

MANIFOLDS WITH A BOUNDARY AND THEIR TRANSFORMATIONS*

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This is the continuation of the paper which appeared in the January, 1926, number of these Transactions.[†] Its chief object is to extend to an M_n , with a boundary the results already obtained for transformations of manifolds without boundary. To these already treated in full we devote a few pages chiefly to elucidate and simplify certain points of importance for the extension. We have succeeded in deriving coincidence and fixed points formulas for the two types of transformations that are alone amenable to anything like a general treatment and extended the formulas of this and the preceding paper to transformations between two different manifolds with or without a boundary. As an incidental acquisition there should be pointed out some highly interesting topological propositions obtained in Parts II, III. Of importance also is the fact that by means of ample use of matrices we have been able to put all coincidence formulas of this and the previous paper in very simple and manageable form.

I. GENERAL REMARKS ON MANIFOLDS

1. **A theorem on intersecting complexes.** Let C_h , C_k , M_p be two complexes and a manifold on M_n , all polyhedral and with C_k a sub-complex of M_p . We wish to prove that *the intersections $C_h \cdot C_k$ taken on M_n and $(C_h \cdot M_p) \cdot C_k$ taken on M_p coincide and when $h+k=n$ the related Kronecker indices are equal*. We assume that as regards all intersections to be considered the restrictions of Tr., No. 15, are fulfilled. The problem is then reduced at once to the case where the complexes are simplexes and the manifolds

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† Referred to in the sequel as Tr. Unless otherwise stated, the notations, terminology, assumptions, etc., of that paper will apply directly here. The only changes will be actually in the definition of an M_n , and using for cycles besides Γ , also γ as in Tr., Part II, and later other letters for special types. Since we shall make considerable use of matrices we may as well give our notations here. All our matrices will have integer terms. Any matrix will be designated as its generic element with position indices omitted. The transverse of a matrix m will be called m' ; when m is square its determinant is denoted by $|m|$ and the sum of the terms in its principal diagonal is called its *trace*.

linear spaces. The natural procedure is as in Tr., Part I, §2, to compare indicatrices. The indicatrix of Tr., No. 7, is now so chosen that $A_0A_1 \cdots A_q$, $q=p+h-n$, be on $C_h \cdot M_p$, $A_0A_1 \cdots A_qA_{h+1} \cdots A_r$, $r=p+q-n$, be on M_p and that multiplied respectively by α_q , α_p , they constitute indicatrices of their complexes. Let also $\alpha'_l A_0A_1 \cdots A_l$ be an indicatrix of $(C_h \cdot M_p) \cdot C_k$ when the intersection is taken on M_p . Then the relations

$$(1.1) \quad \alpha_h \alpha_k \alpha_l \alpha_n = \alpha_h \alpha_p \alpha_q \alpha_n = \alpha_q \alpha_k \alpha_l \alpha_p = 1$$

define the orientations of $C_l = C_h \cdot C_k$ and $C_h \cdot M_n$ taken on M_n , and that of $(C_h \cdot M_p) \cdot C_k$ taken on M_p . Since the α 's are all ± 1 , from (1.1) follows at once

$$(1.2) \quad \alpha_k \alpha'_l = \alpha_p \alpha_q = \alpha_h \alpha_n = \alpha_{k+l},$$

therefore

$$(1.3) \quad \alpha'_l = \alpha_l.$$

The cells of the complexes to be compared are the same, and by (1.3) similarly oriented. Therefore

$$(1.4) \quad C_h \cdot C_k = (C_h \cdot M_p) \cdot C_k,$$

as we wished to prove. As we know from Tr., No. 8, when $l=0$, we merely have a Kronecker index to consider and then

$$(1.5) \quad (C_h \cdot C_{n-h}) = ((C_h \cdot M_p) \cdot C_{n-h}),$$

which may also be established directly as above by comparison of indicatrices.

Of particular interest is the case when the C 's are cycles. The passage through suitable approximations, as in Tr., Part I, §4, will enable us to drop all restrictions as to them provided their intersection does not meet the boundary of M_p . For the latter being polyhedral, the approximations can always be so carried out that C_l remains on it.

2. In the applications that we have especially in view, $p=n-1$, $k=n-h$, so that we deal with a Kronecker index. Furthermore M_{n-1} will there be a subcomplex of the defining C_n of M_n and the preceding result does not apply outright. The extension is, however, easy on this basis: C_h and C_{n-h} are assumed as general as possible and hence their intersections are isolated points, of which any one, say A , is on an E_{n-1} of C_n . As far as the contribution of A to the index is concerned, only the two n -cells E_n , E'_n of C_n incident with E_{n-1} are involved. The situation is then the same as if M_n were reduced to $E_n + E'_n +$ their boundaries. But this system is obviously

homeomorphic with preservation of structure* to a similar one in S_n . Practically this means that we may assume that M_n is an S_n . On the latter, however, we can always choose a defining C_n with A on an n -cell of it. Then M_n will not be a subcomplex of it, or rather its cells through A will not be cells of C_n . Therefore we are back to the case already considered and the conclusion is the same.

3. On the Kronecker index. Until further notice we assume that M_n is *without boundary*. In an important paper† Veblen has had occasion to define so-called *intersection numbers* for associated complexes of special type on M_n .† Since we have applied some of his results it is important to show that his numbers are merely the Kronecker indices of the complexes.

We begin with Poincaré congruences

$$(3.1) \quad C_k \equiv \Gamma_{k-1}, \quad C_{n-k+1} \equiv \Gamma_{n-k},$$

where the cycles have no common points. They give rise as in Tr., No. 18, to the relation

$$(3.2) \quad (C_k \cdot \Gamma_{n-k}) = (-1)^k (\Gamma_{k-1} \cdot C_{n-k+1}),$$

valid without any other restrictions than the one just stated concerning the cycles. That follows at once by passing to polyhedral approximations for which (3.2) holds. Since the indices are defined in each case by means of the approximations the relation is valid for the initial complexes.

* Two sets of simplicial cells, $\{e\}$, $\{e'\}$, are said to have the same structure, whenever to each e there corresponds one and only one e' of same dimensionality and conversely, and when furthermore the incidence relations between any two e 's and the corresponding e' 's are the same.

† These Transactions, vol. 25 (1923), pp. 540–550. Substantially the same results, derived in similar fashion, were also obtained simultaneously but independently by Hermann Weyl, Revista de Matematica Hispano Americana, 1923. Regarding the index, I recently received a communication from Weyl in which he points out that, unknown to me, he had proved its independence from the defining C_n : (a) for $n=2$ in Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 25 (1916), p. 225, also Note to the second edition of *Die Idee der Riemannschen Flächen*; (b) for $(\Gamma_1 \cdot \Gamma_{n-1})$ and any n in the Revista paper. In the same paper he also points out that for $\mu=\frac{1}{2}n$ even, it may not be possible to have a canonical set whose matrix of indices is the identity. The bearing on the coincidence formulas in Tr. is that one must have two associated sets as when $\mu \neq n/2$, with T operating on one and T' on the other. Let these sets be γ^i , γ'^i , with $\gamma'^i \approx \sum g_i \gamma^i$. All matrices g corresponding to such a pair of associated sets are of the form $p g p'$, where p is an arbitrary square matrix of order $R_{n/2}$ with $|p| = \pm 1$. The properties of the whole class of such matrices are invariants of M_n , another form of a remark made by Weyl, loc. cit.

In a letter received in early December, Weyl communicated to me substantially the same derivation as mine of the matrix formulas (10.3) and (39.1) of this paper from those of Tr., Part II (see also the note in the Proceedings of the National Academy of Sciences, for December, 1926). He has thus confirmed my results, at an important point.

4. Let us now adopt the notations of Tr., No. 23. We designate furthermore by \bar{C}_n the dual of C_n , whose cells are subcomplexes of C'_n also. Any one \bar{E}_{n-k} is a sum of cells of C'_n of type (k, p_1, \dots, p_i) , with a common vertex A^k , and the p 's all $> k$. The $(n-k)$ -cells of the sum are of type $(k, k+1, \dots, n)$. A similar statement holds for the cell E_k of C_n that carries A^k , except that now the types are (q_1, \dots, q_i, k) , $q_i < k$ (Coll. Lect.,* p. 89).

Let A^{k-1} be any vertex on a cell E_{k-1} of the boundary of E_k . It is on a cell \bar{E}_{n-k+1} of \bar{C}_n , which as before is the sum of certain cells of C'_n with A^{k-1} for vertex. We assume E_{k-1} positively related to E_k , hence a positive cell of its boundary Γ_{k-1} , and \bar{E}_{n-k+1} so sensed that \bar{E}_{n-k} is a positive cell of its boundary Γ_{n-k} . Now the two cycles have no common points, since these would have to be vertices of C'_n , while the vertices they carry are of the incompatible types A^{k+i} , A^{k-i} , $i \neq 0$. Hence (3.2) is applicable here and

$$(4.1) \quad (E_k \cdot \bar{\Gamma}_{n-k}) = (-1)^k (\Gamma_{k-1} \cdot \bar{E}_{n-k+1}).$$

E_k and $\bar{\Gamma}_{n-k}$ meet at A^k , and nowhere else. For all cells of C'_n of which the cycle is made up are of type (k, p_1, \dots, p_r) , $p_i > k$. Therefore (Tr., No. 23), they are on cells of k dimensions of C_n , unless they are merely vertices and of the same type as A^k . In that case they are on k -cells with one and only one such vertex on each k -cell. Hence the k -cell E_k of C_n can only meet $\bar{\Gamma}_{n-k}$ at a single point which can only be A^k . Therefore

$$(4.2) \quad (E_k \cdot \bar{\Gamma}_{n-k}) = (E_k \cdot \bar{E}_{n-k}).$$

Due to the symmetrical relation of C'_n to the dual complexes, it is not necessary to repeat the discussion for the second index in (4.1) and we infer at once

$$(4.3) \quad (\Gamma_{k-1} \cdot \bar{E}_{n-k+1}) = (E_{k-1} \cdot \bar{E}_{n-k+1}).$$

Therefore in place of (2.1)

$$(4.4) \quad (E_k \cdot \bar{E}_{n-k}) = (-1)^k (E_{k-1} \cdot \bar{E}_{n-k+1}).$$

Now this is precisely the relation proved by Veblen for his intersection numbers in §4 of his paper. Therefore in proving that they are merely Kronecker indices, say for a given k , the latter may be increased or decreased by one unit. Hence we may assume $k = n$. But for this special value Veblen's definition reduces essentially to ours. *Therefore his numbers are Kronecker indices for every k .*

* *Analysis Situs*, by Oswald Veblen (The Cambridge Colloquium, Part II), will be referred to as Coll. Lect. throughout the present paper.

5. We are then justified in taking over Veblen's theorems bodily. Of particular significance are these two:

I. In order that $\gamma_\mu \approx 0$ on M_n without boundary (i.e., that it be a zero-divisor or bounding cycle) it is necessary and sufficient that $(\gamma_\mu \cdot \gamma_{n-\mu}) = 0$ whatever $\gamma_{n-\mu}$.

II. Let $\gamma_\mu^i, \gamma_{n-\mu}^j$ run respectively through the elements of two fundamental sets for their dimensions. Then the rank of

$$(5.1) \quad \|(\gamma_\mu^i \cdot \gamma_{n-\mu}^j)\|$$

is R_μ , and its invariant factors are all unity.

These propositions are proved by Veblen only when in each pair one cycle is a subcomplex of C_n , the other one of the dual \bar{C}_n . As every cycle is homologous to a subcomplex of \bar{C}_n (Tr., No. 27, Remark), and as the index is invariant with respect to homology, the two propositions are true without restrictions.

In our previous paper we actually made use only of the first part of II, that is, of the fact that (5.1) is of rank R_μ , and proved the other part directly. Of I also, the necessary condition alone was needed and established directly in Part I. The only question that could have been raised is then as to whether the R 's in the formulas are the connectivity numbers and not merely the ranks of the matrices (5.1) and we have just answered it in the negative.

