversion of the plane be performed about some circle which has $O$ as center. Since $K + O$ is closed and connected, it follows that $K^*$, the image of $K$, is unbounded, closed, connected, and connected im kleinen. Since the inversion does not act upon the point $O$, and since $D + O$ is uniformly connected im kleinen, it can readily be shown that $D^*$, the image of $D$, is uniformly connected im kleinen with reference to every one of its bounded subsets. Therefore, by Theorem 3, it follows that $K^*$ is an open curve, and hence, that $K + O$ is a simple closed curve. Therefore, condition (1) is necessary. Now since every maximal connected subset $K$ of $B - O$ is a simple closed curve minus one point, every such set $K$ contains an arc of diameter greater than $\frac{1}{2}$ the diameter of $K$. By a theorem of R. L. Wilder's,‡ $B$ cannot contain, for any given positive number $\epsilon$, more than a finite number of mutually exclusive arcs all of diameter greater than $\epsilon$. In view of this result, it follows that for any positive number $\epsilon$, $B - O$ cannot contain an infinite number of maximal connected subsets each of diameter greater than $\epsilon$. Hence, condition (2) is necessary.

The conditions are also sufficient. Suppose $D$ is a bounded domain with connected boundary $B$ which satisfies conditions (1) and (2) in the statement of this theorem. Clearly, $B$ must be a continuous curve. Unless the point $O$ is a cut point of $B$, then $B$ is a simple closed curve and $D$ is its interior. In this case $D$ itself is uniformly connected im kleinen. Hence, unless this theorem is true, $O$ must be a cut point of $B$. No other point is a cut point of $B$. For let $P$ denote any other point of $B$. Let $K$ denote the maximal connected subset of $B - O$ which contains $P$, and let $J$ denote the point set $K + O$. By hypothesis, $J$ is a simple closed curve. Hence, $J - P$ is connected. But $B - K$ is connected, and since the connected sets $J - P$ and

† *Loc. cit.*

$B-K$ have the point $O$ in common, their sum $S$ is connected. But $S = B-P$. Hence, $P$ is not a cut point of $B$. It follows that $O$ is the only cut point of $B$, and therefore, by Theorem 4, $D+O$ is uniformly connected im kleinen.

**Theorem 6.** In order that a continuous curve $M$ should be the boundary of a connected domain it is necessary and sufficient that if $J$ denotes any simple closed curve of $M$, then (1) $M$ is a subset either of $J+I$ or of $J+E$, where $I$ and $E$ denote the interior and exterior respectively of $J$, and (2) if $A$ and $B$ are any two points of $J$, then $M-(A+B)$ is not connected.

Proof. The conditions are necessary. That condition (1) is necessary is evident. Now let $A$ and $B$ denote any two points of $J$, where $J$ is any simple closed curve contained in the boundary $M$ of a complementary domain $D$ of a continuous curve. Since $A$ and $B$ are accessible from $D$, it readily follows that there exists an arc $AXB$ such that $(AXB)$ is a subset of $D$. Now $M+D$ lies wholly either in $J$ plus its interior $I$, or in $J$ plus its exterior $E$, suppose in $J+I$. Then there exists an arc $AYB$ such that $(AYB)$ is a subset of $E$. Let $t$ and $t'$ denote the two arcs of $J$ from $A$ to $B$. Then the simple closed curve $AXBYA$ encloses one of these arcs minus $A+B$, say $t-(A+B)$, and neither contains nor encloses any point of $t'-(A+B)$. Since $M$ has in common with the curve $AXBYA$ only the points $A$ and $B$, it follows that $M-(A+B)$ is not connected. Hence the conditions are necessary.

The conditions are also sufficient. Let $M$ denote a continuous curve which satisfies conditions (1) and (2) of this theorem. Let $K$ denote the unbounded complementary domain of $M$, and let $N$ denote its boundary. Now $N$ contains a simple closed curve $J$, or otherwise $M^+$ is the boundary of $K$ and the theorem is true. By hypothesis, $M$ is a subset either of $J+I$ or of $J+E$, where $I$ and $E$ denote the interior and exterior respectively of $J$.

**Case I.** Suppose $M$ is a subset of $J+E$. I shall show that in this case $N=M$, i.e., that $M$ is the boundary of $K$. Suppose $M$ contains a point $P$ which does not belong to $N$. Then let $R$ denote the complementary domain of $N$ which contains $P$ and let $C$ denote its boundary. By a theorem of R. L. Moore's it follows that $C$ is a simple closed curve. Since $R$ is bounded, $C$ encloses $P$; and $P$ belongs to $E$, the exterior of $J$. Hence, $J$ contains a point $Q$ which does not belong to $C$. The curve $C$ does not enclose $Q$, for $Q$ is a boundary point of $K$, the unbounded complementary domain of $M$.

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‡ Loc. cit., Theorem 4.
Hence Q lies in the exterior of C. But C encloses P and, by hypothesis, M is a subset either of C plus its interior or of C plus its exterior. Thus the supposition that \( N \neq M \) leads to a contradiction. Hence M is the boundary of the connected domain K.

Case II. Suppose M is a subset of \( J+1 \). With the aid of hypothesis (2) it is shown that there exists a point O which does not belong to M and which is within J but is not within any other simple closed curve belonging to M. Let C be a circle having O as center and not enclosing or containing any point of M. Let an inversion of the plane be performed about C. If X is a point set, let \( X' \) denote the image of X under this inversion. Now \( M' \) is a subset of \( J'+I' \), and \( I' \) is the exterior of \( J' \). Let \( K' \) denote the unbounded complementary domain of \( M' \), and let \( N' \) denote its boundary. Then \( N' \) contains \( J' \), and by an argument identical with that used in Case I it can be shown that \( M' \) is the boundary of the connected domain \( K' \). Hence, it follows that M is the boundary of the connected domain K, where K is the point set of which \( K' \) is the image under this inversion of the plane.

**Theorem 7.** If the point P of a continuous curve M belongs to the boundary of no complementary domain of M, then for every positive number \( \epsilon \), M contains a simple closed curve which encloses P and is of diameter less than \( \epsilon \).

Proof. Let P denote a point of a continuous curve M which belongs to the boundary of no complementary domain of M, and let \( \epsilon \) denote any positive number. Let C be a circle having P as center and of diameter less than \( \epsilon/2 \), and such that the exterior of C contains at least one point of M. Let N denote the maximal connected subset of M which contains P and is contained in C plus its interior. By a theorem of H. M. Gehman,* N is a continuous curve. The curve N contains a point A which belongs to C. Any arc whatever from A to P must contain at least one point of N which is distinct from A and from P. For suppose there exists an arc \( t \) from A to P which has only the points A and P in common with N. Since M is connected im kleinen, it readily follows that P is not a limit point of \( M - N \). Hence, there exists a point \( X \) on \( t \) such that the arc \( PX \) of \( t \) has only the point P in common with M. Hence, the connected set \( PX - P \) belongs to some complementary domain of M, and P must be a boundary point of that domain. But P is not a boundary point of any complementary domain of M. It follows, then, that every arc from A to P contains a point of N which is

*Concerning the subsets of a plane continuous curve, Annals of Mathematics, (2), vol. 27 (1925), pp. 29–46, Theorem 4, Lemma B.
distinct from $A$ and from $P$. By a theorem proved by C. M. Cleveland,* it follows that $N$ contains a simple closed curve $J$ which encloses either $A$ or $P$. The curve $J$ cannot enclose $A$, because $A$ belongs to $C$, and $J$ is a subset of $C$ plus its interior. Hence $J$ must enclose $P$. Since it is contained in $C$ plus its interior, $J$ is of diameter less than $\varepsilon$.

