CONCERNING ACYCLIC CONTINUOUS CURVES*

BY

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In this paper, we propose to use the word *acyclic* in place of the phrase *containing no simple closed curve*. That is, an acyclic continuous curve is a continuous curve containing no simple closed curve.

Acyclic continuous curves have been studied by Mazurkiewicz,‡ R. L. Wilder,§ R. L. Moore,|| and the author.¶ As a result of Theorem 1, it follows that certain internal properties of an acyclic continuous curve which have been proved by the above authors for plane curves, are also possessed by curves in n-dimensional space. However, in the present paper, only plane point sets are considered, unless otherwise stated.

THEOREM 1. *Any acyclic continuous curve lying in n-dimensional space can be put into continuous* (1-1) *correspondence with some plane acyclic continuous curve.*

Mazurkiewicz** states that the above theorem is probably true, but he gives no indication of any attempt to prove it. We shall give here a proof based upon the following definition and lemmas, the truth of which is easily established from the definition.

Definition. A continuous (1-1) correspondence between two point sets M_1 and M_2, is said to be *uniformly continuous*, if given any positive number ϵ, there exists a corresponding positive number δ, such that if A_i and B_i are any two points of M_i at a distance apart less than δ, then the correspond-

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‡ S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), pp. 119–130.

§ R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377.

|| R. L. Moore, *Concerning the cut-points of continuous curves and of other closed and connected point sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

¶ H. M. Gehman, *On extending a continuous* (1-1) *correspondence of two plane continuous curves to a correspondence of their planes*, these Transactions, vol. 28 (1926), pp. 252–265.

** Loc. cit., p. 130. *Note added in proof*: Professor J. R. Kline has informed me that this theorem has been proved by T. Wazewski, Annales de la Société Polonaise de Mathématique, vol. 2 (1923).

ing points A_{i+1} and B_{i+1} of M_{i+1} are at a distance apart less than ϵ, where $i=1, 2$, and all subscripts are reduced modulo 2.*

LEMMA A. *A continuous* (1-1) *correspondence between two closed and bounded point sets is uniformly continuous.*

LEMMA B. *A continuous* (1-1) *correspondence between a closed and an open set cannot be uniformly continuous.*

LEMMA C. *A continuous* (1-1) *correspondence between a bounded and an unbounded set cannot be uniformly continuous.*

LEMMA D. *If two open sets are in uniformly continuous* (1-1) *correspondence, then the correspondence can be extended to the closed sets obtained by adding limit points to the two sets.*

LEMMA E. *If any two sets are in uniformly continuous* (1-1) *correspondence, then if any sequence of points of one of the sets has a sequential limit point* (*not necessarily in the set*), *then the sequence of corresponding points in the other set also has a sequential limit point.*

The set of continuous (1-1) correspondences having the property mentioned in Lemma E (i.e., that a sequence having a sequential limit point always corresponds to a sequence having a sequential limit point) includes as a proper subset the set of uniformly continuous (1-1) correspondences as defined above. If a uniformly continuous (1-1) correspondence had been defined by means of this property, all the lemmas would have remained true, and in addition, the words "*and bounded*" may be omitted from Lemma A. Under either definition, the properties of *boundedness* and *closedness* are invariant properties of a point set under the group of uniformly continuous (1-1) correspondences, which is not the case under those (1-1) correspondences which are merely continuous.

The following example will show that under our given definition of a uniformly continuous (1-1) correspondence, a continuous (1-1) correspondence between two closed *and unbounded* point sets is not necessarily uniformly continuous. Let M_1 be the two lines $y=0$, and $y=1$. Let M_2 be the line $y=0$, and the exponential curve $y=e^x$. Let the correspondence between them be such that each point on $y=0$ corresponds to itself, and each point on $y=1$ corresponds to the point with the same abscissa on $y=e^x$. If ϵ is selected as less than 1, then no matter what δ is selected, there are two points of M_2 (i.e., a point of $y=0$ and a point of $y=e^x$ with the same abscissa),

* Compare the definition of uniform continuity given by W. H. Young and G. C. Young, *The Theory of Sets of Points*, 1906, p. 218.

which are at a distance apart less than δ, and yet the corresponding points of M_1 are at a distance apart equal to 1, and therefore greater than ϵ. Therefore this continuous (1-1) correspondence between two closed and unbounded sets is not uniformly continuous.

The proof of Theorem 1 then proceeds as follows: R. L. Wilder† has shown that an acyclic continuous curve M can be expressed as the sum of a set M^* and a totally disconnected set of limit points of M^*, where M^* is a set which (1) is composed of a sequence of arcs $C_1, C_2, C_3, \cdots$, no two of which have in common a point which is an interior point of both, and (2) is such that if n is any positive integer, $C_1+C_2+\cdots+C_n$ is an acyclic continuous curve which is a proper subset of M, and (3) is such that given any positive number ϵ, there exists a number ρ such that if $n>\rho$, the diameter of C_n is less than ϵ, and the diameter of each tree‡ in $M-(C_1+C_2+\cdots+C_n)$ is less than ϵ. Although Wilder's proof is worded for the case of a plane continuous curve, it is evident that with the necessary changes in wording, his proof will hold also for an acyclic continuous curve lying in a space of any number of dimensions.