6. Let us return to the fundamental set $\gamma_\mu^1, \gamma_\mu^2, \dots, \gamma_\mu^{R_\mu}$ relative to the operation \approx as considered in Tr., p. 37. It has the property that any γ_μ of M_n is a combination of the cycles of the set plus a zero-divisor. Hence a fundamental set as to \sim is obtained by merely adding zero-divisors to the γ 's. For example, the set in Coll. Lect., p. 117, is of this very nature, and so are the canonical sets of our first paper, but we need not limit ourselves to these.

Let $\gamma_{n-\mu}^1, \gamma_{n-\mu}^2, \dots, \gamma_{n-\mu}^{R_\mu}$ correspond in analogous fashion to the dimensionality $n-\mu$. As is well known the number of the cycles is again R_μ (Poincaré). Complete the two sets by zero-divisors so as to have fundamental sets for \sim , then form (5.1). By §3, the matrix will merely consist of the square array

$$(6.1) \quad L_\mu = \|(\gamma_\mu^i \cdot \gamma_{n-\mu}^j)\| \quad (i, j = 1, 2, \dots, R_\mu)$$

bordered with rows and columns of zeros. Hence the determinant $|L_\mu| = \pm 1$, for in absolute value it is the product of the invariant factors of L_μ , all equal to one.

7. Continuous transformations. First observation of very general nature: All of Tr. Part I, goes through whether the cells of C_n are simplicial or merely convex (i.e., corresponding to convex polyhedral regions on the representative polyhedron Π_n). The bearing of this becomes clear when we remember that even when the cells of C_p, C_q are simplicial, those of $C_p \times C_q$ are merely convex. Hence it is proper to apply to a product manifold all the approximation work (loc. cit.). Of course the more general type of C_n , practically the same as Veblen's, possesses a regular subdivision of the restricted type (Coll. Lect., p. 85).

This important point settled let us return to the situation of Tr., Part II, §2, particularly as regards the definition of $T\gamma_\mu = \bar{\gamma}_\mu$. It is obtained by means of an approximation $\bar{\Gamma}_n$ to Γ_n , defining cycle of the transformation T . The question arises, however, whether $T\gamma_\mu$ is unique. To show that such is the case let a second defining complex C_n^0 lead to $\bar{\Gamma}_n^0, \gamma_\mu^0, \bar{\gamma}_\mu^0$. We do not exclude the possibility that C_n^0 coincides with C_n . Then by Tr., Part I, §6, if the approximations are sufficiently close,

$$(7.1) \quad \bar{\Gamma}_n^0 \cdot \gamma_\mu^0 \times M'_n \sim \bar{\Gamma}_n \cdot \gamma_\mu \times M'_n \quad (\text{mod } M_n \times M'_n).$$

Let the difference of the two sides bound $C_{\mu+1}$. When the point $A \times B$ describes it, B describes on M'_n a singular image $C'_{\mu+1}$ of the complex, and when $A \times B$ describes the boundary of $C_{\mu+1}$, B describes $\bar{\delta}_\mu - \bar{\delta}_\mu^0$. Hence this last cycle bounds on M'_n , and $\bar{\gamma}_\mu - \bar{\gamma}_\mu^0$ therefore bounds on M_n , that is $\bar{\gamma}_\mu \sim \bar{\gamma}_\mu^0$ (mod M_n), which proves the uniqueness of $T\gamma_\mu$.

8. Throughout our first paper we have assumed that an oriented zero-cell is considered as a cycle of zero dimensions. (See in particular No. 63.) This justifies, for example, the theorem of No. 52, our assertion as to the Euler characteristic, No. 71, etc. Veblen in Coll. Lect., p. 110, does not consider such cycles. The advantage of our procedure is readily perceived. Two points similarly oriented on the same connected piece of a C_n constitute homologous cycles, for their difference is the boundary of an obvious one-cell. Hence R_0 is the number of distinct connected pieces of C_n , and if it is an M_n without boundary, $R_0 = R_n$, and Poincaré's duality formula holds without exception. His formula for the Euler characteristic also takes the simpler form, assumed in Tr., No. 71,

$$(8.1) \quad \sum (-1)^i \alpha_i = \sum (-1)^i R_i.$$

The convention of Tr., No. 58, reduces to the following for $\mu=0$: The transform of $\gamma_0 = A$, point of M_n , is $\alpha_0 A$, where α_0 is as in Tr., No. 56.

Incidentally the orientation of Γ_n , defined (loc. cit.) by $\alpha_0 \geq 0$, is indeterminate when $\alpha_0 = 0$. In that case we orient the cycle arbitrarily. This has no great importance, as a change of orientation merely changes the sign of certain Kronecker indices, whose absolute value, however, is alone of interest.

What matters chiefly is to make sure that the fundamental formula (59.1) holds without exception. The proof goes through in fact even for $\mu = 0$, n , but the direct verification for both cases is very simple. The case $\mu = n$ has already been considered in No. 63, but there is a simpler and more direct verification, as we shall presently see.

Let first $\mu = 0$. Denote again by θ , as in Tr., No. 61, the contribution of a certain point $A \times B$ to α_0 . With the same notations as there used, except that the cycle is $\bar{\Gamma}_n$, if it carries the cell $E_n = A \times B \cdots A_n \times B_n$ then θE_n is its indicatrix as cell of $\bar{\Gamma}_n$. It follows $\alpha_0 = \sum \theta$, where the sum is extended to all points $A \times B$ of $\bar{\Gamma}_n$ corresponding to a fixed A . If \bar{T} is the transformation defined by $\bar{\Gamma}_n$, we may think of them as the points $A \times B$ whose B is the image of some $\bar{T}A$. The verification of (59.1) for $\mu = 0$ requires that

$$(8.2) \quad (\Gamma_n \cdot A \times M'_n) = (\bar{\Gamma}_n \cdot A \times M'_n) = (\bar{T}A \cdot M'_n) = \alpha_0(A \cdot M'_n) = \alpha_0,$$

which is in accordance with the formula in Tr., No. 56.

Let now $\mu = n$, and denote by ϵ the same integer as in Tr., No. 61; $\epsilon B B_1 \cdots B_n$ is then the indicatrix of M'_n at B and ϵ has the sign of $|Y_n|$. Hence as at the end of No. 61, $A \times B$ contributes $(-1)^n \epsilon \theta$ to $(\bar{\Gamma}_n \cdot M_n \times B)$ and therefore

$$(8.3) \quad (\bar{\Gamma}_n \cdot M_n \times B) = (-1)^n \sum \epsilon \theta.$$

Here the sum is extended to all points of Γ_n with a fixed B , or to all points $A \times B$ of the cycle such that among the points $\bar{T}A$ there is one whose M'_n image is B .

Now the image of $\bar{T}C_n$ is a polyhedral γ_n on C_n . We may so subdivide C_n , say into C_n^0 , that $\bar{T}C_n$ be a subcomplex of the subdivision. Then any particular cell E_n of C_n^0 will count say k' times positively and k'' times negatively among the cells of $T\bar{C}_n$, and we shall have from the above, if B is on E_n ,

$$(8.4) \quad k' - k'' = \sum \epsilon \theta = \alpha_n;$$

$$(8.5) \quad \bar{T}M_n = \alpha_n \cdot M_n.$$

The verification of (8.2) for $\mu = n$ requires here that

$$(8.6) \quad (\bar{\Gamma}_n \cdot M_n \times B) = (\bar{\Gamma}_n \cdot M_n \times B) = (-1)^n (\bar{T}M_n \cdot A) \\ = (-1)^n \alpha_n (M_n \cdot A) = (-1)^n \alpha_n,$$

which comparison with (8.3) shows is correct.

9. Invariant form for coincidence and fixed point formulas. The formulas given in Tr. are rather involved and furthermore depend upon a special choice of fundamental sets. It so happens that by making use of matrices there can be derived formulas independent of the particular fundamental sets relative to the operation \approx that may be chosen. To begin with, the first formula on p. 43, Tr., reads*

$$(9.1) \quad (-1)^{\mu(n+1)} L_\mu \epsilon_{n-\mu} L_\mu = \alpha_\mu L_\mu,$$

where L_μ is as defined in §6 of the present paper. Since its determinant is not zero, this gives at once

$$(9.2) \quad (-1)^{\mu(n+1)} L_\mu \epsilon_{n-\mu} = \alpha_\mu,$$

and then

$$(9.3) \quad \epsilon_{n-\mu} = (-1)^{\mu(n+1)} L_\mu^{-1} \alpha_\mu.$$

This solves then the problem of expressing the ϵ 's in terms of the α 's for any choice of fundamental sets as to \approx , and not merely for the canonical sets. The explicit formulas of No. 64, Tr., for canonical sets follow from these simply by replacing the L 's by the form corresponding to each μ .

10. For the coincidence formula the starting point is as in Tr., No. 70, the relation independent of the choice of sets

$$(10.1) \quad (\Gamma_n \cdot \Gamma'_n) = \sum \epsilon_n^{ij} \eta_\mu^{hk} (\gamma_{n-\mu}^i \times \delta_\mu^j \cdot \gamma_\mu^h \times \delta_{n-\mu}^k).$$

By applying Tr., (53.3), and then replacing afterwards the δ 's by γ 's, this becomes

$$(10.2) \quad (\Gamma_n \cdot \Gamma'_n) = \sum (-1)^\mu \epsilon_n^{ij} \eta_\mu^{hk} (\gamma_{n-\mu}^i \cdot \gamma_\mu^h) (\gamma_\mu^j \cdot \gamma_{n-\mu}^k) \\ = \sum (-1)^\mu \text{trace } \epsilon'_{n-\mu} L_{n-\mu} \eta_\mu L'_\mu.$$

By (9.2) or (9.3) applied to both transformations, and recalling that $(ab)' = b'a'$, we have

$$(10.3) \quad \epsilon'_{n-\mu} L_{n-\mu} \eta_\mu L'_\mu = (L_\mu^{-1} \alpha_\mu)' \beta_{n-\mu} L'_\mu \\ = (L_\mu \beta'_{n-\mu} L_\mu^{-1} \alpha_\mu)'.$$

* In that formula $\gamma_{n-\mu}^i$ must be replaced by $\gamma_{n-\mu}^i$.

Since transposed matrices have equal traces, (10.2) may be replaced by

$$(10.4) \quad (\Gamma_n \cdot \Gamma'_n) = \sum (-1)^{\mu} \text{trace } L_\mu \beta'_{n-\mu} L_\mu^{-1} \alpha_\mu.$$

This is the final form that we need for the coincidence formula. It is manifestly more compact and clear than what is found in No. 70, Tr.,* and has also the requisite invariant form most appropriate for manifolds with boundary. For the fixed point formula we choose here the second transformation as the identity. Then the β 's are all unit matrices and (10.4) reduces to

$$(10.5) \quad (\Gamma_n \cdot \Gamma_n^0) = \sum (-1)^{\mu} \text{trace } \alpha_\mu.$$

This is (71.1), Tr., with the two cycles interchanged. However, with the situation chosen there, μ should be replaced by $n-\mu$ in (71.1), or, what is equivalent, the two cycles interchanged on the left. Here again, as in loc. cit., for a transformation of the same class as the identity, or more generally for one which merely adds zero-divisors to any cycle,

$$(10.6) \quad (\Gamma_n^0 \cdot \Gamma_n^0) = \sum (-1)^{\mu} R_\mu,$$

the Euler characteristic.

11. While dealing with transformations, let us bring out the following interesting property: *The transform of a zero-divisor or cycle is also a zero-divisor or cycle.* In signs, if $\gamma_\mu \approx 0$, also $\bar{\gamma}_\mu = T\gamma_\mu \approx 0$. For then $\gamma_\mu \times \delta_{n-\mu}$ is also a zero-divisor or cycle for $M_n \times M_{n'}$, hence, by §3 and Tr. (59.1),

$$(11.1) \quad 0 = (\Gamma_n \cdot \gamma_\mu \times \delta_{n-\mu}) = (-1)^{\mu} (\bar{\gamma}_\mu \cdot \gamma_{n-\mu})$$

for every $\gamma_{n-\mu}$, from which at once $\bar{\gamma}_\mu \approx 0$.