II. CUT POINTS AND END POINTS

In this section, I shall make a study of the properties of the cut points and end points of a given plane continuum. More particularly, I shall study the connected subsets of the set of all the cut points and end points of a continuum, and I shall establish some very fundamental properties of such sets, both internal properties and properties relative to the remainder of the continuum.

Definitions. The term cut point will be used as defined in Part I. The term end point, as applied to a continuous curve, will be used in the sense as defined by R. L. Wilder,† i.e., a point $P$ of a continuous curve $M$ will be called an end point of $M$ provided it is true that if $t$ is any arc of $M$ having $P$ as one of its extremities, then $M - (t - P)$ contains no connected subset which contains $P$. As applied to continua in general, I shall define the term end point as follows. The point $P$ of a continuum $M$ will be called an end point of $M$ provided it is true that if $N$ is any subcontinuum of $M$ which contains $P$, then $P$ is not a limit point of any connected subset of $M - N$.

It is obvious that this definition will allow as many, if not more, points of a continuum to be end points as would the following extension of Wilder’s definition: the point $P$ of a continuum $M$ is said to be an end point of $M$ provided it is true that if $H$ is any subcontinuum of $M$ which contains $P$, then $P$ belongs to no connected subset of $M - (H - P)$. The term acyclic continuous curve will be used, after Gehman, to designate a continuous curve which contains no simple closed curve.

R. L. Moore has shown‡ that no subcontinuum $K$ of a given continuum $M$ can contain an uncountable set of points each of which is a cut point of $M$ but not of $K$. It follows from this theorem that no simple closed curve $K$ can contain more than a countable number of cut points of any continuum.

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* This theorem is to the effect that if $A$ and $P$ are distinct points of a continuous curve $N$ and every arc from $A$ to $P$ contains at least one point of $N$ distinct from $A$ and from $P$, then $N$ contains a simple closed curve which separates $A$ from $P$. Cf. an abstract of a paper by C. M. Cleveland, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 311.

† Concerning continuous curves, Fundamenta Mathematicae, vol. 7 (1925), p. 358.

‡ Concerning the cut points of continuous curves and of other closed and connected point sets, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.
which contains $K$. Extensive use will be made of these results in the proofs given in this section.

Theorem 8. If $H$ is any connected subset of a continuum $M$, then not more than a countable number of points of $\mathcal{H} - H$ are cut points of $M$.

Proof. Let $T$ denote the set of all those points of $\mathcal{H} - H$ which are cut points of $M$. Clearly no point of $T$ is a cut point of $\mathcal{H}$. Hence, by R. L. Moore's theorem quoted above, it follows that $T$ must be countable.

Theorem 9. If $K$ denotes the set of all the cut points and $H$ the set of all the end points of a continuum $M$, and if $T$ is any countable subset of $M$, then every bounded, closed, and connected subset of $K + H + T$ is an acyclic continuous curve.

Proof. Let $N$ denote any bounded continuum which is a subset of $K + H + T$. I shall first show that $N$ is a continuous curve. Suppose $N$ is not a continuous curve. Then by R. L. Moore and R. L. Wilder's* characterization of continua which are not continuous curves it follows that there exist two concentric circles $k_1$ and $k_2$ and that $N$ contains a countable infinity of mutually exclusive continua $N_\omega, N_2, N_3, \ldots$, such that (1) each of these continua contains at least one point on each of the circles $k_1$ and $k_2$, (2) the set $N_\omega$ is the sequential limiting set of the sequence of sets $N_1, N_2, N_3, \ldots$, and (3) there exists a connected subset $L$ of $N$ which contains all of the sets of the sequence $N_1, N_2, N_3, \ldots$, but which contains no point whatever of the set $N_\omega$. Now clearly $L - L$ contains the continuum $N_\omega$. Hence, by Theorem 8, $N_\omega$ can contain not more than a countable number of points of $K$. And since every point of $N_\omega$ is a limit point of $L$, a connected subset of $M - N_\omega$, it follows that no point whatever of $N_\omega$ can belong to $H$. Hence, since $T$ is countable and $N_\omega$ is a subset of $K + H + T$, it follows that $N_\omega$ is countable. But this is absurd. Thus the supposition that $N$ is not a continuous curve leads to a contradiction.

Now suppose $N$ contains a simple closed curve $J$. Then clearly no point of $J$ can belong to $H$. And by R. L. Moore's theorem, only a countable number of points of $J$ can belong to $K$. Hence, since $T$ is countable, $J$ must be countable. But this is impossible. It follows, then, that $N$ is an acyclic continuous curve.

Theorem 10. If \( K \) is any closed and connected subset of the set of all the cut points of a bounded continuum \( M \), and \( H \) is any connected subset of \( M - K \), then \( \overline{H} \) and \( K \) have at most one point in common. And if \( H \) is a maximal connected subset of \( M - K \), then \( \overline{H} \) and \( K \) have exactly one point in common.

Proof. Suppose, on the contrary, that for some closed and connected subset \( K \) of the set of all the cut points of a bounded continuum \( M \), \( M - K \) contains a connected set \( H \) such that \( \overline{H} \) and \( K \) have two points \( A \) and \( B \) in common. Now since, by Theorem 9, \( K \) is a continuous curve, it follows that \( K \) contains an arc \( t \) from \( A \) to \( B \). By Theorem 8, \( t \) contains only a countable number of points of \( \overline{H} \). Hence, \( t \) contains an interior point \( O \) which does not belong to \( \overline{H} \). Let \( C \) denote a circle enclosing \( O \) and not enclosing or containing any point of \( \overline{H} \). Within \( C \) there exist points \( E, G, U, \) and \( W \) on \( t \) in the order \( A, E, U, O, W, G, B \), and arcs \( EFG \) and \( UVW \) having only their end points in common with \( t \) and such that if \( D_1 \) and \( D_2 \) denote the interiors of the closed curves \( EFGWOUE \) and \( UVWOU \) respectively, then \( D_1 \) and \( D_2 \) are mutually exclusive domains each of which lies within \( C \). Let \( N \) denote the continuum \( H + t \). Let \( X \) and \( Y \) denote points of \( D_1 \) and \( D_2 \) respectively, and let \( Z \) denote a point belonging to the unbounded complementary domain of \( M \). It is readily seen that every arc from \( X \) to \( Y \) contains at least one point of \( N \), and that not both \( X \) and \( Y \) can be joined to \( Z \) by an arc which contains no point of \( N \). Let \( v \) denote one of the points \( X, Y \) which cannot be so joined to \( Z \), and let \( u \) denote the other one of the points \( X, Y \). Let \( R_v \) denote that complementary domain of \( N \) which contains \( v \). Let \( \beta \) denote the boundary of \( R_v \). Then let \( R_u \) denote that complementary domain of \( \beta \) which contains \( u \), and let \( \alpha \) denote its boundary. R. L. Moore has shown* that under these conditions \( \alpha \) contains no cut point of itself. But since \( R_v \) contains that one of the domains \( D_1 \) and \( D_2 \) which contains \( v \) and \( R_u \) contains the one which contains \( u \), it readily follows that \( \alpha \) contains the arc \( WOU \) of \( t \). But \( WOU \) belongs to \( K \), and every point of \( K \) is a cut point of \( M \). Hence \( \alpha \) contains an uncountable set of points each of which is a cut point of \( M \) but not of \( \alpha \), and since \( \alpha \) is a continuum, this conclusion is contradictory to R. L. Moore's theorem quoted above. Thus the supposition that \( \overline{H} \) and \( K \) have more than one point in common leads to a contradiction.