Let M_1 be an acyclic continuous curve lying in n-dimensional space, and let M_1 be expressed as described in Wilder's theorem. In a plane S, we can construct an arc D_1 in continuous (1-1) correspondence with C_1, and an arc D_2 such that D_1+D_2 is in continuous (1-1) correspondence with C_1+C_2, preserving the given correspondence between D_1 and C_1, and, in general, an arc D_i such that $D_1+D_2+\cdots+D_i$ is in continuous (1-1) correspondence with $C_1+C_2+\cdots+C_i$, preserving the given correspondence between $D_1+D_2+\cdots+D_{i-1}$ and $C_1+C_2+\cdots+C_{i-1}$. This construction can be performed in such a way that $D_1+D_2+\cdots$ plus limit points of this sequence is an acyclic continuous curve M_2 in S.

By Lemma D, it will be sufficient to prove that $M_1^*=C_1+C_2+\cdots$ and $M_2^*=D_1+D_2+\cdots$ are in uniformly continuous (1-1) correspondence. Suppose that this were not true. Then for some positive number ϵ, there is a sequence of pairs of points X_i, Y_i $(i=1, 2, 3, \cdots)$ of one of the sets, say M_1^*, such that the distance between X_i and Y_i is less than $1/i$, while the distance between the corresponding points X_i' and Y_i' of M_2^* is greater than ϵ. Let us select an integer k so large that the diameter of any tree in $M_2-(D_1+D_2+\cdots+D_k)$ is less than $\epsilon/3$. Then each of the arcs $X_i'Y_i'$ of M_2 contains a subarc $A_i'B_i'$ which lies in $D_1+D_2+\cdots+D_k$, and is of diameter greater than $\epsilon/3$, and therefore contains two points E_i', F_i' whose distance

† Loc. cit., Theorem 15, p. 365.

‡ A *tree* is a *maximal connected subset*. See H. M. Gehman, loc. cit., p. 256.

apart is greater than $\epsilon/3$. Since M_2 was constructed in such a way that the correspondence between $C_1+C_2+\cdots+C_k$ and $D_1+D_2+\cdots+D_k$ is continuous, and therefore, by Lemma A, uniformly continuous, it follows that there exists a constant δ, such that if two points of $C_1+C_2+\cdots+C_k$ are at a distance apart less than δ, the corresponding points of $D_1+D_2+\cdots+D_k$ are at a distance apart less than $\epsilon/3$. Also the set M_1 is uniformly connected im kleinen, and therefore there exists a constant α, such that any two points of M_1 at a distance apart less than α can be joined by an arc in M_1 of diameter less than δ. If we then select an integer i, such that $(1/i)<\alpha$, the diameter of the arc X_iY_i is less than δ. The points E_i, F_i, which are on the arc X_iY_i, are therefore at a distance apart less than δ, and the corresponding points E_i', F_i' are at a distance apart less than $\epsilon/3$, which is contrary to the method of selection of the points E_i', F_i'.

Exactly the same contradiction is obtained if we suppose the points X_i, Y_i to lie in M_2^*. Therefore M_1^* and M_2^* are in uniformly continuous (1-1) correspondence. and Theorem 1 is true.

Definition. If M is a connected point set, and P is a point of M, then if $M-P$ is not connected, P is said to be a *cut point* of M; if $M-P$ is connected, P is said to be a *non-cut point* of M.†

Definition. If M is a continuous curve, and P is a point of M, then P is said to be an *end point* of M, if the maximal connected subset of $M-(A-P)$ containing P consists of P alone, where A is any arc of M having one end point at P.‡ It follows that an end point is always a non-cut point. If M is an acyclic continuous curve, every non-cut point is an end point.

Theorem 2. *If T, the set of all non-cut points of a bounded continuum M lying in n-dimensional space, is a subset of a closed, totally disconnected subset T' of M, then M is an acyclic continuous curve, the set of whose end points is identical with T.*

Suppose that M were not a continuous curve. In that case, M contains a sequence of mutually exclusive continua W, M_1, M_2, $\cdots$ such that W is the sequential limiting set of the sequence M_1, M_2, $\cdots$, and such that there exists a connected subset of M containing $M_1+M_2+\cdots$ but not containing any points of W.§ If to this connected set we add its limit points,

† R. L. Moore, loc. cit., p. 101.

