CONCERNING POINT SETS WHICH CAN BE MADE CONNECTED BY THE ADDITION OF A SIMPLE CONTINUOUS ARC*

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In their paper *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, R. L. Moore and J. R. Kline† prove that it is possible to pass a simple continuous arc through every closed and bounded set M having the property that every closed and connected subset of M is either a single point or an arc t such that no point of t, with the exception of its end points, is a limit point of $M-t$. It is clear, however, that in order that a simple continuous arc may be drawn in such a way as to contain at least one point of every maximal connected subset of a point set M, it is not necessary that the set M be of the particular type satisfying their theorem. In this paper I shall make a study of certain conditions which a point set must satisfy in order that a simple continuous arc or an open curve may be drawn in such a way that the set in question plus that arc or curve will be connected.

LEMMA I. *If M is any closed and bounded point set, then there exists a countable number of arcs $t_1, t_2, t_3, \cdots$, such that for every positive integer n, t_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$.*

Let n denote any definite positive integer. Since M is bounded, there exists a square S which encloses M; S plus its interior can be divided by a finite number of straight lines parallel and perpendicular to the bases of S into a finite number of squares plus their interiors in such a way that the diameter of each of these squares is less than $1/n$ and such that the interiors of no two of them have a point in common. Let G denote this finite set of squares (not including their interiors), and let T denote the point set obtained by adding together all the point sets of the set G. Then since the interior of every square of the set G is of diameter less than $1/n$, every maximal connected subset of M which is of diameter greater than $1/n$ must contain at least one point in common with T. Let F denote the set of all points common to M and T. From each maximal connected subset Y of F select exactly one point

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X; and let P denote the set of all such points (X) thus selected. Then since F is closed and has no continuum of condensation, it follows that $\overline{P}$ is a closed and totally disconnected point set.* It follows, then, from the above mentioned theorem of Moore and Kline† that $\overline{P}$ is a subset of a simple continuous arc t_n. Clearly t_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$.

THEOREM 1. *If M is a closed and bounded point set, a necessary and sufficient condition that there should exist a simple continuous arc which contains at least one point of every maximal connected subset of M is that for every continuum K of M which consists of more than one point there should exist a positive number ϵ_k such that K is not the limiting set of any collection of maximal connected subsets of M each of diameter less than ϵ_k.*

(I). The condition is sufficient. For since M satisfies the conditions of Lemma I, there exists a countable set of arcs $t_1, t_2, t_3, \cdots$, having the same property with respect to M that the corresponding set of arcs in Lemma I has with respect to the set M of Lemma I. Let K_1 denote the set of points common to t_1 and M; let K_2 denote the set common to t_2 and to those maximal connected subsets of M which have no point in t_1; K_3 the set common to t_3 and to those maximal connected subsets of M which have no point in t_1+t_2; in general, let K_n denote the set of points common to t_n and to those maximal connected subsets of M which have no point in $t_1+t_2+t_3+t_4+\cdots+t_{n-1}$. Let K denote the point set $K_1+K_2+K_3+K_4+\cdots$. I will proceed to show that $\overline{K}$ contains no continuum of condensation. Suppose, on the contrary, that $\overline{K}$ contains a continuum of condensation H. Then H is also a continuum of condensation of M; and by hypothesis there exists a positive number ϵ_H such that H is not the limiting set of any set of maximal connected subsets of M each of diameter less than ϵ_H. Now the elements of K have been so selected that for any given positive number, say ϵ_H, there exists a positive number $\delta(\epsilon_H)$ such that for every integer $n>\delta(\epsilon_H)$, K_n contains points of only those maximal connected subsets of M which are of diameter less than ϵ_H. Let i denote an integer greater than $\delta(\epsilon_H)$. Then $\sum_{n=i+1}^{n=\infty} K_n$ contains points of only those maximal connected subsets of M which are of diameter less than ϵ_H. It follows that not every point of H is a limit point of $\sum_{n=i+1}^{n=\infty} K_n$. Let G denote the collection of point sets $K_1, K_2, K_3, \cdots, K_i$. Let

$$A = \sum_{n=1}^{n=i} K_n, \text{ and let } B = \sum_{n=i+1}^{n=\infty} K_n.$$

* In this paper wherever a symbol X is used to denote a point set, the symbol $\overline{X}$ will be used to denote the set X plus all those points which are limit points of X.