Now if \( H \) is any maximal connected subset of \( M - K \), it is clear that \( K \) must contain at least one limit point of \( H \). And in view of the above argument it follows that \( \overline{H} \) and \( K \) must have exactly one point in common.

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* Concerning the sum of a countable number of mutually exclusive continua in the plane, Fundamenta Mathematicae, vol. 6 (1924), p. 190.
Theorem 11. If $L$ denotes the set of all the cut points of a bounded continuum $M$, $T$ is any countable subset of $M$, $K$ is any closed and connected subset of $L + T$, and $H$ is any connected subset of $M - K$, then $K$ contains at most one limit point of $H$.

Theorem 11 may be proved by an argument only slightly different from that given in the proof of Theorem 10.

Theorem 12. In order that the point $P$ of a continuous curve $M$ should be an end point of $M$ it is necessary and sufficient that no arc of $M$ should have $P$ as one of its interior points.

Proof. The condition is sufficient. Let $P$ denote any point of $M$ which is not an end point of $M$. I will show that every such point is an interior point of some arc of $M$. From the definition of an end point it follows that $M$ contains some arc $t$ having extremities at $P$ and some other point $A$ of $M$ and such that $M - (t - P)$ contains a connected set $N$ which contains $P$. Let $X$ denote a point of $N$ which is distinct from $P$. Let $K$ denote the maximal connected subset of $M - t$ which contains $X$. I will show that $P$ is a limit point of $K$. Suppose, on the contrary, that $P$ is not a limit point of $K$. Let $T$ denote the set of points common to $N$ and $K$. Since $M$ is connected im kleinen at every one of its points and $t$ is closed, it readily follows (1) that $N - T$ contains no limit point of $T$, and (2) that $T$ contains no limit point of $N - T$. Hence, $N$ is expressible as the sum of two mutually separated point sets $T$ and $N - T$. But this is impossible, because $N$ is connected. It follows, then, that $P$ is a limit point of $K$. Now $K$ is a domain with respect to $M$, for $t$ is a closed set of points. And the boundary $U$ of $K$ with respect to $M$ is a subset of $t$. Hence $U$ contains no continuum of condensation. By a theorem of R. L. Wilder's it follows that every point of $U$ is accessible in $M$ from $K$. I have just shown that $P$ belongs to $U$. Hence, if $B$ denotes a point of $K$, there exists an arc $BP$ such that $BP - P$ is a subset of $K$. The arcs $t$ and $BP$ have in common only the point $P$. Hence, their sum, $t + BP$, is an arc $APB$ from $A$ to $B$ which contains $P$ as an interior point. I have shown, then, that every point of $M$ which is not an end point of $M$ is an interior point of some arc of $M$. It follows that every point of $M$ which is not an interior point of any arc of $M$ is an end point of $M$.

The condition is also necessary. For suppose some arc $APB$ of $M$ contains as an interior point the point $P$ which is an end point of $M$. Clearly

† Cf. R. L. Wilder, Concerning continuous curves, loc. cit., Section I.
‡ Loc. cit., Theorem 1.
§ R. L. Wilder gives a proof of this part of the theorem for the special case of an acyclic continuous curve. His method of argument, however, is not applicable to the general case here treated.
this is impossible, because the arc $PB$ of $APB$ is a connected subset of $M - (AP - P)$ which contains $P$.

I will remark that Theorem 12 shows the equivalence of Wilder’s definition of an end point of a continuous curve and the following one: the point $P$ of a continuous curve $M$ is said to be an end point of $M$ provided it is true that if $t$ is any arc of $M$ having $P$ as one of its extremities, then $P$ is not a limit point of any connected subset of $M - t$. This latter definition for the case of a continuous curve is analogous to the one I have given above for continua in general.

**Theorem 13.** If $K$ is a connected subset of the set of all the cut points of a continuous curve $M$, then in order that $K$ should be an acyclic continuous curve it is necessary and sufficient that every point of $K$ should be either a cut point or an end point of $M$.

Proof. That the condition is sufficient is a corollary to Theorem 9. I will show that it is necessary. Suppose $K$ is a connected set of cut points of a continuous curve $M$ such that $K$ is an acyclic continuous curve. Let $P$ denote a point of $K$ which is not an end point of $M$. I will show that $P$ is a cut point of $M$. Let $U$ denote a point of $K$ which is distinct from $P$. Then $K$ contains an arc $t$ from $U$ to $P$. Every point of $t$, except possibly the point $P$, is a cut point of $M$. For suppose $t$ contains an interior point $O$ which is not a cut point of $M$. Then $O$ does not belong to $K$. Since, by a theorem of R. L. Wilder’s,* every connected subset of an acyclic continuous curve is arcwise connected, it follows that $K + P$ contains an arc $t_0$ from $U$ to $P$ which does not contain $O$. Then the sum of the arcs $t_0 + t$ contains a simple closed curve, contrary to the hypothesis that $K$ is acyclic. Hence, every point of $t$, except possibly the point $P$, is a cut point of $M$. Now since $P$ is not an end point of $M$, it follows by Theorem 12 that $M$ contains an arc $APB$ having $P$ as one of its interior points. Not both of the arcs $AP$ and $PB$ of $APB$ can contain an interval in common with $t$ which contains $P$, because $P$ is an end point of $t$. Suppose $AP$ has no interval in common with $t$ which contains $P$. Then $AP$ and $t$ have in common only the point $P$. For suppose they have in common a point $V$ which is distinct from $P$. The interval $VP$ of $AP$ contains a point $Q$ which does not belong to $t$. In the order from $Q$ to $P$ and from $Q$ to $A$ respectively on $AP$, let $X$ and $Y$ denote the first points belonging to $t$. The simple closed curve formed by the arc $XY$ of $t$ plus the arc $XQY$ of $AP$ contains a segment $XY$ every point of which is a cut point

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* Concerning continuous curves, loc. cit., Theorem 20.
CONTINUA IN THE PLANE

of $M$. Clearly this is impossible. Hence, it follows that $AP$ and $t$ have in
common only the point $P$.

Now suppose, contrary to this theorem, that $P$ is not a cut point of $M$.
Then by a theorem of R. L. Moore's, $M - P$ contains an arc $b$ from $U$ to $A$.
The sum of the arcs $AP + t + b$ contains a simple closed curve $J$ which con-
tains a segment of $t$, every point of which is a cut point of $M$. This is absurd,
and thus the supposition that $P$ is not a cut point of $M$ leads to a contradic-
tion. It follows, then, that every point of $K$ is either a cut point or an end
point of $M$.

Theorem 14. If $K$ denotes the set of all the cut points of a continuous curve $M$, then for every positive number $\varepsilon$, $K$ contains not more than a finite number of mutually exclusive continua each of diameter greater than $\varepsilon$.

Proof. Suppose Theorem 14 is not true. Then there exists a positive number $\varepsilon$ such that $K$ contains infinitely many mutually exclusive continua each of diameter greater than $\varepsilon$. Since by Theorem 9, every closed and connected subset of $K$ is a continuous curve, it follows that $K$ contains infinitely many mutually exclusive arcs each of diameter greater than $\varepsilon$. Let $t_1, t_2, t_3, \ldots$ denote some sequence of these arcs which has a sequential limiting set $t$. It is evident that $t$ contains two points $A$ and $B$ whose distance apart is $\geq \varepsilon$. Now since $M$ is uniformly connected im kleinen, there exists a positive number $\delta$, such that every two points of $M$ whose distance apart is less than $\delta$, are end points of an arc of $M$ of diameter less than $\frac{1}{2}\varepsilon$. There exists a positive number $d$ such that for every integer $n > d$, $t_n$ contains a point $X_n$ and a point $Y_n$ whose distances from $A$ and $B$ respectively are less than $\frac{1}{4}\delta$. Let $i$ and $j$ denote two integers greater than $d$. Then $X_i$ and $X_j$ and also $Y_i$ and $Y_j$ can be joined by an arc of $M$ of diameter less than $\frac{1}{2}\varepsilon$. Let $X_iX_j$ and $Y_iY_j$ denote these two arcs. It is readily seen that the sum of the arcs $t_i + t_j + X_iX_j + Y_iY_j$ contains a simple closed curve $J$ which contains an interval of the arc $t_i$. But every point of $t_i$ is a cut point of $M$. Thus the supposition that Theorem 14 is false leads to a contradiction.