‡ R. L. Wilder, loc. cit., p. 358. See Theorem 7. For a number of definitions of an end point of a continuous curve which are equivalent to the above, see the author's forthcoming paper *Concerning end points of continuous curves and other continua.*

§ R. L. Wilder, loc. cit., p. 371. See also R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs,* Bulletin of the American Mathematical Society, vol. 29 (1923), p. 296. We shall refer to this hereafter as *Report.*

the resulting continuum R is a subset of M and contains W. It is well known that if to a connected set, we add any set of its limit points, the resulting set is connected. If then we add to the connected set mentioned above, all its limit points save a point P of W, the resulting set $R-P$ is connected. In other words, every point of W is a non-cut point of R.

Under these conditions, only a countable number of points of W can be cut points of M.† Since the set T' which contains all the non-cut points of M is closed and totally disconnected, there is a subcontinuum of W which contains no points of T', and which therefore consists entirely of cut points of M. But a continuum cannot consist of a countable set of points, and we have therefore arrived at a contradiction by supposing that M is not a continuous curve.

Suppose next that the continuous curve M contains a simple closed curve J. By the theorem referred to above, the cut points of M on J are countable. Since T' is closed and totally disconnected, J contains an arc which contains no points of T', and which therefore consists entirely of cut points of M. But an arc cannot consist of a countable number of points, and we have therefore arrived at a contradiction by supposing that M contains a simple closed curve. The continuum M is therefore an acyclic continuous curve, and therefore the set T of non-cut points is identical with the set of end points of M.

Note however, that if T denotes the set of end points of an acyclic continuous curve, it does not follow that there exists a closed, totally disconnected set T' which contains T. For there exist acyclic continuous curves in which every point is a limit point of the set of end points, and in such a case any closed set which contains T contains the acyclic continuous curve and therefore cannot be totally disconnected.

THEOREM 3. *If T' is a closed and totally disconnected subset of a bounded continuum M in n-dimensional space, then a necessary and sufficient condition that M be an acyclic continuous curve, the set of whose end points is a subset of T', is that M be irreducibly connected about T'.*‡

The condition is necessary, as a bounded continuum is always irreducibly connected about any set which contains its non-cut points, as we have proved in a recent paper.§ In this same paper, we prove that a bounded continuum

† R. L. Moore, *Concerning the cut-points*, etc., loc. cit., Theorem B*, p. 102, and *Report*, Theorem D, p. 300.

‡ A set M is said to be *irreducibly connected about a set of points T'*, if M is connected and contains T', but no proper connected subset of M contains T'.

§ H. M. Gehman, *Concerning irreducibly connected sets and irreducible continua*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 544–547.

is not irreducibly connected about any set which does not contain all its non-cut points, and therefore in proving that the given condition is sufficient it follows that T' contains all the non-cut points of M. Then, by Theorem 2, the continuum M is an acyclic continuous curve, whose end points are a subset of T'. This completes the proof of Theorem 3.

Note that this theorem serves to characterize a certain type of acyclic continuous curve by the condition which Lennes* used to define a simple continuous arc.

R. L. Moore and J. R. Kline† have solved the problem of determining the most general plane point set through which an arc may be passed. It is evident, however, that in the space consisting of a plane continuous curve,‡ their conditions are not sufficient. For if the continuous curve K consists of three arcs, AB, AC, AD, having no points in common save the point A, then the point set consisting of B, C, and D satisfies the hypotheses of their theorem, and yet K contains no arc containing B, C, and D.

In Theorem 4, we shall describe the most general closed point set lying in a plane continuous curve K, through which an acyclic continuous curve lying in K can be passed. In our discussion, it is immaterial whether the continuous curve K be a bounded portion of the plane, or some other type of continuous curve.

Lemma F. *If N is a closed bounded set consisting of a collection of connected sets (E), each one of which is a maximal connected subset of N, no one of which separates the plane S, and not more than a finite number of which are of diameter greater than any given positive number, then N cannot separate the plane, and if moreover, a point P of a maximal connected subset e of (E) is accessible from $S-e$, then P is accessible from $S-N$.*

Under these hypotheses, the maximal connected subsets in (E) are mutually exclusive and closed. The collection (E) may contain an uncountable collection of these continua, but in that case those which contain more than one point form a countable collection.

If we add to the collection of continua (E), the collection of mutually exclusive continua each consisting of a single point of $S-N$, the resulting collection (G) forms an upper semi-continuous collection of mutually ex-

* N. J. Lennes, *Curves in non-metrical analysis situs*, American Journal of Mathematics, vol. 33 (1911), p. 308.

† *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, (2), vol. 20 (1919), pp. 218–223.

‡ This idea has been discussed by R. L. Wilder, loc. cit., p. 341, and by R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260.

clusive bounded continua, filling up the entire plane S. Under these conditions R. L. Moore* has proved that if each continuum of (G) be considered as a point, the set of elements of (G) is (from the viewpoint of analysis situs) equivalent to the set of points in a plane S'. The set of points in S' corresponding to the elements of (E) is a closed, totally disconnected set of points, N', through which, by the Moore-Kline theorem, an arc may be passed in S'. Any two points of $S'-N'$ can therefore be joined by an arc having no points in common with N'. Each point of S' on this arc corresponds to a single point in S, and therefore any two points of $S-N$ can be joined by an arc having nothing in common with N. The set N therefore does not separate the plane.

