ON THE "THIRD AXIOM OF METRIC SPACE"*

BY

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1. In his thesis Fréchet† defines a *metric space*‡ or class (D) as a class of elements, the relations among which are established by means of a function of pairs of elements of this class. For any two elements x and y of class (D) this function $\rho(x, y)$ must satisfy three requirements, which we shall call *the axioms of metric space.*

AXIOM I. (Coincidence Axiom.) $\rho(x, y)=0$, *when and only when* $x=y$.

AXIOM II. (Axiom of Symmetry.) $\rho(x, y)=\rho(y, x)$.

AXIOM III. (Triangle Axiom.) $\rho(x, y)=\rho(y, z)+\rho(z, x)$.

It is evident that the metric spaces are cases of the topological spaces of Hausdorff.§ In addition, Fréchet has considered spaces or classes of elements which he calls classes (E). These are classes of elements, the relations among which are established by means of a function $\delta(x, y)$ satisfying the coincidence axiom and the axiom of symmetry. A class (E) which is also a topological space of Hausdorff will be called a *symmetric space.*

It is the purpose of the present paper to present a generalization of the seventh theorem of a joint paper by A. D. Pitcher and E. W. Chittenden‖ concerning the investigation of Axiom III. These authors considered spaces defined by functions $\delta(x, y)$ satisfying Axioms I and II together with one or more of the following three conditions:

(1) (Ch)¶ $\quad \lim \delta(x_n, x) = 0, \quad \lim \delta(x_n, y_n) = 0, \quad \text{imply} \lim \delta(y_n, x) = 0;$

(2) $\quad \lim \delta(x_n, x) = 0, \quad \lim \delta(y_n, x) = 0, \quad \text{imply} \lim \delta(x_n, y_n) = 0;$

(3) $\quad \lim \delta(x_n, y_n) = 0, \quad \lim \delta(y_n, z_n) = 0, \quad \text{imply} \lim \delta(x_n, z_n) = 0;$

where $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences of elements of the given class (E). The theorem cited may be stated in the following form: A *compact* coherent class (E) is a compact metric space.**

* Presented to the Society, September 9, 1926; received by the editors in September, 1926.

† *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906).

‡ The term is due to Hausdorff; *Grundzüge der Mengenlehre*, Leipzig, Veit, 1914, p. 211.

§ Loc. cit., p. 213.

‖ These Transactions, vol. 19 (1918).

¶ A space in which this condition is satisfied is *coherent* in the terminology of Pitcher and Chittenden.

** That is, it is possible to define in terms of the given function $\delta(x,y)$ an infinitesimally equivalent function $\rho(x, y)$ which satisfies the three metric axioms.

The theorem to be proved is as follows.

THEOREM. *A symmetric space in which the condition* (Ch) *is satisfied is a metric space.*

I shall give two demonstrations of this theorem: one, based entirely on the method developed by Pitcher and Chittenden; another, based on the methods and results of the Russian school.

2. LEMMA. *If the condition* (Ch) *is satisfied in a class* (E) *the following condition is also satisfied:**

$$\lim \delta_1(x_n, y_n) = 0, \quad \lim \delta_1(y_n, z_n) = 0, \quad \textit{imply} \lim \delta_1(x_n, z_n) = 0.$$

Let x and y be two arbitrary elements of the given class(E). Let $\delta(x, y) = \eta$. Consider the following: (1) the points of type z' satisfying the conditions

$$\delta(x, z') \leqq \eta, \quad \delta(y, z') > 2\eta;$$

and set $d_1 = \lim\sup \delta(y, z')$; (2) the points of type z'' satisfying the conditions

$$\delta(y, z'') \leqq \eta, \quad \delta(x, z'') > 2\eta;$$

and set $d_2 = \lim\sup \delta(x, z'')$.† If we write $d_0 = (d_1 + d_2)/2$, it follows immediately that $d_0 \geqq \eta$.

Denote d_0 by $\delta_1(x, y)$. It is evident that $\delta_1(x, y) \geqq \delta(x, y)$. Then, to show the equivalence of the functions $\delta_1(x, y)$, $\delta(x, y)$, it is sufficient to prove that

$$\lim \delta(x_n, x) = 0 \quad \text{implies} \quad \lim \delta_1(x_n, x) = 0.$$

Suppose this is not true. Then there is an element x and a sequence $\{x_n\}$ such that

$$\lim \delta(x_n, x) = 0, \quad \delta_1(x_n, x) \geqq \eta > 0 \quad (n = 1, 2, 3, \cdots).$$

Therefore by the definition of $\delta_1(x, y)$ there exist points z_n which are either of type z' for infinitely many integers n or of type z''.