Theorem 15. If $K$ is any closed and connected subset of the set of all the cut points of a continuous curve $M$, then for every positive number $\varepsilon$, $M - K$ contains not more than a finite number of maximal connected subsets of diameter greater than $\varepsilon$.

Proof. Suppose Theorem 15 is not true. Then there exists a positive number $\varepsilon$ such that $M - K$ contains an infinite collection $G$ of maximal

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connected subsets each of diameter greater than \( \varepsilon \). By Theorem 10, \( K \) contains exactly one limit point of each set of the collection \( G \). For each set \( g \) of \( G \) let \( X \) denote the limit point of \( g \) which belongs to \( K \), and let \( H \) denote the set of all such points \( (X) \) thus defined. Now if \( H \) contains infinitely many distinct points, then \( K \) contains a point \( A \) which is a limit point of \( H \). And if \( H \) contains only a finite number of points, then \( H \) contains a point \( A \) which is a limit point of each of an infinite number of distinct sets of the collection \( G \). Let us first suppose that \( A \) is a limit point of \( H \). Then \( H \) contains an infinite sequence of points \( X_1, X_2, X_3, \ldots \) which has \( A \) as its sequential limit point. For every positive integer \( n \), let \( G_n \) denote an element of \( G \) which has \( X_n \) as a limit point. The sequence of sets \( G_1, G_2, G_3, \ldots \) have a sequential limiting set \( L \) which contains \( A \). And since every element of \( G \) is of diameter greater than \( \varepsilon \), it follows that \( L \) contains a point \( B \) whose distance from \( A \) is \( \geq \varepsilon/3 \). Now since \( M \) is connected im kleinen, it can easily be shown that \( B \) must belong to \( K \). Let \( C_1 \) and \( C_2 \) be circles having \( A \) and \( B \) respectively as centers and each of diameter less than \( \varepsilon/10 \). The sequence of points \( X_1, X_2, X_3, \ldots \) contains an infinite subsequence \( X_{n_1}, X_{n_2}, X_{n_3}, \ldots \) every point of which is within \( C_1 \). There exists a circle \( C_b \) having \( B \) as center and such that every point of \( M \) which is enclosed by \( C_b \) can be joined to \( B \) by an arc common to \( M \) and to the interior of \( C_2 \). There exists an integer \( i \) such that \( G_{n_i} \) contains a point \( V \) within \( C_b \). Hence, \( M \) contains an arc \( t \) from \( V \) to \( B \) which lies within \( C_2 \). On \( t \), in the order from \( V \) to \( B \), let \( E \) denote the first point belonging to \( K \). Then \( E \) is a limit point of \( G_{n_i} \). But \( X_{n_i} \) is also a limit point of \( G_{n_i} \), and \( X_{n_i} \) lies within \( C_1 \). Hence, \( K \) contains two distinct limit points of \( G_{n_i} \). But this is contrary to Theorem 10. A similar conclusion is reached when it is assumed that \( A \) is a limit point of each of an infinite number of distinct elements of \( G \). Thus the supposition that Theorem 15 is false leads to a contradiction.

**Theorem 16.** If the bounded continuum \( M \) has the property that every connected subset of \( M \) is arcwise connected, and \( K \) is any maximal connected subset of the set of all the cut points of \( M \), and \( H \) denotes the set of all those limit points of \( K \) which \( K \) does not contain, then every point of \( H \) is an end point of \( M \).

Proof. Suppose, on the contrary, that \( H \) contains a point \( P \) which is not an end point of \( M \). Now by a theorem of R. L. Wilder's,* \( M \) is a continuous curve. Hence by Theorem 12 \( M \) contains an arc \( APB \) having \( P \) as one of its interior points. Let \( U \) denote a point of \( K \). By hypothesis \( K+P \) contains an arc \( t \) from \( U \) to \( P \). Now \( P \) is not a cut point of \( M \), for otherwise it would

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*Concerning continuous curves, loc. cit., Theorem 18.
belong to $K$. In view of this fact, it follows by an argument almost identical with the latter part of the proof of Theorem 13, beginning with the fifteenth sentence, that this situation leads to an absurdity. Hence, every point of $H$ is an end point of $M$.

**Theorem 17.** Under the same hypothesis as in Theorem 16, $K+H$ is an acyclic continuous curve, and every point of $H$ is an end point both of $M$ and of the curve $(K+H)$.

**Theorem 18.** If $K$ is any closed and connected subset of the set of all the cut points of a continuum $M$, then $K$ contains at least one subcontinuum which belongs to the boundary of some single complementary domain of $M$.

Proof. The complementary domains of $M$ are countable. Let them be ordered $D_1, D_2, D_3, \ldots$, and let their respective boundaries be ordered $B_1, B_2, B_3, \ldots$. It is a consequence of a theorem of R. L. Moore's* that $K$ is a subset of the point set $B_1+B_2+B_3+\cdots$. Let $A_1, A_2, A_3, \ldots$ denote the point sets common to $B_1, B_2, B_3, \ldots$ respectively, and to $K$. Then for every positive integer $n$, $A_n$ is a closed point set. Now $K=A_1+A_2+A_3+\cdots$. It is well known that no continuum is expressible as the sum of a countable number of closed point sets each of which is totally disconnected. Hence for some positive integer $i$, $A_i$ is not totally disconnected and therefore contains a continuum $H$. The continuum $H$ belongs to $B_i$, the boundary of $D_i$.

**Theorem 19.** In order that the point $P$ of a bounded continuum $M$ should be a cut point of $M$ it is necessary and sufficient that $P$ should be a cut point of the boundary of some complementary domain of $M$.

Proof. R. L. Moore has shown† that this condition is necessary. I shall show that it is sufficient. Suppose $P$ is a cut point of the boundary $N$ of a complementary domain $D$ of a bounded continuum $M$.

Case I. Suppose $D$ is bounded. Then let $B$ denote the outer boundary‡ of $D$. R. L. Moore has shown§ that $B$ has no cut point. Hence, $B-P$, in case $P$ belongs to $B$, or $B$, in case $P$ does not belong to $B$, must be a subset

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* Concerning the common boundary of two domains, Fundamenta Mathematicae, vol. 6 (1924), pp. 203–213.
† Loc. cit.
‡ If $D$ is a bounded domain, the outer boundary of $D$ is the boundary of the unbounded complementary domain of $D$. If $D_1$ and $D_2$ are mutually exclusive domains, the outer boundary of $D_1$ with respect to $D_2$ is the boundary of that complementary domain of $D_1$ which contains $D_2$. Cf. R. L. Moore, Concerning the separation of point sets by curves, loc. cit., footnote to p. 475.
§ Concerning the sum of a countable number of mutually exclusive continua in the plane, loc. cit.
either of $S_1$ or of $S_2$, where $S_1$ and $S_2$ denote two mutually separated point sets into which, by hypothesis, $N$ is divided by the omission of the point $P$. Suppose it belongs to $S_1$. Then let $R$ denote the complementary domain of the continuum $S_1+P$ which contains $D$. Since no point of $S_2$ belongs to $S_1+P$, and since every point of $S_2$ is a limit point of $D$, it follows that $R$ contains $S_2$. Then $S_2+P$ is a continuum which lies, except for the point $P$, wholly in $R$. By a theorem of R. L. Moore's,* there exists a simple closed curve $J$ which contains $P$, encloses $S_2$, and lies, except for the point $P$, wholly in $R$. The curve $J$ does not enclose or contain any point of $B—P$. Since $J$ encloses $S_2$, it follows that $J—P$ contains a point of $D$. And since $J—P$ is connected and contains no point of $N$, then $J—P$ must be a subset of $D$. Hence, $J—P$ contains no point whatever of $M$. But $S_2$ belongs to the interior of $J$, and $B—P$ to the exterior of $J$, and $J$ contains in common with $M$ only the point $P$. It readily follows that $P$ is a cut point of $M$.