Let us now assume that P is a point of a continuum e of (E), which is accessible from $S-e$. Let PQ be an arc joining P to any point Q not in (E), and having only P in common with e. If this arc has points in common with any continuum of (E) of diameter greater than 1, we can enclose each of these continua by a simple closed curve containing no points of N, such that P and Q are in its exterior, such that every point of the simple closed curve and its interior lies at a distance less than $\frac{1}{2}$ from the continuum that it encloses, and such that each simple closed curve of the set is exterior to each of the others.† In the same way, we can enclose each continuum of diameter greater than $\frac{1}{2}$ (but less than 1) that has points in common with a sub-arc of PQ which is exterior to all the simple closed curves of the first set, by a simple closed curve which has the properties of those of the first set, and which in addition is such that every point of the simple closed curve and its interior lies at a distance less than $\frac{1}{4}$ from the continuum that it encloses. Evidently the diameter of each simple closed curve of this second set is less than $\frac{3}{2}$. If we continue this process, we obtain a point set A consisting of the arc PQ and a countable collection of mutually exclusive simple closed curves, each having points in common with the arc PQ, and such that only a finite number of them are of diameter greater than any given positive number. This set A is a continuum which satisfies a set of conditions which are sufficient that every subcontinuum of A be a continuous curve.‡ Therefore the continuum B consisting of the simple closed curves in A and those points

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* R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428.

† This is an extension of a theorem due to Zoretti. See R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

‡ H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, Annals of Mathematics, (2), vol. 27 (1926), pp. 381–384. See especially Theorem 2, p. 382.

of the arc PQ that are not interior to any simple closed curve in A, is a continuous curve that contains P and Q. Therefore B contains an arc PQ.

The arc PQ in B may contain points of N, but if so, the set of points common to N and PQ is totally disconnected. Let $P_1, P_2, P_3, \cdots$ be a sequence of points on PQ which are not points of N, and which are such that the diameter of the arc P_iP is less than $1/i$. The points of N on QP_1 can be covered by a finite set of mutually exclusive simple closed curves of diameter less than 2, having no points in common with N, and such that P and Q are in the exterior of each simple closed curve of the set. The points of N on P_iP_{i+1} can be covered by a similar finite set of simple closed curves of diameter less than $1/i$ $(i=1, 2, 3, \cdots)$. Then by precisely the same argument as in the preceding paragraph, we obtain an arc PQ which has no points in common with N. The point P is therefore accessible from $S-N$, and the truth of Lemma F is established.

THEOREM 4. *A necessary and sufficient condition that a closed subset N of a continuous curve K be a subset of an acyclic continuous curve lying in K, is that every maximal connected subset of N be an acyclic continuous curve, and that not more than a finite number of these maximal connected subsets be of diameter greater than any given positive number.*

The condition is necessary, because every subcontinuum of an acyclic continuous curve is an acyclic continuous curve,* and therefore not more than a finite number of mutually exclusive subcontinua can be of diameter greater than any given positive number.†

To prove that the condition is sufficient, we shall suppose that a continuous curve K contains a closed subset N satisfying the given condition, and shall show how to construct an acyclic continuous curve M lying in K and containing N.

First we shall enclose all those maximal connected subsets of N that are of diameter greater than 1 by a finite collection (C) of mutually exclusive simple closed curves, each of which encloses one maximal connected subset of N of diameter greater than 1, encloses no other maximal connected subset of N of diameter greater than $\frac{1}{2}$, contains no point of N, and is such that every point of the simple closed curve and its interior is at a distance less than 1 from a point of the subset of N of diameter greater than 1, that it encloses.

Let C_1 be a simple closed curve of (C), and let N_1 be the subset of N of diameter greater than 1, enclosed by C_1. Let AB be a maximal arc of

* S. Mazurkiewicz, loc. cit., Lemma, p. 123.

† H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, (2), vol. 27 (1925), pp. 29–46. See Theorem V, p. 39.