II. MANIFOLDS WITH A BOUNDARY

12. We propose to modify somewhat the definition of manifolds of our earlier paper. The difference, however, pertains only to the boundary and since it has played no direct part there, all results so far obtained will continue to hold. Let C_n be a complex. Consider the star of cells whose center is a given E_k of C_n . Between its cells there take place the same incidence relations as between the elements of a certain C_{n-k-1} : the h -cells of the latter correspond to the $(h+k+1)$ -cells of C_n incident with E_k , or the star of cells of center E_k . Now C_n defines a manifold M_n when C_{n-k-1} is homeomorphic

* In the first formula of No. 70 replace \sim by \times .

(a) with the boundary of an E_{n-k} (sphere of S_{n-k});
or

(b) with an E_{n-k-1} plus its boundary.

E_k is an *interior* or a *boundary* cell of M_n according as (a) or (b) is fulfilled. The set F_{n-1} of all boundary cells is the boundary of M_n .

Due to (b), the C_{n-k-2} image of the star of F_{n-1} with center E_k assumed on F_{n-1} is homeomorphic with the boundary of an E_{n-k-1} . Hence (a) holds for every cell of F_{n-1} , which is thus, itself, an M_{n-1} without boundary.

The manifold conditions may be replaced by others equivalent and often more convenient. Instead of considering the h -cells of the star of center E_k as $(h-k-1)$ -cells of a certain complex, let us think of them as $(h-k)$ -cells of a new system s_{n-k} with a unique zero-cell corresponding to E_k . Then in place of (a) and (b) we may obviously impose the following conditions:

(a') when E_k is an interior cell, s_{n-k} is homeomorphic with an E_{n-k} ;

(b') when E_k is on the boundary, s_{n-k} is homeomorphic with a star of cells of S_{n-k} that constitutes an E_{n-k} plus an E_{n-k-1} on its boundary, or what is the same thing, with a hemispherical region of S_{n-k} plus its flat base.*

The conditions here imposed for an M_n are more stringent for the boundary than Veblen's (Coll. Lect., p. 88). They are, however, in the nature of a certain homogeneity requirement along the boundary and entirely similar to what is imposed on the interior. It will also be observed that they do not demand that the cells of C_n be *simplicial*, but merely that they be *convex*.

13. *When C_n satisfies the manifold conditions so does any subdivision of it, C'_n .* This is proved in outline as follows. With each flat E_k on C_n we associate a system such as $\{e\}$ of Tr., No. 3, where e represents a class of incident $(k+1)$ -cells on the same half S_{k+1} . The system $\{e\}$ is the same for two flat k -cells with a k -subcell in common. The $(h+k+1)$ -cells incident with E_k may be considered as h -cells made up with the similar $(k+1)$ -cells as points, that is with the e 's as points. Hence when E_k is a cell of any particular subdivision C'_n of C_n , $\{e\}$ is homeomorphic with the C'_{n-k-1} similar to C_{n-k-1} of §12. It is then sufficient to show that when C_n behaves as desired, every $\{e\}$ obeys conditions (a), (b).

Let then E_k carry E_{k-1} , with a corresponding system $\{e'\}$. When $\{e\}$ behaves as desired so does $\{e'\}$. For to each e there corresponds a one-cell

* One is tempted to replace (b') by the simpler " s_{n-k} is an E_{n-k} plus an E_{n-k-1} on its boundary." Unfortunately to show that this is equivalent to (b') we need the following theorem: Two h -cells can be homeomorphically transformed into one another in such manner that two $(h-1)$ -cells of their boundaries are similarly transformed. For $h=2$ this goes back to the Jordan curve theorem, but beyond that there is no proof.

of e'' 's with fixed end points (images of E_k itself). Hence the relation between the two systems is like that between a sphere or hemisphere in S_{n-k} (a sphere when E_k is an interior cell, a hemisphere otherwise) and its locus when the space, immersed in an S_{n-k+1} , rotates through an angle π around a diameter. It follows that for our purpose $\{e\}$ may replace $\{e'\}$ and E_k replace E_{k-1} . Ultimately, then, we shall merely have to consider some cell of C_n itself; that is, as asserted, the correct behavior of $\{e\}$ follows from that of every C_{n-k-1} attached to the cells of C_n .

14. From (a) and (b) as applied to $(n-1)$ -cells it follows that every interior $(n-1)$ -cell of C_n separates two n -cells, and every boundary $(n-1)$ -cell is on a unique n -cell of C_n ; F_{n-1} is then the sum of all $(n-1)$ -cells on a unique n -cell of C_n . *We assume again that M_n is orientable.* Then F_{n-1} will also be orientable. For let us orient C_n and then sense each E_{n-1} of F_{n-1} positively in relation to the E_n that it bounds. Between the n - and $(n-1)$ -cells incident with a given E_{n-2} of F_{n-1} we can write down the same Poincaré congruences as for the one- and zero-cells of a polygonal line. Hence the two end $(n-1)$ -cells, which are those of E_{n-1} incident with E_{n-2} , are oppositely related to E_{n-2} , and F_{n-1} is oriented. Its orientation as thus fixed shall be preserved throughout. It corresponds to the congruence $M_n \equiv F_{n-1}$.

15. **The auxiliary manifold V_n .** We assume henceforth that M_n has a boundary F_{n-1} . Take, then, another copy \bar{M}_n of the manifold and piece the two together along corresponding boundary points. *The new configuration V_n so obtained is an M_n without boundary.* (Any element of \bar{M}_n corresponding to a given one of M_n will be called its conjugate and designated by the same letter barred.) If C_n is the basic defining complex of M_n , we use \bar{C}_n for \bar{M}_n and $C_n + \bar{C}_n$ for V_n . Then if E_k is the cell of M_n in §12, when it is not on F_{n-1} , the complex C_{n-k-1} plays the same part for it relative to V_n as to M_n . Hence it behaves then according to (a), and similarly for \bar{E}_k and \bar{C}_{n-k-1} . However, when E_k is on F_{n-1} , in place of C_{n-k-1} we have $C_{n-k-1} + \bar{C}_{n-k-1}$. As this set is composed of two $(n-k-1)$ -cells pieced together along their boundaries, it is homeomorphic to the boundary of an E_{n-k} . This is seen at once by referring to the piecing together of two hemispheres in S_{n-k} into a sphere of that space. Hence E_{n-k} behaves again in accordance with (a), which proves our assertion as to V_n .

If E_n is an n -cell of C_n , E'_n its indicatrix, we sense \bar{E}_n of \bar{M}_n by $-\bar{E}'_n$, hence $V_n = M_n - \bar{M}_n$. The importance of V_n is due to the fact that the solution of the coincidence problem for pairs of transformations of M_n will be reduced to the same problem for pairs of associated transformations of V_n .*

* This or a similar procedure has been followed by other authors dealing with this question. See, for example, Brouwer, Comptes Rendus, vol. 168 (1919), p. 1042; Alexander, these Transactions, vol. 23 (1922), pp. 89–95.

It is evident that its topological properties are really inherent properties of M_n itself. We shall be particularly concerned with the study and disposition of its fundamental sets.

16. Fundamental sets for V_n . As several distinct types of cycles will have to be considered we shall avoid excess of indices by using not only Γ, γ but also G, Δ, D to designate them.

Let $\Gamma_\mu^1, \dots, \Gamma_\mu^{r_\mu^1}$ be a fundamental set for the μ cycles of M_n , and consider all possible homologies

$$(16.1) \quad \sum t_i \Gamma_\mu^i \sim \text{a cycle of } F_{n-1} \pmod{M_n}.$$

By paraphrasing a well known process* we may readily establish that these homologies are sums of multiples of a finite number of the same type which constitute a fundamental set for them. The members of the fundamental set and also the cycles can then be combined in such a fashion as to have a new fundamental set of Γ 's (for which we keep the same designation as above) with fundamental homologies

$$(16.2) \quad \theta_i \Gamma_\mu^{r_\mu^1+i} \sim \text{a cycle of } F_{n-1} \pmod{M_n} \quad (i = 1, 2, \dots, p - r_\mu^1).$$

The operations referred to correspond to elementary transformations on the matrix of the coefficients of the fundamental homologies. The first r_μ^1 cycles are not related by any homology such as (16.1) and in particular they are entirely independent; the remaining cycles have some non-zero multiple homologous to a cycle on the boundary. Between the cycles $\Gamma_\mu^{r_\mu^1+i}$ there may exist homologies mod M_n . Reducing those as above, we shall replace the cycles by a new set $G_\mu^j, j = 1, 2, \dots, p - r_\mu^1$, whose first say r_μ^2 elements are independent while the others are ≈ 0 . Of course $r_\mu^1 + r_\mu^2 = R_\mu$, the μ th connectivity index of M_n . The cycles $\Gamma_\mu^i, i = 1, 2, \dots, r_\mu^1, G_\mu^j, j = 1, 2, \dots, r_\mu^2$, constitute a fundamental set for M_n relative to the operation \approx .

It will be convenient to call a cycle *symmetric* when it is \approx mod M_n to a cycle on F_{n-1} . Then the difference between the cycle and its conjugate is ≈ 0 , mod V_n .

17. Of no less importance than the preceding are the *skew-symmetric* cycles of V_n . We so designate those of type $C_\mu - \bar{C}_\mu$, where C_μ is on M_n . In place of them it would be possible to consider the complexes whose boundary is on F_{n-1} and their properties in regard to what might be called "quasi-homologies" or relations:

$$(17.1) \quad \sum t_i C^i + \text{a complex of } F_{n-1} \sim 0 \pmod{M_n}.$$

* Klein, *Elliptische Modulfunctionen*, vol. 2, p. 543.

The theory resulting therefrom would be largely a paraphrase of the one to be found here, with the mild advantage of being strictly confined to elements of M_n itself.

Since $C_\mu - \bar{C}_\mu$ is a cycle, the boundary of C_μ is on F_{n-1} . Furthermore we can remove from it any μ -cell on F_{n-1} without changing the cycle.

If ρ_μ is the μ th connectivity index of V_n , any $\rho_\mu + 1$ skew-symmetric μ -cycles are dependent, so that Klein's reasoning applies and we find a fundamental set $\Delta_\mu^1, \Delta_\mu^2, \dots, \Delta_\mu^q$, for the type. It is reducible in similar fashion to the above as regards homologies between the intersections with the boundary:

$$(17.2) \quad \sum t_i \Delta_\mu^i \cdot F_{n-1} \sim 0 \quad (\text{mod } F_{n-1}),$$

with a similar conclusion: The set can be replaced by a new one, for which the same designation is preserved, with fundamental homologies for the type (17.2):

$$(17.3) \quad r_j \Delta_\mu^j \cdot F_{n-1} \sim 0 \quad (\text{mod } F_{n-1}) \quad (j = 1, 2, \dots, s \leq q).$$

In short the first s cycles of the new set intersect the boundary of M_n in zero-divisors or bounding cycles of it, while the remaining $q-s$ intersect it in independent cycles of F_{n-1} . By Tr., No. 35, Theorem V, they are independent for V_n as well. There may exist, however, homologies between the first s . Reducing again as regards these we finally obtain a fundamental set consisting of the following:

- (a) r_μ^3 cycles $\Delta_\mu^1, \Delta_\mu^2, \dots, \Delta_\mu^{r_\mu^3}$, independent (mod V_n), but meeting F_{n-1} in cycles ≈ 0 (mod F_{n-1});
- (b) $r_\mu^4 = q-s$ cycles $D_\mu^1, D_\mu^2, \dots, D_\mu^{r_\mu^4}$, independent (mod V_n) and intersecting F_{n-1} in cycles independent (mod F_{n-1});
- (c) a set of at most $q - r_\mu^3 - r_\mu^4$ zero-divisors of V_n .