Case II. Suppose $D$ is unbounded. It is easily shown that there exists a ray $r$ of an open curve which has exactly one point $A$, distinct from $P$, in common with $N$ and lies except for the point $A$ wholly in $D$. Now by hypothesis, $N—P$ is expressible as the sum of two mutually separated point sets $S_1$ and $S_2$, one of which, say $S_1$, contains the point $A$. The set $D—(r—A)$ is connected. Let $R$ denote that complementary domain of the continuum $S_1+r+P$ which contains $D—(r—A)$. The domain $R$ is simply connected and contains $S_2$. Then by R. L. Moore's theorem quoted above there exists a simple closed curve $J$ which encloses $S_2$, contains $P$, and lies except for the point $P$ wholly in $R$. Just as in Case I it follows that $J—P$ is a subset of $D$ and therefore contains no point of $M$. But $J$ encloses $S_2$ and does not enclose or contain the point $A$. It follows that $P$ is a cut point of $M$, and the theorem is proved.

Theorem 20. In order that the point $P$ of a continuous curve $M$ should be an end point of $M$ it is sufficient (but not necessary) that $P$ should be an end point of the boundary of some complementary domain of $M$.

Proof. Let $P$ denote a point of $M$ which is an end point of $N$, the boundary of some complementary domain $D$ of $M$. Suppose, contrary to this theorem, that $P$ is not an end point of $M$. Then, by Theorem 12, $M$ contains an arc $APB$ having $P$ as one of its interior points. Now either (1) each of the segments $(AP)$ and $(PB)$ of $APB$ contains a point of $N$, or (2) one of these segments contains no point whatever of $N$. I will show that in either case $P$ must belong to some simple closed curve of $M$. Suppose case (1) is true.

* Concerning the separation of point sets by curves, loc. cit., Theorem 3.
Then let $X$ and $Y$ denote points of $N$ which belong to the segments $AP$ and $PB$ respectively of $APB$. Since $P$ is not a cut point of $N$, it follows that $N - P$ contains an arc $t$ from $X$ to $Y$. The sum of the arcs $t$ and $APB$ contains a simple closed curve which contains $P$. Now suppose case (2) is true. Let $S$ denote one of the segments $(AP)$, $(PB)$ of the arc $APB$ which contains no point of $N$. Then $S$ belongs to some complementary domain $R$ of $N$. It follows from a theorem of R. L. Moore's* that the boundary of $R$ is a simple closed curve which belongs to $M$. Clearly this curve must contain $P$. Hence, in any case, $M$ contains a simple closed curve $J$ which contains $P$. Let $I$ and $E$ denote the interior and exterior respectively of $J$. Then $D$ is a subset either of $I$ or of $E$, say of $I$. Let $K$ denote the complementary domain of $N$ which contains $E$. By R. L. Moore's theorem just cited, the boundary $C$ of $K$ is a simple closed curve which belongs to $N$. Clearly $C$ must contain $P$. But by hypothesis $P$ is an end point of $N$, and therefore, by Theorem 12, can belong to no simple closed curve of $N$. Thus the supposition that $P$ is not an end point of $M$ leads to a contradiction, and the theorem is proved.

**Theorem 21.** The set of all the end points of a continuous curve is totally disconnected.†

Proof. Let $K$ denote the set of all the end points of a continuous curve $M$. Suppose $K$ contains a connected set $H$ which consists of more than one point. Then from Theorem 12 and Theorem 7 it follows that every point of $H$ must belong to the boundary of some complementary domain of $M$. Let $D_1$ denote a complementary domain of $M$ which has the point $A$ of $H$ on its boundary. Now if $H$ is a subset of the boundary of $D_1$, then by a theorem of R. L. Wilder's,‡ $H$ is arcwise connected, and it easily follows that some point of $H$ must be an interior point of some arc of $M$, contrary to Theorem 12. Hence, there exists a complementary domain $D_2$ of $M$ which has on its boundary a point $B$ of $H$ which does not belong to the boundary of $D_1$. Let $N$ denote the boundary of $D_1$. Let $R$ denote the complementary domain of $N$ which contains $D_2$. By R. L. Moore's theorem mentioned above, the boundary of $R$ is a simple closed curve $J$. It is easily seen that $J$ separates $A$ from $B$. Hence, since $H$ is connected, $J$ must contain a point

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* Concerning continuous curves in the plane, loc. cit., Theorem 4.
† My attention has been called to the fact that K. Menger has recently proved a proposition similar to Theorem 21. However, he uses the term end point in a different sense. Cf. K. Menger, Grundzüge einer Theorie der Kurven, Mathematische Annalen, vol. 95 (1925), pp. 272-306.
‡ Loc. cit., Theorem 20.
of $H$. But this is contrary to Theorem 12. It follows that $K$ is totally disconnected.

**Theorem 22.** If $K$, $H$, and $N$ respectively denote the set of all the cut points, end points, and simple closed curves of a continuous curve $M$, then $K + H + N = M$.

**Proof.** Let $P$ denote a point of $M$, if there be any, which is neither a cut point nor an end point of $M$. I shall show that $P$ belongs to a simple closed curve of $M$ and therefore belongs to $N$. Since $P$ is not an end point of $M$, it follows by Theorem 12 that $P$ is an interior point of some arc $APB$ of $M$. And since $P$ is not a cut point of $M$, it follows by R. L. Moore's theorem mentioned above that $M - P$ contains an arc $t$ from $A$ to $B$. On the arcs $PA$ and $PB$ of $APB$, in the order from $P$ to $A$ and from $P$ to $B$ respectively, let $X$ and $Y$ respectively denote the first points which belong to $t$. The simple closed curve formed by the arc $XY$ of $t$ plus the arc $XPY$ of $APB$ contains the point $P$. Hence, $P$ belongs to $N$, and it follows that $K + H + N = M$.

**Theorem 23.** If $N$ denotes the point set consisting of the sum of all the simple closed curves contained in a continuous curve $M$, then every connected subset of $M - N$ is arcwise connected.