N_1 of diameter greater than 1. If A is a limit point of $N-N_1$, we shall put about A as center a circle C_A of diameter less than $\frac{1}{3}$, whose exterior contains B and C_1, and a circle C_A' of diameter half that of C_A. Under the given hypotheses, there are only a finite number of maximal connected subsets of N that have points interior to C_A' and points exterior to C_A. Therefore we can put about A as center a circle C_A'' whose exterior contains every maximal connected subset of $N-N_1$ that contains points exterior to C_A. Since K is connected im kleinen at A, there exists a circle C_A^* about A as center and interior to C_A'', such that every point of K within C_A^* can be joined to A by an arc in K lying within C_A''. Since A is a limit point of $K-N_1$, we can select a point X of $K-N_1$ lying within C_A^* and can select a definite arc XA in K lying within C_A''. Let Y be the first point which this arc has in common with N_1. The arc XY together with those maximal connected subsets of $N-N_1$ that have points in common with XY forms a continuum M, every subcontinuum of which is a continuous curve,† which has only the point Y in common with N_1, and which lies entirely within C_A. The maximal connected subsets of $N-N_1$ that are of diameter >0 and lie in M can be arranged in a sequence $H_1, H_2, H_3, \cdots$, such that given any positive number ϵ, there exists an integer n, such that for $i>n$, H_i is of diameter less than ϵ. Let X_1 and Y_1 be respectively the first and last points that the arc XY has in common with H_1, and let M_1 denote the arc X_1Y_1 of H_1, the arcs XX_1 and Y_1Y of the original arc XY, plus those sets of the sequence H_2, H_3, $H_4, \cdots$ which have points in common with either XX_1 or Y_1Y. Let H_{n_2} be the first member of the sequence $H_2, H_3, H_4, \cdots$ that lies in M_1, and let X_2 and Y_2 be respectively the first and last points that the arc XX_1Y_1Y has in common with H_{n_2}, and let M_2 denote the arc X_2Y_2 of H_{n_2}, the arcs XX_2 and Y_2Y of the arc XX_1Y_1Y, plus those sets of the sequence H_{n_2+1}, $H_{n_2+2}, \cdots$ which have points in common with either XX_2 or Y_2Y. The method of defining $M_3, M_4, \cdots$ is obvious. The common part of the sequence $M, M_1, M_2, \cdots$ is a new arc XY in K having only Y in common with N_1, and lying within C_A, and therefore lying within C_1 and of diameter less than $\frac{1}{3}$. Also from the method in which it has been constructed, the arc XY has the property that if it has two points P, Q in common with a maximal connected subset of N, the arcs PQ of XY and of N are identical. (The arcs $B'Z$, $G'V$, X_iY_i, etc., to be constructed later, are also to have this property.) Since N_1 is a maximal connected subset of N, the arc XY contains a point of $K-N$. Let A' be such a point.

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† H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, loc. cit., Theorem 2, p. 382.

In case A is not a limit point of $N-N_1$, let $A=Y=A'$. In either case, if we add to N_1 the arc $A'Y$ plus all maximal connected subsets of N that have points in common with $A'Y$, we obtain a new acyclic continuous curve N_2 that contains N_1.

In the same way, if the point B is a limit point of $N-N_1$, there is a point B' of $K-N$, and an arc $B'Z$ of K of diameter less than $\frac{1}{3}$, which is interior to C_1 and has no points in common with the arcs $A'Y$ and AY, and has only Z in common with N_2. In case B is not a limit point of $N-N_1$, let $B=B'=Z$. If we add to N_2 the arc $B'Z$ and all maximal connected subsets of N that have points in common with $B'Z$, we obtain a new acyclic continuous curve N_3, containing N_1 and N_2, and such that $A'B'$ is a maximal arc of N_3 whose end points are not limit points of $(N+N_3)-N_3$. The arc $A'B'$ has the arc YZ in common with N_1, and is therefore of diameter greater than $\frac{1}{3}$, because, the diameters of the arcs AY and BZ of N_1 being each less than $\frac{1}{3}$, the diameter of YZ is greater than $\frac{1}{3}$.

By Lemma F, each point of N_3 is accessible from $S-(N+N_3)$. We can therefore join* A' to a point D of C_1 by an arc in S having only A' in common with $N+N_3$, and only D in common with C_1. We can also join B' to a point E of C_1 $(E\neq D)$ by an arc having only B' in common with $N+N_3$, only E in common with C_1, and having no points in common with the arc $A'D$.

If the set $N_3-A'B'$ contains a tree T of diameter greater than $\frac{1}{5}$, it is necessarily a tree of N_1-YZ. Let the foot of the tree be F, and let FG be a maximal arc of $T+F$ of diameter greater than $1/10$. If G is a limit point of $N-N_1$, as before we can construct an arc $G'V$ in K, within a circle about G of diameter less than $1/30$ whose exterior contains C_1 and the arcs DA', $A'B'$, $B'E$, the arc $G'V$ being such that G' is a point of $K-(N+N_3)$, and V is the only point that it has in common with N_3. Since the diameter of $G'V$ is less than $1/30$, the diameter of the arc VF of N_3 is greater than $2/30$.

If G is not a limit point of $N-N_1$, let $G=G'=V$. Let N_3 plus $G'V$ plus all maximal connected subsets of N that have points in common with $G'V$ be denoted by N_4, which is also an acyclic continuous curve. As before, $N+N_4$ satisfies the hypotheses of Lemma F, and therefore the point G' can be joined to a point H of C_1 $(D\neq H\neq E)$ by an arc having only G' in common with $N+N_4$, only H in common with C_1, and having no points in common with either $A'D$ or $B'E$.