If infinitely many of the points z are of type z', a sequence $\{z_{n_i}\}$ exists such that

$$\lim \delta(x_{n_i}, z'_{n_i}) = 0, \quad \lim \delta(x_{n_i}, x) = 0,$$
$$\delta(z'_{n_i}, x) \geqq \eta > 0 \qquad (i = 1, 2, 3, \cdots).$$

But this contradicts the condition of the lemma.

* Relative to a different function $\delta_1(x, y)$, equivalent to $\delta(x, y)$.

† In case there are no points z', z'' of these types, let $\delta_1(x, y) = \delta(x, y)$.

If infinitely many of the points z_n are of the type z'' there exists a sequence $\{z'_{n_i}\}$ such that

$$\lim \delta(x_{n_i}, x) = 0,\ \lim \delta(z''_{n_i}, x) = 0,\ \delta(x_{n_i}, z''_{n_i}) \geqq \eta > 0 \qquad (i = 1,2,3, \cdots).$$

But this is also impossible by condition (Ch) and a theorem of Pitcher and Chittenden.*

The equivalence of the functions $\delta_1(x, y)$ and $\delta(x, y)$ is established.

It will now be shown that the function $\delta_1(x, y)$ satisfies the condition of the lemma. Suppose that it does not, then two cases may occur.

(i) There are sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$, which do not satisfy the requirement of the theorem either by the old definition of the distance function or by the new one. Then we have

$$\lim \delta(x_n, y_n) = 0, \quad \lim \delta(y_n, z_n) = 0, \quad \delta(x_n, z_n) > \eta \qquad (n = 1,2,3, \cdots);$$

$$\lim \delta_1(x_n, y_n) = 0, \quad \lim \delta_1(y_n, z_n) = 0, \quad \delta_1(x_n, z_n) > \eta \qquad (n = 1,2,3, \cdots).$$

Let us take N large enough to have

$$\delta_1(x_n, y_n) \leqq \eta/2, \qquad \delta_1(y_n, z_n) \leqq \eta/2 \qquad (n \geqq N).$$

The same inequalities are satisfied by $\delta(\leqq \delta_1)$, that is,

$$\delta(x_n, y_n) \leqq \eta/2, \quad \delta(y_n, z_n) \leqq \eta/2 \qquad (n \geqq N).$$

There are two sub-cases:

$$(1) \qquad \delta(x_n, y_n) \leqq \delta(y_n, z_n); \qquad (2)\ \delta(x_n, y_n) > \delta(y_n, z_n).$$

In the first case we denote $\delta(y_n, z_n)$ by ϵ' and obtain

$$\delta(y_n, z_n) = \epsilon', \ \delta(x_n, y_n) \leqq \epsilon', \ \delta(x_n, z_n) > \eta \geqq 2\epsilon'.$$

It then follows from the definition of the function $\delta_1(x, y)$ that $\delta_1(x_n, y_n) > \eta$, a contradiction.

If we denote $\delta(x_n, y_n)$ in the second case by ϵ'' a similar contradiction is obtained.

(ii) There exist sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, which satisfy the conditions of the theorem by the old definition of the distance function, but not by the new one. Then we should have

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* Loc. cit., Theorem 1. This theorem is equivalent to the following statement: In every coherent class (E) it is possible to define the function $\delta(x, y)$ so that condition (2) is satisfied.

$\lim \delta(x_n, y_n) = 0, \ \lim \delta(y_n, z_n) = 0, \ \lim \delta(x_n, z_n) = 0;$

$\lim \delta_1(x_n, y_n) = 0, \ \lim \delta_1(y_n, z_n) = 0, \ \delta_1(x_n, z_n) > \eta > 0 \quad (n = 1, 2, 3, \cdots).$