Every skew-symmetric cycle is \approx to a sum of Δ 's and D 's (these notations are henceforth reserved for cycles (a) and (b)). Also there can be no homology involving both D 's and Δ 's, as we see at once by reference to their intersections with F_{n-1} . Since there is none involving each type alone, they constitute $r_\mu^3 + r_\mu^4$ independent cycles of V_n , and therefore a fundamental set as regards \approx and skew-symmetric cycles.

Let $\gamma_\mu \approx \Delta_\mu + D_\mu$ be a skew-symmetric cycle, Δ_μ and D_μ being sums of cycles (a) or (b). Let $\delta_{n-\mu}$ be any cycle of F_{n-1} . From §1 follows, with indices computed as to F_{n-1} ,

$$(17.4) \quad ((\gamma_\mu \cdot F_{n-1}) \cdot \delta_{n-\mu}) = ((D_\mu \cdot F_{n-1}) \cdot \delta_{n-\mu}).$$

Hence, by §5 and the definition of the D 's, in order that $\gamma_\mu \cdot F_{n-1} \approx 0 \pmod{F_{n-1}}$ it is necessary and sufficient that $\gamma_\mu \approx \Delta_\mu$ alone.

18. Let $\gamma_{\mu-1}$ be a cycle of F_{n-1} not ≈ 0 on that manifold, but bounding C_μ on M_n . Then $C_\mu - \bar{C}_\mu$ is a skew-symmetric cycle of type D not $\approx 0 \pmod{V_n}$, since its intersection γ_μ with F_{n-1} is not $\approx 0 \pmod{F_{n-1}}$. Hence to a set of t independent μ -cycles of F_{n-1} that bound on V_n correspond as many independent skew-symmetric cycles of type D , and conversely. Therefore r_μ^4 is the *number of distinct μ cycles F_{n-1} that bound on M_n* .

Another interesting property is the following: *Every Δ has a multiple which is the sum of a cycle on M_n and of a cycle on \bar{M}_n .* Let Δ_μ be the cycle, Δ'_μ a polyhedral approximation of maximum generality intersecting F_{n-1} in $\gamma_{\mu-1}$. By §17,

$$(18.1) \quad \gamma_{\mu-1} \approx 0 \quad (\text{mod } F_{n-1}).$$

Since Δ'_μ is of maximum generality it has no μ -cells on F_{n-1} , hence we may write

$$(18.2) \quad \Delta'_\mu = C_\mu - \bar{C}'_\mu,$$

where the first complex is on M_n , the second on \bar{M}_n , and both have the common boundary $\gamma_{\mu-1}$. From (18.1) we infer that there exists on F_{n-1}

$$(18.3) \quad C''_\mu \equiv t\gamma_{\mu-1}, \quad t \neq 0,$$

therefore

$$(18.4) \quad t\Delta_\mu \sim t\Delta'_\mu = (tC_\mu - C''_\mu) - (t\bar{C}'_\mu - C''_\mu).$$

Each parenthesis at the right is a cycle, the first on M_n , the second on \bar{M}_n , which proves our assertion.

19. **Theorems.** I. *Every cycle of V_n is the sum of a skew-symmetric cycle and of one on M_n .*

Let γ_μ be the cycle. It may be assumed polyhedral and the sum of two complexes C_μ and \bar{C}'_μ , the first on M_n , the second on \bar{M}_n . But

$$(19.1) \quad \gamma_\mu = (C_\mu + C'_\mu) + (\bar{C}'_\mu - C'_\mu).$$

The second parenthesis is a skew-symmetric cycle, the first a complex on M_n , the difference of two cycles, hence also a cycle, and the theorem is therefore proved.

II. *A γ_μ of M_n not $\approx 0 \pmod{M_n}$ cannot be skew-symmetric.*

For let it be $\approx \delta_\mu$, skew-symmetric. Then

$$(19.2) \quad \delta_\mu - \gamma_\mu \approx 0 \approx \bar{\delta}_\mu - \bar{\gamma}_\mu \approx +\delta_\mu + \bar{\gamma}_\mu \quad (\text{mod } V_n),$$

Therefore

$$(19.3) \quad t(\gamma_\mu + \bar{\gamma}_\mu) \sim 0 \quad (\text{mod } V_n); \quad t \neq 0.$$

First let γ_μ be symmetric, with $\theta\gamma_\mu, \theta \neq 0$, homologous (mod M_n) to γ_μ^0 on F_{n-1} . Then

$$(19.4) \quad \theta\gamma_\mu \sim \theta\bar{\gamma}_\mu \sim \gamma_\mu^0 \quad (\text{mod } V_n),$$

and therefore by (19.3)

$$(19.5) \quad 2t\gamma_\mu^0 \sim 0 \quad (\text{mod } V_n).$$

There exists then on V_n a

$$(19.6) \quad C_{\mu+1} \equiv 2t\gamma_\mu^0.$$

Let $C'_{\mu+1}$ be the subcomplex of $C_{\mu+1}$ that includes all its cells on M_n (logical intersection of $C_{\mu+1}$ and M_n) and set $C_{\mu+1} - C'_{\mu+1} = \bar{C}_{\mu+1}''$, complex made up of all $(\mu+1)$ -cells of $C_{\mu+1}$ interior to \bar{M}_n plus their boundaries. Since the boundary of $\bar{C}_{\mu+1}''$ can only be on F_{n-1} , it coincides with that of $C_{\mu+1}''$. Hence

$$(19.7) \quad C'_{\mu+1} + C_{\mu+1}'' \equiv 2t\gamma_\mu^0,$$

for $C_{\mu+1}$ has the same boundary as the complex at the left. But the latter is on M_n , hence

$$(19.8) \quad \theta\gamma_\mu \sim \gamma_\mu^0 \approx 0; \quad \gamma_\mu \approx 0 \quad (\text{mod } M_n),$$

contrary to assumption.

Assume now that γ_μ is not symmetric. It is reducible to a polyhedral cycle whose μ -cells are all interior to M_n . The first part of this double assertion is established as in Coll. Lect., pp. 95, 118, the second as in Tr. No. 24, (b). Furthermore a complete proof, independent of the present discussion, is given below (§23). The left member of (19.3), polyhedral and without μ -cells on F_{n-1} , will also bound a polyhedral $C_{\mu+1}'$ (Coll. Lect., p. 120). Let again the sum of its cells on M_n be called $C'_{\mu+1}$. This last complex has for total boundary $t\gamma_\mu$ plus a cycle γ'_μ on F_{n-1} . Therefore

$$(19.9) \quad t\gamma_\mu \sim -\gamma'_\mu \quad (\text{mod } M_n),$$

and γ_μ is a symmetric cycle, — a new contradiction, and II is proved.

In the second part of the discussion we have established that if (19.3) holds then γ_μ is symmetric. The identical reasoning holds for the more general homology, in which γ_μ is still on M_n ,

$$(19.10) \quad t\gamma_\mu + \theta\bar{\gamma}_\mu \sim 0 \quad (\text{mod } V_n).$$

which shows that

III. *If γ_μ of M_n is dependent upon its conjugate γ_μ then it is a symmetric cycle.*

IV. *When γ_μ of M_n is not ≈ 0 , mod M_n , then it is also not ≈ 0 , mod V_n .*

This is a special case of II corresponding to $\delta_\mu = 0$.

20. From the preceding propositions follows at once the all important

THEOREM. *The cycles Γ_μ^α , G_μ^α , Δ_μ^α , D_μ^α ($\alpha_1=1, 2, \dots, r_\mu^i$) constitute a fundamental set for V_n and the operation \approx . To obtain a fundamental set for \sim it is only necessary to add zero-divisors of M_n and skew-symmetric zero-divisors of V_n .*

From their definition we know that the Γ 's and G 's are independent, and similarly for the Δ 's and D 's. From II follows that the four sets are independent in their totality. Then again from I and the fact that the Γ 's and G 's constitute a fundamental set as to \approx and M_n , and similarly the Δ 's and D 's for the skew-symmetric cycles and V_n , the theorem follows in its completeness.

COROLLARY. *The μ th connectivity index of V_n is $\rho_\mu = r_\mu^1 + \dots + r_\mu^4$.*

21. Our present object is to show that with a suitable choice of associated sets for the dimensionalities μ and $n-\mu$ certain indices are, or can be made to be, zero, which will naturally lead to the canonical sets.

22. **LEMMA.** *Let E_n be a cell, Γ_μ a cycle on E_n , both polyhedral. Then there exists a polyhedral $C_{\mu+1}$ on E_n bounded by Γ_μ .*

Since Γ_μ is on E_n , it has no points on the boundary of the cell. Hence there exists a cell E'_n which together with its boundary lies on E_n , and also carries Γ_μ . Let σ be the least distance between points of the boundaries of the cells. Cover E_n with a polyhedral complex whose cells are all of diameter $< \sigma$, and remove all its n -cells with a boundary point on the boundary of E_n . The n -cells that are left together with their boundaries constitute a C_n on E_n and carrying E'_n . The cycle Γ_μ bounds on E'_n , hence also on C_n , and on that complex it bounds a polyhedral $C_{\mu+1}$ (Coll. Lect., p. 120) which proves the lemma.

23. Let C_μ be a complex on M_n with its boundary on F_{n-1} , and let us follow step by step the approximation described in Tr., Part I, §4. We first apply the Alexander-Veblen process, and obtain $C_{\mu+1}$, polyhedral with an

associated congruence

$$(23.1) \quad C_{\mu+1} \equiv C_\mu - C'_\mu + C_\mu^0.$$

By reference to Coll. Lect., pp. 95, 118, we find that we may approximate each vertex of the boundary of C_μ by a vertex of the boundary of C_n , that is, by a point of F_{n-1} . Then C_μ^0 and the boundary of C'_μ will be on F_{n-1} . Except for that, the rest of the work of approximation is directly applicable here. However, No. 23, Tr., does not apply to non-boundary cells of C'_μ on F_{n-1} , and necessitates a slight modification.

We take C_n such that C'_μ is now a subcomplex of it (Tr., No. 14, Lemma II), and as a first move, reduce its boundary on F_{n-1} as described in Tr., No. 23. We thus obtain a polyhedral cycle $\Gamma'_{\mu-1}$ of F_{n-1} whose $(\mu-i)$ -cells are all on cells of no less than $n-i$ dimensions of C_n . Furthermore $\Gamma_{\mu-1} - \Gamma'_{\mu-1}$ bounds a polyhedral C''_μ , and both new cycles and complex are as near as we please to C'_μ , hence to C_μ . It follows that in (23.1), C'_μ and C_μ^0 may be replaced by $C'_\mu + C''_\mu$ and $C_\mu^0 + C''_\mu$, without altering the situation. Therefore we may start with a complex C'_μ whose boundary is already reduced as indicated. To extend then the reduction of Tr., No. 23, all we need to do is to replace C'_μ by a complex whose non-boundary cells are interior to M_n , since the reduction in question can be applied to these. The reader will verify with ease that the situation pertaining to (23.1) remains unaffected by any step to be taken presently. Furthermore, the reduction will be more thorough than in Tr., No. 23, in that the new elements introduced first here, then by the process of Tr., No. 23, will be throughout interior to M_n . Hence the complex as finally reduced will have all non-boundary cells interior to the manifold.

Let first E_μ be a simplicial cell of C'_μ on F_{n-1} and on the boundary of the cell E_n of C_n . Draw rectilinear segments from a fixed point of E_n to all points of E_μ . The resulting simplicial cell has for boundary a Γ_μ whose μ -cells other than E_μ (which is one of them) are on E_n . There is a $t \neq 0$ such that $C'_\mu - t\Gamma_\mu \sim C'_\mu$ is a complex which no longer includes E_μ . The cell E_μ has then been replaced by cells on E_n , or interior cells of M_n . Thus we can reduce C'_μ to a similar complex whose μ -cells are interior to M_n . Assume then that all cells of more than h dimensions, $h < \mu$, of C'_μ are interior cells. I say that the reduction can be extended to the h -cells as well.