If the set $N_4-(A'B'+FG')$ contains a tree of diameter greater than $\frac{1}{5}$, it is necessarily a tree of $N_1-(YZ+FV)$, and as we have done before we

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* The construction from this point follows in the main that used in the proof of Theorem 1 of my paper *On extending a continuous* (1-1) *correspondence*, etc., loc. cit.

can construct an arc within C_1 from a point of C_1 to a point of $K-(N+N_4)$ or a point of N_4 (depending upon whether the end point of N_4 first chosen is or is not a limit point of $N-N_1$), and obtain as before an acyclic continuous curve N_5 which contains N_4 as a subset.

After a finite number k of steps, $N_k-(A'B'+FG'+\cdots)$ will contain no tree of diameter greater than $\frac{1}{5}$, otherwise N_1 would contain an infinite set of mutually exclusive arcs of diameter greater than 2/30 which is impossible.

Therefore, after the kth step, the interior of C_1 has been expressed as the sum of a finite number of domains, plus boundary points of these domains, where the boundary of each domain is a simple closed curve consisting of an arc of N_k, and an arc having only its end points in common with $N+N_k$. The arc of N_k lying in the boundary is a maximal arc of N_k, and its end points are not limit points of $(N+N_k)-N_k$.

Let J denote one of these simple closed curves, PQ the arc of N_k in J, N' the points of N_k contained in and enclosed by J. We can select a finite set of points $P_1, P_2, \cdots, P_n$ on PQ, such that (a) $P_1=P$, (b) $P_n=Q$, (c) P_i precedes P_{i+1} on PQ, for $i=1, 2, \cdots, n-1$, (d) the diameter of each arc P_iP_{i+1} is less than $\frac{1}{5}$, (e) no tree in $N'-PQ$ has its foot at any of the points P_i.

If P_i is a limit point of points of $(N+N_k)-N_k$ interior to J, then as before, there is a point X_i of $K-(N+N_k)$, and an arc X_iY_i in K having only Y_i in common with N_k, interior to J save possibly for Y_i, of diameter less than $\frac{1}{5}$, such that no two arcs of the set X_iY_i have points in common or contain points of the same maximal connected subset of $(N+N_k)-N_k$, and such that the arc Y_iP_i of N_k does not contain any other points of the sets $P_1, P_2, \cdots, P_n$ and $Y_1, Y_2, \cdots, Y_n$. If P_i is not a limit point of $(N+N_k)-N_k$, then $P_i=X_i=Y_i$. Let N' plus the arcs $X_1Y_1, X_2Y_2, \cdots, X_nY_n$ plus all maximal connected subsets of N that have points in common with one of these arcs, be denoted by N''.

As in the paper referred to in the previous footnote, there exists a set of arcs $X_1X_2, X_2X_3, \cdots, X_{n-1}X_n$ in S, which have no points in common save end points of consecutive arcs of the sequence, are interior to J save possibly for end points, are of diameter less than 2, and have no points in common with $N+N_k+N''$ save their end points. These arcs form with the corresponding arcs of N'' a set of simple closed curves $J_1, J_2, \cdots, J_{n-1}$ which contain or enclose all points of N''. The diameter of the arc X_iX_{i+1} of N'' is less than 1, and therefore the diameter of each of the simple closed curves J_i is less than 3.

If the above construction is made in each of the simple closed curves J, we have a finite set of simple closed curves (similar to J_i) each of diameter less than 3, which contain or enclose all points of N_k, and are such that each is composed of an arc in K and an arc no point of which is a limit point of points of N. No two of these simple closed curves have an interior point in common.

We shall now make the same construction in each of the simple closed curves of the set (C) that we have made within C_1. Let (D) denote the collection of all the simple closed curves similar to J_i. If any maximal connected subsets of N are exterior to (D), we shall cover each such set by a simple closed curve containing no points of N or of (D), enclosing no points of (D), and such that every point of the simple closed curve and its interior is at a distance less than 1 from a point of the subset of N that it encloses. Since any subset of N exterior to (D) is of diameter less than 1, it follows that each simple closed curve of this set is of diameter less than 3. If this set is infinite, we can select from it a finite subset (E) which encloses all points of N exterior to (D). If any two simple closed curves of (E) have points in common, the collection (E) may be replaced by a finite collection (F) of simple closed curves enclosing all points of N exterior to (D), each simple closed curve being of diameter less than 3, and having the additional property that no two have a point or an interior point in common.*

Let R denote one of the simple closed curves in $(D)+(F)$. Any maximal connected subset of K contained in R plus its interior is a continuous curve, and no more than a finite number of these continuous curves are of diameter greater than any given positive number.† If R is in (F), then since there is some constant d which is less than the distance from a point of N to a point of R, it follows that the points of N in the interior of R lie in a finite number of mutually exclusive continuous curves which are subsets of K. Similarly, if R is in (D), then, since there is some constant d' which is less than the distance from a point of N to the arc that has no points in common with N (and since the other arc of (R) lies in K), it follows that the points of N in the interior of R and on R, lie in a finite number of mutually exclusive continuous curves which are subsets of K.