**Proof.** Let $L$ denote any definite connected subset of $M - N$. It follows from Theorem 22 that every point of $L$ is either a cut point or an end point of $M$. And since, by Theorem 21, the set of all the end points of $M$ is totally disconnected, $L$ must contain at least one point $P$ which is a cut point of $M$. By the part of Theorem 19 established by R. L. Moore, $P$ belongs to the boundary $B$ of some complementary domain $D$ of $M$. I will show that $L$ is a subset of $B$. Suppose, on the contrary, that $L$ contains a point $Q$ which does not belong to $B$. Then $Q$ lies in a complementary domain $R$ of $B$. By R. L. Moore's theorem, the boundary $J$ of $R$ is a simple closed curve which belongs to $B$. Since $L$ contains no point of $N$, $J$ contains neither $P$ nor $Q$. Now $R$ is either the interior or the exterior of $J$. And if $R$ is the interior [exterior] of $J$, then $Q$ belongs to the interior [exterior] of $J$, and $P$ belongs to the exterior [interior] of $J$. Hence, in any case $P$ and $Q$ are separated by $J$. Therefore, $L$ contains a point of $J$, contrary to hypothesis. Thus the supposition that $L$ contains a point which does not belong to $B$ leads to a contradiction. Hence, $L$ is a subset of $B$, and by a theorem of R. L. Wilder's* it follows that $L$ is arcwise connected.

* Loc. cit.
Theorem 24. Under the same hypothesis as in Theorem 23, if $L$ is any connected subset of $M - N$, then $\overline{L}$ is an acyclic continuous curve which belongs to the boundary of some single complementary domain of $M$, and every point of $\overline{L}$ is either a cut point or an end point of $M$.

Proof. From the proof of Theorem 23 it follows that $\overline{L}$ belongs to the boundary $B$ of some complementary domain $D$ of $M$. Now since, by R. L. Wilder's theorem, every connected subset of $B$ is arcwise connected, and since every point of $\overline{L} - L$ is a limit point of $L$ by definition, it can easily be shown by methods identical with those used in the proof of Theorem 16 that every point of $\overline{L} - L$ is either a cut point or an end point of $B$. Now, by Theorem 19, every cut point of $B$ is a cut point also of $M$; and by Theorem 20, every end point of $B$ is an end point also of $M$. Hence, since by Theorem 22, every point of $L$ is either a cut point or an end point of $M$, every point of $\overline{L}$ is either a cut point or an end point of $M$. By Theorem 9 and the above argument, it follows that $\overline{L}$ is an acyclic continuous curve which satisfies all the conditions of Theorem 24.

Theorem 25. If $M$ is the complete boundary of two mutually exclusive domains $D_1$ and $D_2$, then no point of $M$ is an end point of any continuum which contains $M$.

Proof. It is sufficient to show that $M$ contains no end point of itself. Suppose, on the contrary, that there exists a point $P$ which is an end point of $M$. Then $P$ belongs to no continuum of condensation of $M$. For let $H$ be any subcontinuum of $M$ which contains $P$. R. L. Moore has shown* that $M - H$ is connected. Hence, since, by supposition, $P$ is an end point of $M$, $P$ is not a limit point of $M - H$. Therefore $P$ belongs to no continuum of condensation of $M$. By a theorem of R. L. Wilder's† it follows that $P$ is accessible from each of the domains $D_1$ and $D_2$. Hence, if $A$ and $B$ are points of $D_1$ and $D_2$ respectively, there exist arcs $AP$ and $BP$ such that $AP - P$ and $BP - P$ are subsets of $D_1$ and $D_2$ respectively. Since for any continuum $H$ of $M$ which contains $P$, $P$ is not a limit point of $M - H$, it can easily be shown that there exists an arc $EFG$ from a point $E$ of $AP - P$ to a point $G$ of $BP - P$ which contains no point whatever of $M$. This is impossible, because $E$ belongs to $D_1$, and $G$ belongs to $D_2$, and $D_1$ and $D_2$ are mutually exclusive domains by hypothesis. Thus the supposition that $M$ contains an end point leads to a contradiction, and the theorem is proved.

Theorem 26. No end point of a continuum $M$ can be a boundary point of more than one complementary domain of $M$.

* Concerning the common boundary of two domains, loc. cit., Theorem 2.
† Loc. cit., Theorem 2.
Proof. Suppose, on the contrary, that an end point $P$ of $M$ belongs to the boundary of each of two mutually exclusive domains $D_1$ and $D_2$ which are complementary to $M$. Let $N$ denote the outer boundary of $D_2$ with respect to $D_1$. By a theorem of R. L. Moore's* $N$ is the complete boundary of each of two mutually exclusive domains $R_1$ and $R_2$ which contain $D_1$ and $D_3$ respectively. And since $P$ is a limit point both of $R_1$ and $R_2$, $P$ must belong to $N$. But $P$ is an end point of $M$, and by Theorem 25, it cannot belong to any point set which belongs to $M$ and is the complete boundary of two mutually exclusive domains. Thus the supposition that $P$ belongs to the boundary of more than one complementary domain of $M$ leads to a contradiction.

Theorem 27. The collection $G$ of all the continua $(X)$ contained in the boundary $M$ of a simply connected bounded domain $D$ such that $X$ is the complete boundary of some two mutually exclusive domains, is countable.

Proof. Let $K$ denote the unbounded complementary domain of $M$, and let $B$ denote its boundary. For every element $X$ of $G$, I shall define a domain $R_x$ as follows. (1) When $X = B$, let $R_x = K$. (2) For every element $(X)$ of $G$ such that $B$ is a subset of $X$, the unbounded complementary domain of $X$ contains $D$. For every such element $X$ of $G$ let $R_x$ denote one bounded domain which has $X$ as its boundary. (3) For every element $(X)$ of $G$ such that $B \not\subset X$ but such that $B$ is a subset of $X$, it is true that $X$ is the complete boundary of at least two bounded mutually exclusive domains, because for every such element $X$, the unbounded complementary domain of $X$ is identical with $K$, and $X$ is not the complete boundary of $K$. Not both of these bounded domains can contain points of $D$. Then for every such element $X$ of $G$ let $R_x$ denote one of the bounded domains of which $X$ is the boundary and which contains no point whatever of $D$.

Clearly, for every element $X$ of $G$ there corresponds a domain $R_x$ as above defined. It is evident that for every element $X$, $R_x$ is a complementary domain of $M$. It is well known that the collection $T$ of all such domains ($R_x$) is countable. Since every element of $G$ is the boundary of at least one domain of the collection $T$, it follows that $G$ is countable.

I will remark here that Theorem 27 is a generalization of a theorem of R. L. Wilder's to the effect that the collection of all the simple closed curves contained in the boundary of a complementary domain of a continuous curve is countable.

* Loc. cit., Theorem 1.
Theorem 28. If $K$ denotes the set of all the cut points of a bounded continuum $M$, $G$ denotes the collection of all the continua $(X)$ of $M$ such that $X$ is the complete boundary of two mutually exclusive domains, and $T$ denotes the point set obtained by adding together all the point sets of the collection $G$, then the set of points common to $K$ and $T$ is countable.

Proof. Let $H$ denote the set of points common to $K$ and $T$. Let the complementary domains of $M$ be ordered $D_1, D_2, D_3, \ldots$, and their boundaries denoted by $B_1, B_2, B_3, \ldots$, respectively. Now by the part of Theorem 19 established by R. L. Moore, $K$ is a subset of the point set $B_1 + B_2 + B_3 + \cdots$. Hence, if for every $i$, $A_i$ denotes the set of points common to $H$ and $B_i$, then $H = A_1 + A_2 + A_3 + \cdots$. I will show that for every positive integer $i$, $A_i$ is a countable set of points. Let $P$ denote a point of $A_i$. Then $P$ belongs to some element $X$ of $G$, and $X$ is the complete boundary of two domains $R_1$ and $R_2$. One of these domains, say $R_1$, contains no point whatever of $D_i$. Let $Y$ denote the outer boundary of $D_i$ with respect to $R_1$. Then $Y$ is an element of $G$ which contains $P$ and is a subset of $B_i$. Let $G_i$ denote the collection of all those elements of $G$ which are subsets of $B_i$. Then by Theorem 27, $G_i$ is countable. It was just shown that every point of $A_i$ belongs to some element of $G_i$. Since by R. L. Moore's theorem, no element of $G$ contains any cut point of itself, it follows that no element of $G_i$ contains more than a countable number of points of $K$. It follows, then, that $A_i$ is countable, and therefore $H$ is countable.