Let E_h of C'_μ be on F_{n-1} . Introduce new interior vertices on C_n so chosen that for the new complex C'_n the star of cells of center E_n carries no boundary cells of C'_μ on its own. Since the star attached to C_n carries no other boundary cells of C'_μ than those on E_h or its boundary, the construction offers no

difficulty. The related C_{n-h-1} of Tr., No. 3, which is merely the boundary of the star, will then intersect C'_μ in a $(\mu-h-1)$ -cycle on E_{n-h-1} . By our lemma this cycle bounds a polyhedral $C_{\mu-h}$ on E_{n-h-1} , having then no point on F_{n-1} . Let $E_h = A_0 A_1 \cdots A_h$, and let also $A_{h+1} \cdots A_i$ be an arbitrary cell of $C_{\mu-h}$; then denote by $\bar{C}_{\mu+1}$ the complex sum of the cells $A_0 A_1 \cdots A_i$. It has in common with C'_μ all cells incident with E_h , and except for these cells and their boundary points it is entirely interior to M_n . Let us sense its boundary Γ'_μ so that the indicatrix of any cell is obtained by naming first the vertices of E_h , next those of the related cell on C_{n-h-1} so that both sets be vertices of an indicatrix for their cell. Then C'_μ and Γ'_μ will have the same cells incident with E_h each counted with the same multiplicity for both complexes. Hence $C'_\mu - \Gamma'_\mu \sim C'_\mu$ will have lost E_h without acquiring new cells on F_{n-1} . This shows that the reduction can also be extended to h cells of C'_μ , hence to its boundary cells.

Combining the whole discussion we have the important

THEOREM. *Let C_n be an assigned defining complex of M_n and C_μ a complex on the manifold, whose boundary is on F_{n-1} . Then there is a corresponding congruence (23.1) with (a) C'_μ polyhedral, as near as we please to C_μ , with its boundary on F_{n-1} and as near as we please to that of C_μ , also with its $\mu-i$ cells on cells of no less than $n-i$ dimensions of C_n , the non-boundary cells being interior to M_n ; (b) C_μ^0 on F_{n-1} and as near as desired to the boundary of C_μ .*

COROLLARY I. *Every cycle of M_n is homologous to an interior cycle behaving in accordance with the theorem.*

COROLLARY II. *Every skew-symmetric cycle is homologous to one of form $C'_\mu - \bar{C}'_\mu$, where C'_μ behaves in accordance with the theorem.*

For let $\gamma_\mu = C_\mu - \bar{C}_\mu$. Reduce C_μ as above with the congruence (23.1). Then also

$$(23.2) \quad \bar{C}_{\mu+1} \equiv \bar{C}_\mu - \bar{C}'_\mu + C_\mu^0 ;$$

therefore

$$(23.3) \quad \gamma_\mu - (C'_\mu - \bar{C}'_\mu) \sim 0,$$

as was to be proved.

Unless otherwise stated we shall always assume the cycles of M_n and the skew-symmetric cycles reduced as far as allowed by theorem and corollaries. Practically everywhere in the sequel, a cycle of one of these two types may be replaced by one which is \sim , mod M_n and V_n respectively, and then

if it is not already reduced, we shall be at liberty to reduce it without further discussion.

24. Theorems on indices. I. $(\Gamma_\mu^i \cdot G_{n-\mu}^j) = (G_\mu^i \cdot G_{n-\mu}^j) = 0$.

There exists $G_{n-\mu}$ of F_{n-1} , $\sim t G_{n-\mu}^j$, mod M_n , $t \neq 0$. If we replace $G_{n-\mu}^j$ by $G_{n-\mu}$ we merely multiply the indices by t . Hence we need only show that

$$(24.1) \quad (\Gamma_\mu^i \cdot G_{n-\mu}) = (G_\mu^i \cdot C_{n-\mu}) = 0,$$

which is obvious since Γ_μ^i and G_μ^i may be chosen interior to M_n and then they will not meet $G_{n-\mu}$.

Explicitly, and since $(\Gamma \cdot G) = \pm (G \cdot \Gamma)$, the indices $(\Gamma \cdot G)$, $(G \cdot \Gamma)$, $(G \cdot G)$ are all zero.

II. *The index of two skew-symmetric cycles is zero.*

Let $\delta_\mu = C_\mu - \bar{C}_\mu$, $\delta_{n-\mu} = C_{n-\mu} - \bar{C}_{n-\mu}$ be the two cycles in the reduced form in position of maximum generality. Then for evident reasons of symmetry the two indices $(C_\mu \cdot C_{n-\mu})$ and $(\bar{C}_\mu \cdot \bar{C}_{n-\mu})$ taken with M_n and \bar{M}_n as the carrying manifolds are equal. Hence, taken with $V_n = M_n - \bar{M}_n$ as the carrying manifold, they are opposite. But C_μ does not meet $\bar{C}_{n-\mu}$ and $C_{n-\mu}$ does not meet \bar{C}_μ . Hence

$$(24.2) \quad \begin{aligned} (\delta_\mu \cdot \delta_{n-\mu}) &= (C_\mu - \bar{C}_\mu)(C_{n-\mu} - \bar{C}_{n-\mu}) \\ &= (C_\mu \cdot C_{n-\mu}) - (\bar{C}_\mu \cdot \bar{C}_{n-\mu}) = 0. \end{aligned}$$

III. $(G_\mu^i \cdot \Delta_{n-\mu}) = 0$.

In the proof G may be replaced by $t G$, $t \neq 0$. Since there is a $t G$ on F_{n-1} , we may assume that G itself is on the boundary. Then (§ 18) there is an $s \neq 0$ such that

$$(24.3) \quad s\Delta \sim \Delta' + \bar{\Delta}'',$$

where Δ' is a cycle on M_n and $\bar{\Delta}'$ a cycle on \bar{M}_n . By I and since Δ' and Δ'' depend upon the G 's and Γ 's,

$$(24.4) \quad (G \cdot \Delta') = (G \cdot \Delta'') = - (G \cdot \bar{\Delta}'') = 0;$$

therefore

$$(24.5) \quad (G \cdot \Delta) = 0 = (\Delta \cdot G).$$

CONCLUSION. All indices $(\Gamma \cdot G)$, $(\Delta \cdot G)$, $(\Delta \cdot \Delta)$, $(\Delta \cdot D)$, and those obtained by permuting the cycles, are zero.

25. The matrix L_μ will then have the form

$$L_\mu = \begin{array}{c|cccc} & \Gamma_{n-\mu} & \Delta_{n-\mu} & G_{n-\mu} & D_{n-\mu} \\ \hline A & B & 0 & C & \\ H & 0 & 0 & 0 & \\ 0 & 0 & 0 & F & \\ M & 0 & N & 0 & \end{array} \begin{array}{c} \Gamma_\mu \\ \Delta_\mu \\ G_\mu \\ D_\mu \end{array}$$

where the terms are all matrices; for example, $H = ||(\Gamma_\mu^i \cdot \Delta_{n-\mu}^i)||$, $||(\Delta_\mu^i \cdot \Delta_{n-\mu}^i)|| = 0$, etc. We shall now study L_μ and in particular show that by proper choice of cycles in each group it can be reduced to a much simpler form, with only one indeterminate matrix, namely A . As an incidental result we shall obtain very interesting duality theorems regarding the integers r_μ , theorems quite similar to Poincaré's relation for the connectivity numbers of manifolds without boundary.

26. I. *F and N are square.* The permissible operations of adding a multiple of a cycle of a fundamental set to another or permuting two of them, applied to the groups G_μ , $D_{n-\mu}$ will amount to the noted elementary transformations as applied to F . Hence F may be reduced to the well known form

$$(26.1) \quad \begin{array}{c|ccccc|ccccc} & e_1 & 0 & \cdots & 0 & 0 & \overbrace{\cdots}^q & 0 & & \\ & 0 & e_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & \cdots & \\ & 0 & \cdots & \cdots & \cdots & e_s & 0 & \cdots & 0 & \\ & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & \\ p & \cdots & \\ 0 & \cdots & 0 \end{array}$$

where the e 's are the invariant factors of F and there are p rows and q columns of zeros. Assume this done and continue to call the reduced matrix F . We must show that $p=q=0$. Evidently, $p=0$, for otherwise L_μ , as reduced, would have a whole row of zeros, whereas its determinant is ± 1 (§ 6). Then if $q \neq 0$ there is a $D_{n-\mu}$ such that $(G_\mu \cdot D_{n-\mu}) = 0$ for every symmetrical cycle G_μ and in particular for every one on F_{n-1} . Hence (§§5, 17), $D_{n-\mu}$ is dependent upon the cycles $\Delta_{n-\mu}$, which is untrue. Therefore $q=0$, F is square and so similarly is N . Their determinants are integers and factors of $|L_\mu| = \pm 1$, hence they are both ± 1 .

In the canonical form the e 's for both matrices will be $+1$. When $\mu \neq \frac{1}{2}n$, they may both be separately reduced to that form. Denoting generically by I_k the unit matrix of order k , we have, when the reduction is carried

out, $F = I_{\mu^2}$, $N = I_{\mu^4}$. When $\mu = \frac{1}{2}n$, we have at once, by Tr., No. 8, $N = (-1)^{n/2}F'$. The reduction of F brings about that of N . For $\frac{1}{2}n$ even, the final form is as above; for $\frac{1}{2}n$ odd, it is $F = I_{\mu^2} = -N$.

As an important result proved incidentally we have $r_\mu^2 = r_{n-\mu}^4$ or

II. (FIRST DUALITY THEOREM.) *The number of distinct symmetric μ cycles is equal to the number of independent $n-\mu$ cycles of F_{n-1} that bound on M_n .*

Since F is a unit matrix and since a $\Gamma - G$ is also a Γ , we can subtract from every Γ_μ of the fundamental set a G_μ cycle so chosen as to reduce the corresponding row of C to zero. This means that we can so select the set of Γ_μ 's as to have $C = 0$. If $\mu \neq \frac{1}{2}n$, we may operate similarly on the $\Gamma_{n-\mu}$'s and reduce M to zero, while when $\mu = \frac{1}{2}n$, we shall have $M = \pm C = 0$. Hence

III. *The fundamental sets can be so selected as to give L_μ one of the two forms*

$$\begin{array}{|ccc|} \hline A & B & 0 \\ H & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline \end{array}, \quad \begin{array}{|ccc|} \hline A & B & 0 \\ H & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ \hline \end{array},$$

according as μ is not or is $\frac{1}{2}n$ and odd. To simplify we have merely indicated the unit matrices by 1.

IV. *A is a square matrix.* It may be reduced to the type (26.1) by the two operations of the beginning of this section applied to Γ 's alone. As reduced to that type we shall show again that $p = q = 0$. For evident reasons of symmetry it is sufficient to show that $q = 0$. For every Γ_μ we have $(\Gamma_\mu \cdot \Gamma_{n-\mu}^{s+1}) = 0$. Then if $\delta_{n-\mu} = \Gamma_{n-\mu}^{s+1} - \bar{\Gamma}_{n-\mu}^{s+1}$, we have, for every G_μ ,

$$(26.2) \quad (G_\mu \cdot \delta_{n-\mu}) = (G_\mu \cdot \Gamma_{n-\mu}^{s+1}) - (G_\mu \cdot \bar{\Gamma}_{n-\mu}^{s+1}) = 0,$$

for the last two indices are equal in absolute value and the first is zero (§ 24, Theorem I). Also, as we may assume Γ_μ interior to M_n and $\bar{\Gamma}_{n-\mu}^{s+1}$ interior to \bar{M}_n , and hence that the two are without common points,

$$(26.3) \quad (\Gamma_\mu \cdot \delta_{n-\mu}) = -(\Gamma_\mu \cdot \bar{\Gamma}_{n-\mu}^{s+1}) = 0.$$

And finally, since $\delta_{n-\mu}$ is skew-symmetric,

$$(26.4) \quad (\Delta_\mu \cdot \delta_{n-\mu}) = (D_\mu \cdot \delta_{n-\mu}) = 0.$$

In short, $(\gamma_\mu \cdot \delta_{n-\mu}) = 0$ for every μ cycle of the fundamental set, therefore for every μ cycle of V_n . Hence $\delta_{n-\mu} \approx 0$. There exists, then, a $t \neq 0$ such that $t\Gamma_{n-\mu}^{s+1} \sim t\bar{\Gamma}_{n-\mu}^{s+1}$. Hence (§ 19, Theorem III), $\Gamma_{n-\mu}^{s+1}$ is a symmetric cycle,

contrary to assumptions. There exists then no $\Gamma_{n-\mu}$ whose upper index $>s$, which means that $q=0$ and proves IV.