† R. L. Moore and J. R. Kline, loc. cit.

Then $K=A+B$. Let P denote a point of H which is not a limit point of B; and let C be a circle enclosing P and not enclosing or containing any point whatever of $\overline{B}$. From a theorem due to Janiszewski,* it follows that C plus its interior contains a subcontinuum D of H. Then D is a subset of the closed set $\overline{A}$. Let K_a and K_b denote any two elements of G, K_a denoting the one of lower subscript. I will show that $\overline{K}_a$ and $\overline{K}_b$ have at most a closed and totally disconnected set in common. Suppose, on the contrary, that $\overline{K}_a$ and $\overline{K}_b$ have in common a continuum t which consists of more than one point. Then t is a subset both of t_a and of t_b; hence t is an arc. Let E and F denote the end points of t. Since t is a subset of t_a, and since t_a precedes t_b, then no point of t can belong to K_b. And since no point of t except the points E and F can be a limit point of t_b-t, then no points of t except E and F can belong to $\overline{K}_b$. But by supposition, t is a subset of $\overline{K}_b$. It follows that $\overline{K}_a$ and $\overline{K}_b$ have at most a closed and totally disconnected set in common. Let U denote the set of all points (X) of $\overline{A}$ such that for some two elements K_a and K_b of G, X is common to $\overline{K}_a$ and $\overline{K}_b$. Since U is the sum of a finite number of closed and totally disconnected point sets, U itself must be closed and totally disconnected. Hence D, a continuum consisting of more than one point, cannot be a subset of U. Therefore, there exists a point P of D such that for some element K_p of G, P belongs to $\overline{K}_p$ and is not a limit point of $\overline{A}-\overline{K}_p$. It follows from the above mentioned theorem of Janiszewski's† that $\overline{K}_p$ contains a continuum l of D such that no point of l is a limit point of $\overline{A}-\overline{K}_p$. But l is a subset of t_p. Hence l is an arc, and no points of l except its end points can be limit points of $\overline{K}_p-l$. Hence if O is an interior point of l, O is not a limit point of $\overline{K}-l$. But l, by supposition, belongs to H, a continuum of condensation of $\overline{K}$. Thus the supposition that $\overline{K}$ contains a continuum of condensation leads to a contradiction.

Now from each maximal connected subset Y of $\overline{K}$ let us select exactly one point X. Let N denote the set of all the points (X) thus selected. Since $\overline{K}$ contains no continuum of condensation, it readily follows that $\overline{N}$ is a closed and totally disconnected set. It is clear that $\overline{N}$ contains at least one point of every maximal connected subset of M which is of diameter greater than 0. Let Q denote the set of all those maximal connected subsets of M which have no point in common with $\overline{N}$. Then since every maximal connected subset of Q is a single point, it follows from our hypothesis that $\overline{Q}$ is a closed and totally disconnected point set. Let R denote the point set $\overline{N}+\overline{Q}$.

* *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, (2), vol. 16 (1912), p. 109.

† Loc. cit.

Clearly R is closed and totally disconnected; accordingly, there exists a simple continuous arc T_0 which contains R; T_0 contains at least one point of every maximal connected subset of M.