Theorem 29. If $K$ denotes the set of all the cut points and $N$ denotes the point set consisting of the sum of all the simple closed curves of a continuous curve, then the set of points common to $K$ and $N$ is countable.

Theorem 29 is a corollary to Theorem 28.

Theorem 30. Every continuum $M$ in a plane $S$ is connected im kleinen at every one of its end points which is accessible from some point of $S - M$.

Proof. Suppose $P$ is any end point of $M$ which is accessible from $S - M$. There exists an arc $t$ having $P$ as one of its extremities and such that $t - P$ is a subset of $S - M$. Suppose, contrary to this theorem, that $M$ is not connected im kleinen at $P$. Then there exists a circle $C'$ having center at $P$ and such that every circle which is concentric with $C'$ encloses a point $X$ which belongs to $M$ but which lies in no connected subset of $M$ which contains $P$ and is enclosed by $C'$. Let $C$ be a circle concentric with $C'$ and of diameter less than $\frac{1}{3}$ the diameter of $C'$ and also less than $\frac{1}{3}$ the diameter of $t$. Then $M$ contains a countable infinity of continua $M_m, M_1, M_2, M_3, \ldots$
such that (1) each of these continua has at least one point on $C$ and is contained in $C$ plus its interior, (2) no two of these continua have a point in common, and, indeed, no one of them, save possibly $M_\infty$, is a proper subset of any connected point set common to $M$ and to $C$ plus its interior, (3) no point of the set $M_1 + M_2 + M_3 + \cdots$ lies together with $P$ in any connected subset of $M$ which is enclosed by $C'$, and (4) $M_\infty$ contains the point $P$ and is the sequential limiting set of the sequence of continua $M_1, M_2, M_3, \cdots$.\* Let $I$ denote $M_\infty$ plus all the bounded complementary domains of $M_\infty$. It is clear that $I$ is a maximal connected subset of the closed point set $I + M_1 + M_2 + M_3 + \cdots$, and that $I$ neither separates the plane nor contains any point of $t - P$. Hence, by a theorem of R. L. Moore's\† there exists a simple closed curve $J$ which encloses $I$, contains no point of the point set $M_1 + M_2 + M_3 + \cdots$, is a subset of the interior of $C'$, and is such that its exterior contains at least one point $A_0$ of $t$. Let $B$ (see Fig. 2) denote a point

\* For indications of the proof of this statement see papers by R. L. Moore: Continuous sets which have no continuous sets of condensation, Bulletin of the American Mathematical Society, vol. 25 (1919), pp. 174–176, and A characterization of Jordan regions by properties having no reference to their boundaries, loc. cit.

\† Concerning the separation of point sets by curves, loc. cit.
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which is common to \( M \) and \( C \). The point set \( M \) contains* a continuum \( H \) which is irreducible between \( P \) and \( B \). Let \( H' \) denote the point set obtained by adding to \( H \) all of its bounded complementary domains. Now in the order from \( P \) to \( A_0 \) on \( t \), let \( A \) denote the first point belonging to \( J \). It is readily shown that there exists an arc \( BOE \) from \( B \) to a point \( E \) of \( J \) such that \( (BOE) \) is common to the interior of \( J \) and to the exterior of \( C \). Let \( AXE \) and \( AYE \) respectively denote the two arcs of \( J \) from \( A \) to \( E \). The continuum consisting of \( H' \) plus the arc \( PA \) of \( t \) plus the arc \( BOE \) divides the interior of \( J \) into just two domains \( D_1 \) and \( D_2 \). One of these domains, say \( D_1 \), has \( AXE \) in its boundary, and the other, \( D_2 \), has \( AYE \) in its boundary. It follows that one of these domains, say \( D_1 \), contains infinitely many of the continua \( M_1, M_2, M_3, M_4, \ldots \).

Now let us consider the maximal connected subsets of \( M - H \). It is evident that each of the continua \( M_1, M_2, M_3, \ldots \) must belong to one such subset of \( M - H \). And since \( P \) is an end point of \( M \), it follows that no maximal connected subset of \( M - H \) can contain more than a finite number of these continua. Hence, it is true that there exists an infinite sequence of distinct maximal connected subsets of \( M - H \), each of which contains at least one of the continua \( M_1, M_2, M_3, \ldots \). Let one such sequence be ordered \( K_1, K_2, K_3, \ldots \). For every positive integer \( i \), \( H \) contains at least one limit point of \( K_i \). Let \( C_1 \) be a circle having \( P \) as center which lies entirely within \( J \) and is of diameter less than \( \frac{1}{2} \) the diameter of \( C \). From a theorem of Janiszewski's† it follows that \( H \) contains a continuum \( L_1 \) which contains \( P \) and a point of \( C_1 \) and which is the maximal connected subset of \( H \) which contains \( P \) and which belongs to \( C_1 \) plus its interior. By a theorem of Miss Mullikin's‡ the continuum \( H \) contains a connected set \( Q \) which contains neither the point \( B \) nor any point of \( L_1 \) but which has \( B \) for a limit point and has at least one limit point in \( L_1 \). Now since \( H \) is irreducible between \( P \) and \( B \), it readily follows that if \( H_1 \) denotes the point set \( Q + B \), then \( H = H_1 + L_1 \). Since \( P \) is an end point of \( M \), it follows (1) that \( P \) is not a limit point of \( H_1 \), and (2) that for not more than a finite number of positive integers \( (i) \) does \( H_1 \) contain a limit point of \( K_i \). Hence, there exists a positive integer \( n_1 \) such that \( H_1 \) contains no limit point of \( K_{n_1} \). Hence since \( H \) contains at least one limit point of \( K_{n_1} \), \( L_1 \) must contain a point \( X_1 \) which is a limit point of \( K_{n_1} \). Now from condition (3), above, which the sequence of sets \( M_1, M_2, M_3, \ldots \) satisfies, it follows that for every positive integer \( i \), \( K_i \) contains at least one point in common with \( J \). Hence, by Miss Mullikin's

* Cf. Janiszewski, loc. cit.
† Loc. cit.
‡ These Transactions, vol. 24 (1922), pp. 144–162.
theorem mentioned above, \( K_n \) contains a connected set \( N_1^0 \) which contains no point of either of the continua \( L_1 \) and \( J + BOE \) but is such that each of these continua contains at least one limit point of \( N_1^0 \). Clearly, \( N_1^0 \) is a subset either of \( D_1 \) or of \( D_2 \). And since \( D_1 \) contains infinitely many of the continua \( M_1, M_2, M_3, \ldots \), of which only a finite number can contain points in common with \( N_1^0 \), it can be shown that \( N_1^0 \) cannot belong to \( D_1 \), and therefore, must belong to \( D_2 \). Let \( N_1 \) denote the point set obtained by adding to \( N_1^0 \) all of its limit points. It has already been shown that \( N_1 \) must be a subset of \( D_2 + L_1 + \text{the arc } AYEOB \). It is evident that \( N_1 \) divides \( D_2 \) into at least two domains, one of which must have the arc \( AP \) of \( t \) in its boundary. Let \( R_1 \) denote the one which has \( AP \) in its boundary. It is clear, then, that no point of \( \bar{H}_1 - L_1 \cdot \bar{H}_1 \) is a limit point of \( R_1 \) and, therefore, that the boundary of \( R_1 \) is a subset of \( N_1^0 + L_1 + \text{the arc } PAYEOB \).