Since A is square, $r_\mu^1=r_{n-\mu}^1$ and therefore

V. (SECOND DUALITY THEOREM.) *The number of cycles of M_n of which no combination is a cycle of F_{n-1} is the same for the dimensions μ and $n-\mu$.*

VI. *B and H are square matrices and their determinants are ± 1 .* We reduce again say B to the form (26.1) and show that $p=q=0$. If $q \neq 0$ there exists a $\Delta_{n-\mu}$ such that $(\gamma_\mu \cdot \Delta_{n-\mu})=0$ whatever γ_μ , hence $\Delta_{n-\mu}$ is a zero-divisor contrary to assumptions, and $q=0$. Assume now $p \neq 0$. Then the number of distinct $\Delta_{n-\mu}$'s is $< r_\mu^1 = r_{n-\mu}^1$, the order of A . Consider, however, the cycles $\Gamma_{n-\mu}^i - \bar{\Gamma}_{n-\mu}^i$. Since the Γ 's may be taken interior to M_n , these skew-symmetric cycles do not intersect F_{n-1} and therefore are dependent upon the Δ 's. On the other hand they are independent, or there would exist a $\Gamma_{n-\mu} \approx \bar{\Gamma}_{n-\mu}$, and hence symmetric (§ 19, Theorem III) in contradiction to the assumptions on the Γ type. Hence there are at least r_μ^1 distinct $\Delta_{n-\mu}$'s and $p=0$. Therefore B is a square matrix and so is H . Here again their determinants are integers and factors of $|L_\mu| = \pm 1$, therefore also $= \pm 1$.

From VI it follows that $r_\mu^1 = r_{n-\mu}^1 = r_\mu^3 = r_{n-\mu}^3$. Hence

VII. (THIRD DUALITY THEOREM.) *The number of distinct skew-symmetric cycles that intersect F_{n-1} in zero-divisors or bounding cycles is the same for the dimensions μ and $n-\mu$ and equal to the number of distinct cycles of M_n of μ or $n-\mu$ dimensions that are independent of the cycles of F_{n-1} .*

27. It follows from the above that for $\mu \neq \frac{1}{2}n$ we can reduce both B and H to $I_{\mu,1}$. For $\mu = \frac{1}{2}n$, $H = (-1)^n B$ and the reduction of B will bring about that of H .

For the computation to follow it is advisable to select fundamental sets thus: when $\mu < \frac{1}{2}n$ the four groups of cycles are taken in the order Γ, Δ, D, G ; when $\mu \geq \frac{1}{2}n$ in the order Γ, Δ, G, D . Then

$$(27.1) \quad \begin{aligned} \mu \neq \frac{1}{2}n, \quad L_\mu &= \left\| \begin{array}{c|c} A & 1 \\ \hline 1 & 0 \\ \hline 0 & 1 \\ \hline 0 & 1 \end{array} \right\|; \\ \mu = \frac{1}{2}n, \quad L_\mu &= \left\| \begin{array}{c|c|c|c} A & 1 & 0 & \\ \hline (-1)^{n/2} \cdot 1 & 0 & 0 & \\ \hline 0 & (-1)^{n/2} \cdot 1 & 0 & \end{array} \right\| \end{aligned}$$

where the 1 stands in each case for a unit matrix whose order is not essential (see Remark below). What matters chiefly is that for the dimensions μ and $n - \mu$ those in the same places are of the same orders. In the upper left corner they are both of order r_μ^1 , in the right corner of orders r_μ^2 and r_μ^4 . Concerning A we have no available information, but fortunately it disappears entirely from our formulas and therefore need not concern us further.

Remark. Consider for a moment two matrices written in the form

$$(27.2) \quad \alpha = \|\alpha_{ij}\|, \quad \beta = \|\beta_{ij}\|,$$

with elements α_{ij} , β_{ij} themselves matrices. The ordinary multiplication rule

$$(27.3) \quad \alpha\beta = \|\sum \alpha_{ik}\beta_{kj}\|$$

is directly applicable (taking care not to interchange factors in $\alpha_{ik}\beta_{kj}$) provided that (a) the number of columns in α is the same as the number of rows in β ; (b) the sequence of the number of columns for the elements in a row of α is the same for all rows and also the same as for β' . These two conditions are fulfilled if α and β are square with their diagonal elements α_{ii} , β_{ii} also square and of equal order for the same i . In that case not only $\alpha\beta$ but also $\beta\alpha$ may be obtained by the usual rule. This is the precise situation that we shall face throughout, where we shall find products of matrices all of the same structure as L_μ .

Let us recall incidentally that, for example, $\alpha' = \|\alpha'_{ij}\|$, that is, the transposed of α is obtained by interchanging rows and columns and replacing each individual term by its transposed. All this goes back to the rule for matrix multiplication.

III. CONTINUOUS TRANSFORMATIONS OF MANIFOLDS WITH A BOUNDARY

28. Just as in the no-boundary case the definition of a continuous transformation is best given by reference to $M_n \times M'_n$, where M'_n is a copy of M_n . However, before discussing the transformations we shall show that the product is also a manifold. The M'_n image of any M_n configuration will be denoted throughout by the same letter accented.

Let then E_h be any cell of C_n , defining complex of M_n , s_{n-h} the corresponding system such as appears in § 12 in connection with conditions (a'), (b'). We have

$$(28.1) \quad s_{n-h} = E_{n-h} + tE_{n-h-1},$$

where $t=0$ or 1 according as E_h is or is not an interior cell of C_n . To any cell E_k of C'_n will similarly correspond

$$(28.2) \quad s_{n-k} = E_{n-k} + t'E_{n-k-1},$$

where all terms have an obvious meaning. But referring to the definition of the s 's we find that $s_{n-h} \times s_{n-k}$ corresponds in a similar manner to $E_h \times E_k$ as a cell of $C_n \times C'_n$. Now

$$(28.3) \quad \begin{aligned} s_{n-h} \times s_{n-k} &= E_{n-h} \times E_{n-k} + tE_{n-h-1} \times E_{n-k} \\ &\quad + t'E_{n-h} \times E_{n-k-1} + tt'E_{n-h-1} \times E_{n-k-1}. \end{aligned}$$

Each term represents a cell whose dimensions are the sum of those of the factors. Now $E_h \times E_k$ is a boundary cell only when one of the t 's is not zero. When both are zero, (a') is manifestly satisfied, and when, say, $t=1, t'=0$, (b') is satisfied, as they should be. Let, then, $t=t'=1$. We must show that $E_{n-h-1} \times E_{n-k} + E_{n-h} \times E_{n-k-1} + E_{n-h-1} \times E_{n-k-1}$ is homeomorphic to a $2(n-h-k-1)$ -cell. It will be remembered that, by condition (b'), E_{n-k} is homeomorphic to the interior of a hemisphere in S_{n-k} with E_{n-k-1} as the flat base of the hemisphere. It follows that the first term is of the same type for an $S_{2n-h-k-1}$ and similarly for the second with the third as the common flat base. The sum is then homeomorphic to the interior of a sphere in the same space, that is to a cell, which completes the proof.

Remarks. I. The same proof holds for a product $M_p \times M_q$, $p \neq q$.

II. Since the factors are orientable this is also true for the product (Tr., No. 49).

29. We now define a continuous transformation T of M_n as in Tr., No. 56, by the condition that the set $\{A \times B\} = K_n$ be a C_n of $M_n \times M'_n$ with its boundary on that of the product. Let T' be another transformation, K'_n its complex. The problem is again to determine $(K_n \cdot K'_n)$ (number of signed coincidences) in terms of the transformations induced by T, T' on the cycles, or of similar data (i. e., information naturally at hand when T, T' are known).

No result of any generality is to be expected unless K_n and K'_n are so restricted that the boundaries of suitably defined approximations do not intersect. Indeed, unless this is so, $(K_n \cdot K'_n)$ ceases to be an invariant of classes of transformations. We assume of course throughout that M_n is connected, but F_{n-1} need not be so. Let $F_{n-1}^1, \dots, F_{n-1}^p$ be its connected parts. By Tr., (49.2), the boundary of $M_n \times M'_n$ is the sum of the products $F_{n-1}^i \times M'_n$, $(-1)^i M_n \times F'_{n-1}^i$, each of which is a manifold (§ 28). The

boundary of each K consists of subcomplexes distributed among some or all of these 2^p manifolds. The least that we can exact is that none carry boundary points of both K 's in its interior. However, this is still beyond the reach of the method to be used here, at least in this general form. *We shall therefore restrict our discussion to pairs of transformations such that the boundary of one K is on $F_{n-1} \times M'_n$, while that of the other is on $M_n \times F_{n-1}$, one of the boundaries being actually interior to the carrying manifold.* This will greatly simplify matters, sufficiently indeed to compensate amply for whatever may be lost in generality. In § 32 we shall give a topological interpretation of these types of transformations by means of certain approximating transformations.

30. We shall reduce the coincidence and fixed points problems for M_n to similar problems for certain associated transformations of V_n that we now define.

Case I: *T is of the first type*, that is with the boundary Γ_{n-1} of K_n on $F_{n-1} \times M'_n$. According to § 23, Corollary I, it is homologous thereon to Γ'_{n-1} interior to $F_{n-1} \times M'_n$ and satisfying in all respects the conditions there stated. If beyond this point we apply the same reductions as before (loc. cit.) to K_n itself, we shall reduce it to a complex H_n bounded by Γ'_{n-1} with its non-boundary cells all interior to $M_n \times M'_n$ and behaving in every respect in accordance with the theorem of § 23. If C_μ of M_n has its boundary on F_{n-1} we first reduce it as in § 23; then its transform TC_μ is determined as in Tr., No. 58, with C_μ in place of γ_μ and H_n in place of Γ_n (loc. cit.). As a special case C_μ may be a γ_μ ; then it is first reduced to the interior of M_n and $T\gamma_\mu$ is then determined in the same way.

Since $V_n \times V'_n = M_n \times V'_n - \bar{M}_n \times V'_n$, it may be derived from $M_n \times V'_n$ as V_n from M_n . The two parts of the manifold are now matched along their common boundary $F_{n-1} \times V'_n$, and their points associated in conjugate pairs $A \times B$ and $\bar{A} \times B$. To K_n and H_n there correspond in this fashion associated skew-symmetric cycles $K_n - \bar{K}_n \sim H_n - \bar{H}_n = \Gamma_n$, and the latter will serve to define the transformation T_1 of V_n associated with T .

Since the B points describe on V'_n the images of the transforms of the loci of the A points, we have, by Tr., No. 58,

$$(30.1) \quad T_1 \bar{C}_\mu = T_1 C_\mu = T C_\mu ; \quad T_1 (C_\mu - \bar{C}_\mu) = 0.$$

Furthermore, $T_1 \gamma_\mu = T \gamma_\mu$ since K'_n alone comes into play in determining the two transforms. Hence T_1 transforms cycles of M_n into cycles of M_n and skew-symmetric cycles into bounding cycles.