(II). The condition is also necessary. Suppose, on the contrary, that there exists a closed and bounded point set M and a simple continuous arc T such that T contains at least one point of every maximal connected subset of M, but such that M does not satisfy the condition of Theorem 1. Then M contains some continuum K consisting of more than one point and such that for every positive number ϵ, K is the limiting set of a set of maximal connected subsets of M each of diameter less than ϵ. I will show that every point of K must be a limit point of $T-K\cdot T$. For suppose K contains a point P which is not a limit point of $T-K\cdot T$. Let C be a circle having P as center and not enclosing any point of $T-K\cdot T$ and of radius less than $\frac{1}{3}$ of the diameter of K. Let r denote the radius of C. By hypothesis there exists a set L of maximal connected subsets of M each of which is of diameter less than $\frac{1}{4}r$ such that K is the limiting set of L. Since P belongs to K, there exists an element g of L which contains a point whose distance from P is less than $\frac{1}{4}r$; and since g is of diameter less than $\frac{1}{4}r$, g must lie wholly within C. But g must contain at least one point Q of T. Now since K is of diameter $\geqq 3r$, K cannot be an element of L. Hence Q does not belong to K, and therefore must belong to $T-K\cdot T$. But Q lies within C, and C, by supposition, encloses no point of $T-K\cdot T$. It follows, then, that every point of K is a limit point of $T-K\cdot T$. It is easily seen that K must be a subset of T; and since K is closed and connected and consists of more than one point, K must be an arc. And if O denotes an interior point of K, then O is not a limit point of $T-K$. But we have just shown that every point of K is a limit point of $T-K$. Thus the hypothesis that the condition of Theorem 1 is not necessary leads to a contradiction, and the theorem is proved.

Definition. A point set M will be said to satisfy Condition L provided it is true that if K is any continuum whatever consisting of more than a single point, then there exists a positive number ϵ_K such that K is not a subset of the limiting set of any collection of maximal connected subsets of M each of diameter less than ϵ_K.

THEOREM 2. *If M is any closed point set, then in order that there should exist a simple continuous arc which contains at least one point of every maximal connected subset of M it is necessary and sufficient* (1) *that there should exist a bounded portion of the plane which contains at least one point of every maximal connected subset of M, and* (2) *that M should satisfy Condition* L.

It follows by an argument similar to part (II) of the proof of Theorem 1 that the conditions are necessary. I will proceed to show that they are sufficient. By hypothesis it follows that there exists a circle C such that C plus its interior contains at least one point of every maximal connected subset of M. Let R denote the interior of C, and let N denote the set of points common to M and to $R+C$. It readily follows that N satisfies Condition L; and since N is closed and bounded, it follows from Theorem 1 that there exists an arc T which contains at least one point of every maximal connected subset of N. But every maximal connected subset of N belongs to a single maximal connected subset of M, and each maximal connected subset of M contains at least one maximal connected subset of N. If follows, then, that T contains at least one point of every maximal connected subset of M.

THEOREM 3. *If M is a closed point set which satisfies conditions* (1) *and* (2) *of Theorem* 2, *and if K is a closed and bounded subset of M having the property that every subcontinuum of K is either a single point or an arc t such that no point of t, with the exception of its end points, is a limit point of $M-t$, then there exists an arc T which contains K and which contains at least one point of every maximal connected subset of M.*

By an argument almost identical with part (I) of the proof of Theorem 1, it follows that there exists a closed, bounded, and totally disconnected point set R which contains at least one point of every maximal connected subset of M. Let N denote the point set $K+R$. Then clearly N satisfies all the conditions of the above mentioned theorem of Moore and Kline.* Accordingly, there exists a simple continuous arc T which contains N; T, then, contains K and also contains at least one point of every maximal connected subset of M.

It is interesting to note that Theorem 3 is a generalization of Moore and Kline's theorem. It reduces to their theorem in case $K=M$.

THEOREM 4. *If M is a bounded point set such that the totality of all those limit points of M which do not belong to M is a closed set, then in order that there should exist a simple continuous arc which contains at least one point of every maximal connected subset of M it is necessary and sufficient that M should satisfy Condition* L.