It follows that a sequence $\{w_{n_i}\}$ exists such that

$$\lim \delta(x_{n_i}, w_{n_i}) = 0, \quad \delta(w_{n_i}, z_{n_i}) > 0 > \eta \quad (i = 1, 2, 3, \cdots),$$

or vice versa. Because of the complete symmetry, we need consider only the first case. About the sequence $\{w_{n_i}\}$ we can make the two following suppositions:

$$(1) \qquad \lim \delta(w_{n_i}, y_{n_i}) = 0; \quad (2) \ \delta(w_{n_i}, y_{n_i}) > \eta > 0 \ (i = 1, 2, 3, \cdots).$$

Let us consider them separately. In the first case it follows readily from the definition of $\delta_1(x, y)$ that

$$\limsup \delta_1(y_{n_i}, z_{n_i}) \geqq \eta > 0,$$

which contradicts the hypothesis. In the second case we have similarly

$$\limsup \delta_1(x_{n_i}, y_{n_i}) \geqq \eta > 0.$$

This completes the proof of the lemma.

Pitcher and Chittenden have proved that the distance function of the lemma is equivalent to a uniformly regular écart.* It therefore follows from the lemma and the fundamental result of Chittenden† on the equivalence of "écart régulier" and distance that the theorem proposed is completely proved.‡

3. In this section the theorem of § 1 is demonstrated by means of new methods and with a new formulation of special interest.

Definition. *A topological space R satisfies the local axiom of the triangle, if for every element x and positive number ϵ a number η_x may be found such that*

$$\text{(F)} \qquad \delta(x, y) \leqq \eta_x, \quad \delta(x, y) \leqq \eta_x \ \textit{imply} \ \delta(x, z) \leqq \epsilon.$$

To facilitate the following proof we write the condition (F) in the following form.

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* Theorem 4, loc. cit.

† These Transactions, vol. 18 (1917).

‡ In the statement of this theorem we have supposed that the class (*E*) is topological in the sense of Hausdorff, but we have not used this condition. It is not difficult to prove the following theorem: *A class* (*E*) *satisfying the condition* (Ch) *is a topological space of Hausdorff.*

For every element x and positive number ϵ there exists an η_x and a region G_x such that if $\delta(x, y) < \eta_x$ and $\delta(y, z) < \eta_x$, then*

$$z \subset G_x \subset S(x, \epsilon). \dagger \qquad \text{(N)}$$

THEOREM. *Every symmetric space R in which the local axiom of the triangle is satisfied is a metric space.*

From condition (N) there exists for every point of the space R a sequence of spheres

$$S_1^x, S_2^x, S_3^x, \cdots, S_k^x = S(x, \epsilon_k^x), \cdots \qquad (\lim \epsilon_k^x = 0),$$

satisfying the following condition:

(B) *Every sphere S_i^x contains a region G_i^x such that for every point y of S_{i+1}^x, $S(y, \epsilon_{i+1}^x) \subset G_i^x$.*

The families of regions

$$\Pi_1 = (G_1^x), \Pi_2 = (G_2^x), \cdots, \Pi_k = (G_k^x), \cdots$$

form a sequence of coverings of the space R whose properties we shall investigate.‡

Let z be an element of $G_k^x \cdot G_k^y$. Then for the corresponding spheres S_k^x and S_k^y, we have $z \subset S_k^x \cdot S_k^y$. Let the radius of S_k^x be γ_1, and the radius of S_k^y be γ_2, and let the notation x, y be chosen so that $\gamma_2 \leqq \gamma_1$.

By the axiom of symmetry we have

$$\delta(z, x) < \gamma_1, \qquad \delta(z, y) < \gamma_2 \leqq \gamma_1.$$

If we describe about the point z a sphere of radius γ_1 it will contain the points x and y. From condition (B) we have

$$S_{k-1}^x \supset S_k^x + S(z, \gamma_1).$$

Let the radius of S_{k-1}^x be γ_3 $(\gamma_2 \leqq \gamma_1 < \gamma_3)$. Describe about the point y a sphere of radius γ_3. This sphere will by construction include the sphere S_k^y. A second application of condition (B) shows that

$$S_{k-2}^x \supset S_{k-1}^x + S(y, \gamma_3).$$

* By a *region* we understand the complement of a closed set.

† Consider the sphere $S(x, \epsilon)$ of center x and radius ϵ. As the space R is topological in the sense of Hausdorff, there exists a region G_x which contains x and is a subset of $S(x, \epsilon)$. Likewise, G_x contains a sphere $S(x, \epsilon')$. Let us choose, in accordance with condition (F), a number η_x such that if $\delta(x, y) \leqq \eta_x$, $\delta(y, z) \leqq \eta_x$, then $\delta(x, z) < \epsilon'$. The number η_x evidently satisfies condition (N).