Let \( C_1 \) be a circle concentric with \( C \) which encloses and contains no point of either \( N_1 \) or \( H_1 \) and which is of diameter less than \( \frac{1}{2} \) the diameter of \( C_1 \). Let \( L_2 \) be a subcontinuum of \( H \) which bears the same correspondence to \( C_2 \) as \( L_1 \) bears to \( C_1 \). Let sets \( H_2, K_n, N_2^0, \) and \( N_2 \) be selected and defined with respect to \( C_2 \) and \( L_2 \) just as the corresponding sets \( H_1, K_n, N_1^0, \) and \( N_1 \) were defined with respect to \( C_1 \) and \( L_1 \). Again, \( N_2^0 \) must be a subset of \( D_2 \). Hence, \( N_2 \) contains a point \( A_2 \) on the arc \( AYEOB \). And since \( N_1 \) and \( N_2 \) can have no point in common, it can easily be shown that on \( AYEOB \), in the order from \( A \) to \( B \), \( A_2 \) precedes every point which belongs to \( N_1 \). Hence, \( N_2^0 \) is a subset of \( R_1 \). Let \( R_2 \) denote that complementary domain of the continuum \( L_2 + N_2 + \text{the arc } PAYEOB \) which is a subset of \( R_1 \) and has the arc \( PA \) of \( t \) in its boundary. Again, \( \bar{H}_2 - L_2 \cdot \bar{H}_2 \) contains no limit point of \( R_2 \). This process may be continued indefinitely, and it follows that there exists an infinite sequence of continua \( N_1, N_2, N_3, \ldots \), having the properties as above indicated. Also there exists a sequence of domains \( R_1, R_2, R_3, \ldots \), such that for every positive integer \( n \), \( R_n \) has the arc \( PA \) of \( t \) in its boundary, contains \( R_{n+1} \), and contains \( N_n + N_{n+1} + \cdots \). And there exist two sequences of connected point sets \( L_n, L_2, L_3, \ldots \) and \( H_1, H_2, H_3, \ldots \), such that for every positive integer \( n \), \( L_n + \bar{H}_n = C \), such that if \( r \) denotes the radius of \( C \), then \( L_n \) contains \( P \) and is of diameter less than \( 3r/n \), and such that \( \bar{H}_n - L_n \cdot \bar{H}_n \) contains no limit point whatever of \( R_n \).

Let \( N \) denote the limiting set of the sequence of continua \( N_1, N_2, N_3, \ldots \). It readily follows from the above properties of this sequence that \( N \) contains \( P \) but contains no other point whatever of \( H \). The set \( N \) contains at least one point \( U \) of \( J \). Now \( N \) is a continuum. Let \( N_u \) denote the maximal connected subset of \( N - P \) which contains \( U \). Then clearly \( P \) is a limit point of \( N_u \). But \( P \) is an end point of \( M \), and is, therefore, not a limit point.
of any connected subset of $M - H$. Thus the supposition that $M$ is not connected im kleinen at $P$ leads to a contradiction, and the theorem is proved.

The following example demonstrates that the conclusion of Theorem 30 does not necessarily remain valid if the restriction that the end point of $M$ in question shall be accessible from $S - M$ is removed. Let $I$ be the straight line interval from $(0,0)$ to $(1,0)$. And for every integer $n$ such that $n = (2)^i$, where $i$ takes on all positive integral values from 1 to $\infty$, let $L_i$ denote the broken line through the points $(1/n,0), (1/n,-1/n), (-1/n,1/n), (1,1/n), (1,3/4n),$ and $(0,3/4n)$ in the order named. (See Fig. 3.)

![Fig. 3]

If $M$ denotes the continuum $I + L_1 + L_2 + L_3 + \cdots$, and $P$ denotes the point $(0,0)$, then $P$ is an end point of $M$, but $M$ is not connected im kleinen at $P$.

**Theorem 31.** If a continuum $M$ is irreducible between some pair of points $A, B$, then $M$ is connected im kleinen at every one of its end points.

**Proof.** Let $P$ denote an end point of $M$. Let us first suppose that either $P = A$ or $P = B$, say $P = B$. Then by Janiszewski’s theorem mentioned above it follows that if $C$ denotes any circle having $P$ as center, then $C$ encloses a subcontinuum $H$ of $M$ which consists of more than one point and which contains $B$ but not $A$. From Miss Mullikin’s theorem it follows immediately that $M - H$ contains a connected set $N$ which contains $A$ and which has at least one limit point in $H$. Since $M$ is irreducible between $A$ and $P$, clearly $M = H + N$. And since $P$ is an end point of $M$, $P$ is not a limit point of $M - H$. Hence, there exists a circle $K$ concentric with and within $C$ which encloses no point of $M - H$. Any point of $M$ which is interior to $K$ lies together with $P$ in a closed and connected subset of $M$ which is enclosed by $C$, namely, in $H$ itself. Hence, $M$ is connected im kleinen at $P$.

Now in case neither $P = A$ nor $P = B$, then $M$ is the sum of two continua $K_a$ and $K_b$, irreducible between $A$ and $P$, and $B$ and $P$ respectively. By the
above argument, both $K_a$ and $K_b$ are connected im kleinen at $P$. It follows that their sum, $M$, is connected im kleinen at $P$.

In his paper *Concerning the cut points of continuous curves and of other closed and connected point sets*, R. L. Moore proves the following results.

I. In order that a bounded continuum $M$ should be an acyclic continuous curve it is necessary and sufficient that every subcontinuum of $M$ should contain uncountably many points each of which is a cut point of $M$.

II. In order that the continuous curve $M$ should contain no simple closed curve it is necessary and sufficient that if $K$ denotes the set of all those points of $M$ that are not cut points of $M$, then no subset of $K$ disconnects $M$ even in the weak sense.

In Theorem 32, below, I shall establish a generalization of R. L. Moore's result (II) quoted here.

**Theorem 32.** In order that the bounded continuum $M$ should be an acyclic continuous curve it is necessary and sufficient that if $K$ denotes the set of all those points of $M$ which are not cut points of $M$, then no subset of $K$ disconnects $M$ even in the weak sense.

Proof. The condition is sufficient. For suppose a bounded continuum $M$ satisfies the condition but is not an acyclic continuous curve. Then by result (I), above, of R. L. Moore's, it follows that $M$ contains a subcontinuum $N$ which does not contain more than a countable number of cut points of $M$. Let $A$ and $B$ denote two points of $N$. By hypothesis, $M - [K - (A+B)]$ is connected in the strong sense. Hence, it contains a continuum $H$ which contains $A$ and $B$. Since every point of $H$, except possibly the points $A$ and $B$, is a cut point of $M$, it follows by Theorem 9 that $H$ is a continuous curve. Therefore, $H$ contains an arc $t$ from $A$ to $B$. Since $N$ contains not more than a countable number of cut points of $M$, there exist points $E$ and $F$ on $t$ in the order $A, E, F, B$ such that the interval $EF$ of $t$ contains no point whatever of $N$. Since $N$ contains both $A$ and $B$, it follows by Miss Mullikin's theorem that $N$ contains a connected set $Q$ containing no point of $t$ and such that each of the intervals $AE$ and $FB$ of $t$ contains at least one limit point of $Q$. But $t$ is a continuum every point of which, save possibly two, is a cut point of $M$, and $Q$ is a connected subset of $M - t$. Hence, by Theorem 11, $t$ can contain at most one limit point of $Q$. Thus the supposition that $M$ is not an acyclic continuous curve leads to a contradiction.

It follows by R. L. Moore's theorem II quoted above that the condition is necessary.

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