Therefore all points of N lie in a finite collection (G) of continuous curves which are subsets of K and which are of diameter less than 3. Let $G_1, G_2, \cdots, G_a$ be the finite set of maximal connected subsets of (G), and let P_i be a point in G_i. We can join P_1 to P_2 by an arc in K, and on this arc, we

* Moore and Kline, loc. cit., Lemma 1, p. 219.

† H. M. Gehman, *Concerning the subsets of a plane continuous curve*, loc. cit., Lemma B, p. 34. See also Theorem 6 of H. M. Gehman, *Some relations between a continuous curve and its subsets*, Annals of Mathematics, (2), vol. 28 (1927), pp. 103–111.

can select all possible arcs AB, such that A is in G_1 and B is in one of the sets $G_2, G_3, \cdots, G_a$, and AB has no other points in common with any of the sets G_i. If there is more than one arc AB such that B is a point of G_i (for a fixed value of i), then select one definite arc AB from G_1 to G_i. The set G_1 plus the arcs AB plus the sets of $G_2, G_3, \cdots, G_a$ that contain one of the points B, we shall call H_1. The other sets of $G_2, G_3, \cdots, G_a$ we shall call $H_2, H_3, \cdots, H_b$. Note that $b \leqq a-1$. Let Q_i be a point of H_i, and construct an arc in K from Q_1 to Q_2, and then proceed as with the arc P_1P_2.

After a finite number of steps we obtain a continuous curve K_1 which is a subset of K, contains N, contains no point exterior to the simple closed curves in $(D)+(F)$ except points of the arcs AB, and has the property that there is only one "path" composed of arcs AB and sets G_i, joining any two points of K_1.

If we now consider each member G_i of the finite collection (G) of continuous curves of diameter less than 3 which are subsets of K and contain all of N, we can perform the same type of construction with G_i within and on the simple closed curve that encloses and contains G_i, as we have just performed with K, in such a way as finally to cause all the points of N that are contained in G_i to lie in a finite number of continuous curves that are subsets of G_i (and therefore of K), and are of diameter less than 1. Having done that with each member of (G), we can construct arcs AB in K_1, so as to obtain a continuous curve K_2 which is a subset of K_1, contains N, contains no point save points of the arcs AB exterior to the finite set of simple closed curves that enclose all of N, and has the property that there is only one "path" joining any two points of K_2.

Continuing this process, the set of points common to $K, K_1, K_2, \cdots$ is a bounded continuum M, which is a subset of K and contains N. We shall now show that M is an acyclic continuous curve.

Sierpinski* has shown that a bounded continuum M is a continuous curve if M can be expressed as the sum of a finite collection of continua each of diameter less than a preassigned positive number. Let us then select a positive number d. There is an integer n such that $1/3^n < d$. The set K_{n+2} consists of a finite collection of continuous curves each of diameter less than $1/3^n$ plus a finite collection of arcs, each of which may be expressed as the sum of a finite collection of arcs each of diameter less than $1/3^n$. The set M consists of these arcs plus some points of the continuous curves, i.e., for each continuous curve in K_{n+2}, M contains only the points common to the

* W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fundamenta Mathematicae, vol. 1 (1920), pp. 44–60.

continuous curve and all the continua of the sequence K, K_1, K_2, $\cdots$. Since this common part is always a continuum, the part in M is a continuum which is of diameter less than $1/3^n$, and therefore M is a continuous curve.

Suppose that M contains a simple closed curve J. Since N contains no simple close curves, J contains a point P which is not a point of N. If the distance from P to a point of N is greater than $1/3^m$, it is evident that P cannot be a point of a continuous curve in K_{m+2}, and therefore P is a point of an arc AB of K_{m+2}. If X and Y are two points on this arc, such that the arc XPY contains no point of N, then since J lies in K_{m+2} there are evidently two "paths" joining X to Y, contrary to our method of constructing the sets K_i. Therefore M is an acyclic continuous curve.

THEOREM 5. *A necessary and sufficient condition that a bounded continuum K be a continuous curve is that every closed, totally disconnected subset T' of K be a subset of some subcontinuum M of K which is irreducibly connected about T'.*

In view of Theorem 3, Theorem 5 is equivalent to the following theorem:

THEOREM 5'. *A necessary and sufficient condition that a bounded continuum K be a continuous curve is that every closed, totally disconnected subset T' of K be a subset of an acyclic continuous curve M in K, the set of whose end points is a subset of T'.*

The condition is necessary, because T' satisfies the conditions of Theorem 4, and therefore an acyclic continuous curve M can be constructed in K containing T', by the method used in the proof of that theorem. As was pointed out in the final paragraph of the proof of Theorem 4, a point of M that is not a point of T' is a point of an arc introduced at a certain step, and is therefore a cut point of each of the continuous curves K_i after that step, and also a cut point of M. Therefore the set of end points of M is a subset of T'.