Let the transformation matrix for the cycles Γ_μ, G_μ of the fundamental sets be

$$P_\mu = \begin{array}{c|cc} \Gamma_\mu & G_\mu \\ \hline P_{11}^\mu & P_{12}^\mu \\ P_{21}^\mu & P_{22}^\mu \end{array} \Gamma_\mu \quad G_\mu.$$

It is the transformation matrix for the cycles of M_n and T , the transformed cycles being given as \approx to a combination of the initial cycles. The similar matrix for V_n is

$$\mu < \frac{1}{2}n, \alpha_\mu = \begin{array}{c|ccc} \Gamma & \Delta & D & G \\ \hline P_{11}^\mu & 0 & 0 & P_{12}^\mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ P_{21}^\mu & 0 & 0 & P_{22}^\mu \end{array} \Gamma \quad \begin{array}{c|ccc} \Gamma & \Delta & D & G \\ \hline P_{11}^\mu & 0 & P_{12}^\mu & 0 \\ 0 & 0 & 0 & 0 \\ P_{21}^\mu & 0 & P_{22}^\mu & 0 \\ 0 & 0 & 0 & 0 \end{array} \Delta ; \quad \mu \geq \frac{1}{2}n, \alpha_\mu = \begin{array}{c|ccc} \Gamma & \Delta & G & D \\ \hline P_{11}^\mu & 0 & P_{12}^\mu & 0 \\ 0 & 0 & 0 & 0 \\ P_{21}^\mu & 0 & P_{22}^\mu & 0 \\ 0 & 0 & 0 & 0 \end{array} \Gamma \quad \begin{array}{c|ccc} \Gamma & \Delta & G & D \\ \hline P_{11}^\mu & 0 & P_{12}^\mu & 0 \\ 0 & 0 & 0 & 0 \\ P_{21}^\mu & 0 & P_{22}^\mu & 0 \\ 0 & 0 & 0 & 0 \end{array} \Delta \quad \begin{array}{c|ccc} \Gamma & \Delta & G & D \\ \hline P_{11}^\mu & 0 & P_{12}^\mu & 0 \\ 0 & 0 & 0 & 0 \\ P_{21}^\mu & 0 & P_{22}^\mu & 0 \\ 0 & 0 & 0 & 0 \end{array} G \quad \begin{array}{c|ccc} \Gamma & \Delta & G & D \\ \hline P_{11}^\mu & 0 & P_{12}^\mu & 0 \\ 0 & 0 & 0 & 0 \\ P_{21}^\mu & 0 & P_{22}^\mu & 0 \\ 0 & 0 & 0 & 0 \end{array} D$$

31. Case II: T is of the second type, or with its boundary on $M_n \times F'_{n-1}$. In this case H_n will have its boundary interior to $M_n \times F'_{n-1}$. Then since $V_n \times V'_n = V_n \times M'_n - V_n \times \bar{M}'_n$, the left side is to be considered as obtained by matching the two manifolds at the right whose common boundary is $V_n \times F'_{n-1}$, and $A \times B, A \times \bar{B}$ constitute the conjugate pairs. Again $K_n - \bar{K}_n \sim H_n - \bar{H}_n = \Gamma_n$, cycle which serves to define T_1 .

We find now that $T_1 C_\mu$ consists of $C'_\mu = T C_\mu$ and $-\bar{C}'_\mu$, the second complex having a minus sign because its orientation is determined by means of $-\bar{K}'_n$. Also no cell on \bar{M}_n has any transform. It follows that every cycle of V_n is transformed into a skew-symmetric cycle by T_1 . More explicitly, let the cycle first polyhedrally approximated be $\gamma_\mu = C'_\mu + \bar{C}''_\mu$, where the first complex is on M_n , the second on \bar{M}_n . Then $T_1 \gamma_\mu = T C'_\mu - (\bar{T} C'_\mu)$.

The transformation matrix is given below:

$$\mu < \frac{1}{2}n, \alpha_\mu = \begin{array}{c|cccc} \Gamma & \Delta & D & G \\ \hline 0 & R_{11}^\mu & R_{12}^\mu & 0 \\ 0 & Q_{11}^\mu & Q_{12}^\mu & 0 \\ 0 & Q_{21}^\mu & Q_{22}^\mu & 0 \\ 0 & R_{21}^\mu & R_{22}^\mu & 0 \end{array} \Gamma \quad \begin{array}{c|cccc} \Gamma & \Delta & D & G \\ \hline 0 & R_{11}^\mu & R_{12}^\mu & 0 \\ 0 & Q_{11}^\mu & Q_{12}^\mu & 0 \\ 0 & Q_{21}^\mu & Q_{22}^\mu & 0 \\ 0 & R_{21}^\mu & R_{22}^\mu & 0 \end{array} \Delta ; \quad \begin{array}{c|cccc} \Gamma & \Delta & D & G \\ \hline 0 & R_{11}^\mu & R_{12}^\mu & 0 \\ 0 & Q_{11}^\mu & Q_{12}^\mu & 0 \\ 0 & Q_{21}^\mu & Q_{22}^\mu & 0 \\ 0 & R_{21}^\mu & R_{22}^\mu & 0 \end{array} D \quad \begin{array}{c|cccc} \Gamma & \Delta & D & G \\ \hline 0 & R_{11}^\mu & R_{12}^\mu & 0 \\ 0 & Q_{11}^\mu & Q_{12}^\mu & 0 \\ 0 & Q_{21}^\mu & Q_{22}^\mu & 0 \\ 0 & R_{21}^\mu & R_{22}^\mu & 0 \end{array} G$$

$$\mu \geq \frac{1}{2}n, \quad \alpha_\mu = \begin{array}{c} \begin{array}{cccc} \Gamma & \Delta & G & D \\ \hline 0 & R_{11}^\mu & 0 & R_{12}^\mu \\ 0 & Q_{11}^\mu & 0 & Q_{12}^\mu \\ 0 & R_{21}^\mu & 0 & R_{22}^\mu \\ 0 & Q_{21}^\mu & 0 & Q_{22}^\mu \end{array} \\ \Gamma \\ \Delta \\ G \\ D \end{array}$$

The matrix $Q = ||Q_{ij}^\mu||$ is the transformation matrix for skew-symmetric cycles, and alone will appear in the final formulas.

32. The complex H_n defines for both types an approximation \bar{T} to the given T such that (a) when T is of the first type, $\bar{T}M_n$ is wholly interior to M_n ; (b) when T is of the second type, $\bar{T}F_{n-1}$ does not exist, i.e., the boundary belongs to the set of points that have no \bar{T} transform. This may be considered as a geometric characterisation of the two types.

33. Let us now assume that T is of type I, T' of type II and represent the cycle and complexes attached to T' by the same letters as for T with primes. We are explicitly assuming (§ 29) that K and K' intersect only in interior points of $M_n \times M'_n$. Hence $(K_n \cdot K'_n)$ is perfectly determined and by definition equal to $(H_n \cdot H'_n)$ (Tr., No. 35). Therefore, at once,

$$(33.1) \quad (\Gamma_n \cdot \Gamma'_n) = ((H_n - \bar{H}_n) \cdot (H'_n - \bar{H}'_n)) = (H_n \cdot H'_n) = (K_n \cdot K'_n),$$

for when a term of a pair H, H' is barred the two complexes do not meet. Hence *the coincidence problem for T, T' is reduced to the same problem for T_1, T'_1* .

IV. COINCIDENCE AND FIXED POINTS FORMULAS

34. We have just shown that these formulas are the same for M_n as for the associated transformations of V_n . We apply then (10.4), taking for T_1 the transformation corresponding to the α 's, for T'_1 that corresponding to the β 's, and we have

$$(34.1) \quad (K_n \cdot K'_n) = (\Gamma_n \cdot \Gamma'_n) = \sum (-1)^\mu \text{trace } L_\mu \beta'_{n-\mu} L_\mu^{-1} \alpha_\mu.$$

We now assume the fundamental sets in the canonical form of § 27 and carry out the computation. The matrices in the product have the same structure, with four square submatrices in the principal diagonal in the same order. L_μ^{-1} for $\mu \neq \frac{1}{2}n$ odd, is like L_μ except that

$$\begin{vmatrix} A & 1 \\ 1 & 0 \end{vmatrix} \text{ is replaced by } \begin{vmatrix} 0 & 1 \\ 1-A & \end{vmatrix}.$$

For $\mu = \frac{1}{2}n$ odd,

$$(34.2) \quad L_{n/2}^{-1} = \begin{vmatrix} 0 & -1 & & \\ 1 & A & & \\ & & 0 & \\ & & 0 & -1 \\ \hline 0 & & 0 & \\ & 1 & & 0 \end{vmatrix}.$$

By way of illustration, we examine the computation for $\mu < \frac{1}{2}n$. Then, dropping the indices μ and $n-\mu$ for the present,

$$(34.3) \quad \begin{aligned} L\beta' &= \begin{vmatrix} A & 1 & 0 & \\ 1 & 0 & & \\ & & 1 & \\ \hline 0 & & & \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & 0 & 0 \\ R'_{11} & Q'_{11} & R'_{21} & Q'_{21} \\ 0 & 0 & 0 & 0 \\ R'_{12} & Q'_{12} & R'_{22} & Q'_{22} \end{vmatrix} \\ &= \begin{vmatrix} R'_{11} & Q'_{11} & R'_{21} & Q'_{21} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ R'_{12} & Q'_{12} & R'_{22} & Q'_{22} \end{vmatrix}; \end{aligned}$$

$$(34.4) \quad \begin{aligned} L^{-1}\alpha &= \begin{vmatrix} 0 & 1 & 0 & \\ 1 & -A & & \\ & & 1 & \\ \hline 0 & & & \end{vmatrix} \cdot \begin{vmatrix} P_{11} & 0 & 0 & P_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ P_{21} & 0 & 0 & P_{22} \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 & 0 & \\ P_{11} & 0 & 0 & P_{12} \\ 0 & 0 & 0 & 0 \\ P_{21} & 0 & 0 & P_{22} \end{vmatrix}. \end{aligned}$$

Hence

$$(34.5) \quad L\beta'L^{-1}\alpha = \begin{vmatrix} Q'_{11}P_{11} + Q'_{21}P_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Q'_{12}P_{12} + Q'_{22}P_{22} \end{vmatrix}$$

where the terms not in the main diagonal are omitted and need not be computed, as unnecessary for the trace. It follows that

$$(34.6) \quad \text{trace } L_\mu \beta'_{n-\mu} L_\mu^{-1} \alpha_\mu = \text{trace } \sum Q'_{ij} n^{-\mu} P_{ij}^\mu = \text{trace } Q'_{n-\mu} P_\mu,$$

therefore

$$(34.7) \quad (K_n \cdot K'_n) = \sum (-1)^\mu \operatorname{trace} Q'_{n-\mu} P_\mu$$

is the desired coincidence formula. Its similarity to (10.4) for the non-boundary case is very striking.

35. The preceding formula corresponds to highly specialized fundamental sets. Let us consider the more general case where they are merely composed of the same four types Γ, Δ, G, D , in the same order as before for each μ , but no further specialized than is demanded by the condition that all indices $(\Gamma \cdot D), (D \cdot \Gamma)$ be zero. This amounts to requiring that $C = M = 0$, where the two matrices are as in § 25. The passage from a canonical set to the more general type is by means of transformations operating separately on each of the four types. Then

$$(35.1) \quad \begin{aligned} L &= \left\| \begin{array}{cc|c} A & B & 0 \\ H & 0 & \\ \hline 0 & F & 0 \\ 0 & 0 & N \end{array} \right\| \quad (\mu < \frac{1}{2}n), \\ &= \left\| \begin{array}{cc|c} A & B & 0 \\ H & 0 & \\ \hline 0 & N & 0 \\ 0 & 0 & F \end{array} \right\| \quad (\mu > \frac{1}{2}n), \\ &= \left\| \begin{array}{cc|c} A & B & 0 \\ (-1)^{n/2}B & 0 & \\ \hline 0 & 0 & F \\ 0 & (-1)^{n/2}F & 0 \end{array} \right\| \quad (\mu = \frac{1}{2}n), \end{aligned}$$

where all terms have the subscript μ omitted for the sake of simplicity. It turns out that the only terms needed are

$$(35.2) \quad B_\mu = \|(\Gamma_\mu^i \cdot \Delta_{n-\mu}^j)\|, \quad F_\mu = \|(D_\mu^i \cdot G_{n-\mu}^j)\|.$$

Then, either by computing directly as before or else from (34.7), we derive the equivalent invariant form:

$$(35.3) \quad \begin{aligned} (K_n \cdot K'_n) &= \sum (-1)^\mu \operatorname{trace} (B_\mu Q'_{n-\mu} B_\mu^{-1} P_{11} \\ &\quad + BQ'_{21} F^{-1} P_{21} + FQ'_{12} B^{-1} P_{12} + FQ'_{22} F^{-1} P_{22}), \end{aligned}$$

where the dimensionality indices, the same for all four terms at the right, are written only for the first. This is the desired generalization of (34.7).