‡ P. Alexandroff and P. Urysohn (Comptes Rendus, vol. 177, p. 1274) have defined a *covering* as a collection of regions such that every point of the space belongs to at least one of them.

Furthermore, there exists a region G_{k-2}^x such that

$$\text{(K)} \qquad S_{k-2}^x \supset G_{k-2}^x \supset S_k^x + S_k^y \supset G_k^x + G_k^y.$$

We now consider the chain of coverings

$$\Pi_1', \Pi_2', \cdots, \Pi_k', \cdots$$

obtained by setting

$$\Pi_k' = \Pi_{2k+1}.$$

It will be shown that this chain is regular and complete.* That the chain is regular follows from its construction. Let us prove it complete.

Let $G_1, G_2, \cdots, G_k, \cdots$ be a sequence of regions such that G_k contains a point x for every value of k, and G_k belongs to the covering Π_k'. Since each of the sets G_k is a region, every G_k contains a neighborhood of the point x. Suppose that U is a neighborhood of x which contains no set G_k. Consider the sequence of spheres $S_k^{y_k}$ of radius ϵ_k ($\lim \epsilon_k = 0$), such that

$$G_k^{y_k} \subset S_k^{y_k}.$$

Then $\lim \delta(x, y) = 0$, obviously, and therefore y_k must, for sufficiently large values of k, belong to U. Let z_k be a point of G_k, which does not belong to U. Since U is a neighborhood of the point x it contains a sphere $S(x, \epsilon)$. We have $\delta(z_k, x) \geqq \epsilon$. But $\lim \delta(x, y_k) = 0$, $\lim \delta(y_k, z_k) = 0$. This contradicts condition (F).

Since P. Alexandroff and P. Urysohn have shown† that every topological space of Hausdorff which admits a complete and regular chain of coverings is a metric space the proof of the theorem is complete.

To complete the proof of the theorem of § 1 it is required to show that the condition (Ch) is equivalent to the local axiom of the triangle.

If condition (Ch) is fulfilled, condition (F) is also satisfied. Otherwise there exist a point x and a positive number ϵ, such that for every small positive number η there will exist points y and z satisfying the inequalities

$$\delta(x, y) < \eta, \quad \delta(y, z) < \eta, \quad \delta(x, y) \geqq \epsilon.$$

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* P. Alexandroff and P. Urysohn (loc. cit.) have called a chain of coverings *complete* if for any point x of the space R, and any regions $V_1, V_2, \cdots, V_n, \cdots$ containing x and belonging to $\Pi_1, \Pi_2, \cdots, \Pi_n, \cdots$ respectively the sequence $\{V_n\}$ defines the point x in R, that is, it is a complete system of neighborhoods of the point x; and a chain of coverings *regular* if the following condition is fulfilled: for every integer n and for arbitrary regions V_n and W_n of a covering Π_n there exists in Π_{n-1} a region V_{n-1} containing both V_n and W_n.

† Loc. cit.

Choose $\eta_1, \eta_2, \eta_3, \cdots, \eta_n, \cdots$ ($\lim \eta_n = 0$), $y_1, y_2, \cdots, y_n, \cdots$ and $z_1, z_2, \cdots, z_n, \cdots$ correspondingly, so that we will have

$$\limsup \delta(x, y_n) = 0, \quad \limsup \delta(y_n, z_n) = 0, \quad \limsup \delta(x, z_n) \geqq \epsilon.$$

Then the condition (Ch) is not satisfied as we assumed.

Suppose that condition (F) is satisfied and that condition (Ch) is not. Then there are a point x, a positive number ϵ, and a pair of sequences $\{y_n\}$, $\{z_n\}$ of points such that

$$\limsup \delta(x, y_n) = 0, \quad \limsup \delta(y_n, z_n) = 0, \quad \limsup \delta(x, z_n) \geqq \epsilon.$$

For this value of ϵ one cannot choose a number η_x to satisfy condition (F), contrary to hypothesis.

This completes the proof.

But the investigation of the third axiom is not completed, for those topological conditions which imply the local axiom of the triangle are left undetermined. I intend to investigate this question more closely in my next paper.

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