The condition is sufficient. For if K were not a continuous curve, it would contain a sequence of continua W, M_1, M_2, $\cdots$, having the properties mentioned in the proof of Theorem 2, and each having at least one point in common with each of two concentric circles, C_1 and C_2, and lying entirely in the point set H composed of the circles and all points of the plane between them, each set (save possibly W) being a maximal connected subset of K in H.

Let P be a point of W lying between C_1 and C_2, and P_1, P_2, $\cdots$ a sequence of points such that P_i is a point of M_i, and such that P is the sequential limit of the sequence. Let $P+P_1+P_2+\cdots$ be T'. By hypothesis, K contains a continuous curve M containing T'. Therefore all but a finite number of the points of T' can be joined to P by an arc in K and in H, which contradicts our supposition that each set in the original sequence (except possibly W) was a maximal connected subset of K in H. Therefore K is a continuous curve, and the condition is sufficient.

THEOREM 6. *Every closed, bounded, totally disconnected point set is identical with the set of end points of some acyclic continuous curve.*

Let T' be a closed, bounded, totally disconnected set. We shall show how to construct an acyclic continuous curve, the set of whose end points is identical with T'.

An arc AB may be passed through T', by the Moore-Kline theorem, in such a way that its end points are points of T'. Then a set of points belonging to $AB-T'$ can be selected which divide AB into a finite number of intervals, $I_1, I_2, \cdots, I_n$, each of diameter less than 1. If I_r contains an infinite number of points of T', we shall denote the first point of T' in I_r by A_r, and the last point by D_r. We shall denote by B_r and C_r two points of T' on the arc A_rD_r, such that the arc B_rC_r contains no other points of T', and such that the four points occur on I_r in the order $A_rB_rC_rD_r$.

Let P be a point of the plane not on AB. Let us construct arcs PA and PB having only P in common, and having only A and B respectively in common with AB, thus forming a simple closed curve which we shall denote by J. Then construct a finite set of arcs, each one interior to J save for its end points, and no pair having any points in common save P, by joining P to each point of $T'-(A+B)$ in those intervals of AB that contain only a finite number of points of T', and by joining P to each of the points A_r and D_r in the remaining intervals. In case I_1 contains an infinite number of points of T', let the arc PA_1 be the arc PA of J; in case I_n contains an infinite number of points of T', let PD_n be the arc PB of J.

For each value of r for which the points A_r, B_r, C_r, D_r have been defined, we next make the following construction. We enclose the arc A_rB_r by a simple closed curve J_r, whose exterior contains the arc C_rD_r and all the arcs that have been constructed from P to points of AB (excepting of course, the arc PA_r), and which is such that every point of J_r and its interior is at a distance less than 1 from a point of A_rB_r. The diameter of J_r is therefore less than 3. If X is the first point that A_rP has in common with J_r, then B_r can be joined to any point E_r of $A_rX(A_r\neq E_r\neq X)$ by an arc interior to J_r, interior to the simple closed curve PA_rD_r save for its end points, having only E_r in common with A_rP and only B_r in common with AB. The arcs B_rE_r, E_rA_r, and A_rB_r are each of diameter less than 3, and form a simple closed curve which is also of diameter less than 3.

By a similar construction, we obtain an arc C_rF_r, where F_r is a point of $D_rP(D_r \neq F_r \neq P)$, such that the arc has only C_r in common with AB, only F_r in common with PD_r, has no points in common with any other of the arcs joining P to points of AB, has no points in common with B_rE_r, is interior to PA_rD_r save for its end points, is of diameter less than 3, and forms with the arcs F_rD_r and C_rD_r a simple closed curve of diameter less than 3.

Let us denote by K_1 the set consisting of all arcs joining P to points of AB, plus the set of simple closed curves $A_rB_rE_r$ and $C_rD_rF_r$ and their interiors. This set has all the properties of the set K_1 obtained in the proof of Theorem 4. If we continue the same method of construction within each of the simple closed curves $A_rB_rE_r$, $C_rD_rF_r$ that we have performed within J, replacing P by E_r(or F_r), A by A_r(or C_r), and B by B_r(or D_r), dividing the arc A_rB_r (or C_rD_r) into intervals of diameter less than $\frac{1}{3}$, we obtain a set K_2 which is a subset of K_1, and has all the properties of the set K_2 obtained in the proof of Theorem 4.

Continuing this process, the set of points common to K_1, K_2, $\cdots$ is an acyclic continuous curve M, as was shown in the proof of Theorem 4. It is evident that M consists of a connected set consisting of a countable infinity of arcs obtained by the process outlined above, plus limit points of these arcs. The limit points are points of T' which are not end points of arcs of the countable set, and all such points are non-cut points of M. Also the points of T' which are end points of arcs are non-cut points of M, and therefore every point of T' is a non-cut point (and therefore an end point) of M.

The points of $M-T'$ are not points of AB, and are cut points of M, as was shown in the proof of Theorem 4. Therefore the set T' is identical with the set of end points of M.

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