36. We pass now to the derivation of the *fixed point formulas*. We assume the fundamental sets general in the sense of § 35.

I. *T is of the first type.* Since the identity is of each of the two types, we now assign it to the second type, approximate its K , say K_n^0 , by H_n^0 with its boundary on $M_n \times F_{n-1}'$, and interpret the problem as the determination of $(K_n \cdot K_n^0)$. It is in fact exactly that when T actually reduces F_{n-1} to the interior of M_n .

We shall so sense K_n^0 that the corresponding integer θ of § 8, here the same at every point of K_n^0 , is +1. Then it is immediately seen that the corresponding $Q_\mu = I$.

Whence

$$(36.1) \quad Q_{12}^{n-\mu} = Q_{21}^{n-\mu} = 0, \quad Q_{11}^{n-\mu} = Q_{22}^{n-\mu} = 1.$$

Therefore

$$(36.2) \quad (K_n \cdot K_n^0) = \sum (-1)^\mu \text{trace } P_\mu.$$

II. *T is of the second type.* The discussion is now the same, with the identity ascribed to the first type. H_n^0 determines a deformation of M_n into an interior part of itself, and $(K_n^0 \cdot H_n)$ is then the number of points where T operates as that infinitesimal deformation. In this case

$$(36.3) \quad P_\mu = 1 = P_\mu^{11} = P_\mu^{22}; \quad P_\mu^{12} = P_\mu^{21} = 0.$$

Hence

$$(36.4) \quad (K_n^0 \cdot K_n') = \sum (-1)^\mu \text{trace } (B_\mu Q'_{11}{}^{n-\mu} B_\mu^{-1} + F_\mu Q'_{22}{}^{n-\mu} F_\mu^{-1}).$$

But by well known properties of matrices,

$$(36.5) \quad \text{trace } uv'u^{-1} = \text{trace } v' = \text{trace } v.$$

This together with a change of μ into $n - \mu$ gives

$$(36.6) \quad (K_n^0 \cdot K_n') = (-1)^n \sum (-1)^\mu \text{trace } Q_\mu.$$

37. A particularly interesting case of (36.2) corresponds to the fixed points of a deformation T (infinitesimal or otherwise) of M_n into an interior part of itself. Then the actual number of signed fixed points when finite is given by

$$(37.1) \quad (K_n'^0 \cdot K_n^0) = \sum (-1)^\mu R_\mu,$$

where $K_n'^0$ corresponds to the deformation. Here R_μ is the sum of the traces of P_{11}^μ and P_{22}^μ when they both reduce to the identity, that is, the sum of their orders. This is also the number of distinct Γ 's and G 's in the fundamental set

for the dimension μ ; hence R_μ is the μ th connectivity index of M_n itself, and the sum in (37.1) is its Euler characteristic. This shows that (10.6) holds for an M_n with a boundary also, of course with the K 's in place of the Γ 's. Therefore

THEOREM. *For every M_n , with or without boundary, the number of signed fixed points of a deformation is the Euler characteristic.*

It is understood of course that when M_n has a boundary the deformation reduces M_n to part of itself. It is necessary to point out that instead of a deformation we may equally well consider a T reducing M_n to an interior part of itself and merely adding zero-divisors to the cycles of the manifold.

The question of the *singular points of a vector distribution* is equivalent to the determination of the fixed points of an infinitesimal deformation. The authors dealing with it, notably Brouwer, Birkhoff, and recently Hopf,* have always restricted their manifolds more than in this paper. Hopf, for example, assumes that at every vertex of the defining C_n of M_n the incident cells constitute a star with the same structure as some star embedded in an S_n . Then, whatever the point A , there exists a region containing it wherein any two points may be joined by a uniquely defined polygonal line, image of a rectilinear segment in the S_n region. Until it is actually proved, as may be done for $n=2, 3$, that *every star of cells which is an n -cell has the same structure as some star in S_n* , our manifolds must be considered as much more general, and our results as having a notably wider range. On the other hand it must be stated that all analytical manifolds are of the more restricted type, so that for various applications the restriction may actually not be important.

38. The fixed point formulas derived from the general coincidence formula may also be obtained directly and in a very simple manner. Indeed, instead of associating with the identity the transformations of §§ 30, 31, we may associate with it the identical transformation for V_n also. The corresponding cycle Γ_n^0 is on $M_n \times M'_n$ and $\bar{M}_n \times M'_n$, hence at once for type I,

$$(38.1) \quad (K_n \cdot K_n^0) = (\Gamma_n \cdot \Gamma_n^0),$$

and for type II

$$(38.2) \quad (K_n^0 \cdot K'_n) = (\Gamma_n^0 \cdot \Gamma'_n).$$

* Mathematische Annalen, vol. 96 (1926), pp. 225–250. He has derived (37.1) for a vector distribution on the general M_n of the type that he considers. His seem to be the only investigations on general manifolds along the line of this paper that are to be found in the literature. In Mathematische Annalen, vol. 95 (1925), he has generalized in an interesting manner the well known topological property of the total Gaussian curvature.

Then from (10.5) the desired results follow easily.

39. Transformation of one manifold into another. The transformations of M_n into a new manifold M'_n are treated exactly like those of M_n into itself. First let both be without boundary and let \bar{L}_μ be the analogue of L_μ for M'_n . The transformation of M_n into M'_n will be defined by cycles Γ_n, Γ'_n on $M_n \times M'_n$ and (59.2), Tr., will hold.

The expression for $(\Gamma_n \cdot \Gamma'_n)$ will be again (10.4) with the first factor, L , in each term replaced by \bar{L} , and the others unchanged or*

$$(39.1) \quad (\Gamma_n \cdot \Gamma'_n) = \sum (-1)^\mu \text{trace } \bar{L}_\mu \beta'_{n-\mu} L_\mu^{-1} \alpha_\mu.$$

When there are boundaries, the types of transformations must again be restricted as previously. If \bar{B}, \bar{F} correspond to \bar{L} as B, F to L , we find here the same coincidence formula (35.3) as before, except that B, F are replaced by \bar{B}, \bar{F} , everything else (notably B^{-1}, F^{-1}) remaining unchanged. It does not seem necessary to write the formula explicitly.

40. A different type of coincidence formula. In these Transactions, vol. 25 (1923), Alexander has derived for 1- τ transformations of surfaces a formula of a different type from ours. The difference consists in the fact that in place of the transformation matrices there appear everywhere the Kronecker index matrices for the intersections of the $n-\mu$ cycles with the transforms of the μ cycles. Let us show that such formulas can be derived from those of the present paper.

Let first M_n be without boundary and introduce for T the matrix

$$(40.1) \quad \xi_\mu = \| (T \gamma_\mu^i \cdot \gamma_{n-\mu}^j) \|$$

with a similar matrix η_μ for T' . The problem is to express the number of signed coincidences in terms of the ξ 's and η 's. We have

$$(40.2) \quad \begin{aligned} (T \gamma_\mu^i \cdot \gamma_{n-\mu}^j) &= \left(\sum \alpha_\mu^{ih} \gamma_\mu^h \cdot \gamma_{n-\mu}^j \right) \\ &= \sum \alpha_\mu^{ih} (\gamma_\mu^h \cdot \gamma_{n-\mu}^j), \end{aligned}$$

therefore

$$(40.3) \quad \xi_\mu = \alpha_\mu L_\mu; \quad \eta_\mu = \beta_\mu L_\mu.$$

From this and the readily verified relation

$$(40.4) \quad L'_{n-\mu} = (-1)^{\mu(n+1)} L_\mu,$$

* See footnote, §3.

follows in place of (10.6) the desired formula

$$(40.5) \quad (\Gamma_n \cdot \Gamma'_n) = \sum (-1)^{n\mu} \operatorname{trace} \eta'_{n-\mu} L_\mu^{-1} \xi_\mu L_\mu^{-1}.$$

For the fixed points, assume the second transformation to be the identity. Then directly from (10.5) or else from (40.5) and

$$(40.6) \quad \eta_{n-\mu} = L_{n-\mu} = (-1)^{\mu(n+1)} L'_\mu,$$

we find the number of signed fixed points

$$(40.7) \quad (\Gamma_n \cdot \Gamma_n^0) = \sum (-1)^\mu \operatorname{trace} \xi_\mu L_\mu^{-1}.$$

Let $n=2$ as in Alexander's paper. The terms corresponding to $\mu=0, 2$ are the same as in (10.6), namely $\alpha_0 + (-1)^n \alpha_n = \alpha_0 + \alpha_2$. The term $\mu=1$ alone needs to be computed. When the fundamental set is canonical, L_1 is the well known matrix $\|l_{ij}\|$ ($i, j=1, 2, \dots, 2p=R_1$) with $l_{2i-1, 2i} = -l_{2i}, l_{2i-1} = 1$ and all other terms zero. Let $\gamma_1, \dots, \gamma_{2p}$ be the retrosections of the Riemann surface. If we recall that for any two cycles γ, δ on the surface, $(\gamma \cdot \delta) = -(\delta \cdot \gamma)$, and observe that $L'_1 = -L_1$, we find that (40.7) becomes, with $\bar{\gamma} = T\gamma$ as usual,

$$(40.8) \quad (\Gamma_2 \cdot \Gamma_2^0) = \alpha_0 + \alpha_2 + \sum (\bar{\gamma}_{2i-1} \cdot \gamma_{2i}) + (\gamma_{2i-1} \cdot \bar{\gamma}_{2i}),$$

which generalizes Alexander's formula (4), loc. cit., and except for the notation reduces to it when, as he does, we consider $1-\tau$ transformations ($\alpha_0=1, \alpha_2=\tau$).

The treatment for manifolds with a boundary is along the same line. We find

$$(40.9) \quad \begin{aligned} (K_n \cdot K'_n) = & \sum (-1)^{n\mu} \operatorname{trace} (\sigma'_{11}^{n-\mu} B_\mu^{-1} \pi_{11}^\mu B_\mu^{-1} \\ & + \sigma'_{21} N^{-1} \pi_{12} B^{-1} + \sigma'_{12} B^{-1} \pi_{21} N^{-1} \\ & + \sigma'_{22} N^{-1} \pi_{22} N^{-1}), \end{aligned}$$

where the sequence of dimensional indices is the same for all four terms under the sum and therefore has been written down only for the first. As for the various letters their meaning is as follows:

$$(40.10) \quad \begin{aligned} B_\mu &= \|(\Gamma_\mu^i \cdot \Delta_{n-\mu}^j)\|, \quad N_\mu = \|(G_\mu^i \cdot D_{n-\mu}^j)\|, \\ \pi_{11}^\mu &= \|(T\Gamma_\mu^i \cdot \Delta_{n-\mu}^j)\|; \end{aligned}$$

π_{12} , π_{21} , π_{22} are as π_{11} with the pair $\Gamma\Delta$ replaced by $G\Delta$, ΓD and GD respectively, while σ_{ii} is derived from π_{ii} , by substituting for T and μ , T' and $n-\mu$ and permuting the cycles of the corresponding pair.

For the fixed points of T of first type, we take $T'=1$, and it turns out that (40.7) holds also provided that ξ and L correspond to Kronecker index matrices for the fundamental sets of M_n alone, that is, for the intersections of the Γ 's and G 's and their transforms. For fixed points of T' of second type we take $T=1$ and find

$$(40.11) \quad (K_n^0 \cdot K'_n) = (-1)^n \sum (-1)^\mu \operatorname{trace} (B_\mu^{-1} \sigma_{11}^\mu + N_\mu^{-1} \sigma_{22}^\mu).$$

This completes the derivation of the formulas of Alexander's type from those of this paper. The type to be used in any particular case will depend upon the available data on the transformations.

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