That the condition is necessary follows by an argument identical with part (II) of the proof of Theorem 1. I shall show that the condition is sufficient. Let M' denote the totality of all those limit points of M which M

* Loc. cit.

does not contain. For any definite positive integer n, let the sets S, G, T, F, and P be selected exactly as was done in the proof of Lemma I. Then $\overline{P}$ is totally disconnected. For suppose $\overline{P}$ contains a continuum H consisting of more than a single point. Then every point of H is a limit point of a set of points of P which belong to H. And since M' is closed, it readily follows from Janiszewski's theorem mentioned above that H contains a continuum D which consists of more than one point and which is a subset of M. Since D is a subset of a finite number of arcs, then D must contain at least one arc t such that only the end points of t are limit points of $T-t$. But since t belongs to only one maximal connected subset of F, then P contains only one point at most of t. And since only the end points of t can be limit points of P, $\overline{P}$ can contain at most three points of t. Thus the supposition that $\overline{P}$ is not totally disconnected leads to a contradiction. It follows, then, that there exists a simple continuous arc t_n which contains $\overline{P}$, and therefore contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$. Hence, there exists a countable set of arcs $t_1, t_2, t_3, \cdots$, such that for every positive integer n, t_n contains at least one point of every maximal connected subset of M of diameter greater than $1/n$.

Now let the sets $K_1, K_2, K_3, \cdots, K, N$, and R be selected exactly as in the proof of Theorem 1. It can then be shown that R is totally disconnected. For suppose R contains a continuum H consisting of more than one point. Then either (1) H belongs wholly to M', or (2) H contains a subcontinuum D which belongs wholly to M and which consists of more than a single point. In either case, H is a continuum of condensation of $\overline{K}$, and either of the two cases can be shown to lead to a contradiction by the same method as was used in the proof of Theorem 1 to show that the set $\overline{K}$ contained no continuum of condensation. It follows, then, that R is closed and totally disconnected; consequently, there exists a simple continuous arc which contains R and which therefore contains at least one point of every maximal connected subset of M.

THEOREM 5. *In order that a closed point set M (which is not itself an open curve) should be a subset of an open curve, it is necessary and sufficient* (1) *that every subcontinuum of M should be either a single point or a set t such that t is either an arc or a ray of an open curve having the property that no point of t, with the exception of its end point* (*s*), *is a limit point of $M-t$, and* (2) *that if M contains two rays r_1 and r_2, then $M-(r_1+r_2)$ is a bounded point set.*

The conditions are evidently necessary. I shall show that they are sufficient. There exists a circle C with center O such that C plus its interior contains no point of M. By an inversion of the whole plane about the circle

C, M is thrown into a bounded point set M^* which is closed except possibly for the point O. It is easily shown that the image under this inversion of every arc t of M is an arc t^* of M^*, and that the image of every ray r of M is an arc minus one end point in M^*, that end point in every case being the point O itself. Since M contains not more than two mutually exclusive rays, then O is an end point of not more than two arcs of M^*+O which have in common only the point O; and if O is an end point of two such arcs, i.e., if O is an interior point of any arc of M^*+O, then O is not a limit point of M^*+O minus that maximal connected subset of M^*+O to which O belongs. It readily follows, then, that M^*+O is a closed and bounded point set which satisfies all the conditions of Moore and Kline's theorem quoted above. Accordingly, there exists a simple continuous arc t which contains M^*+O. Let A and B denote the extremities of t. There exists an arc t_0 from A to B having only the points A and B in common with t. Let l^* denote the simple closed curve $t+t_0$. It can easily be shown that the point set of which l^*-O is the image is an open curve which contains M.

THEOREM 6. *If M is a closed point set (bounded or not), then in order that there should exist an open curve which contains at least one point of every maximal connected subset of M it is necessary and sufficient that M should satisfy Condition* L.

That the condition is necessary follows by an argument almost identical with part (II) of the proof of Theorem 1. I shall proceed to show that it is sufficient.

Proof I (depending on Theorem 4). Let C denote a circle having center O and not enclosing or containing any point of M. By an inversion of the plane about the circle C, M is thrown into a bounded point set M^* which is closed except possibly for the point O. I shall show that M^* satisfies Condition L. Suppose the contrary is true; then there exists a continuum K consisting of more than one point and such that for every positive number ϵ, K is the limiting set of a set of maximal connected subsets of M^* each of diameter less than ϵ. Let P be a point of K which is different from the point O, and let J^* be a circle having P as center and not containing or enclosing O. It is a consequence of Janiszewski's theorem† that J^* plus its interior contains a subcontinuum H^* of K which consists of more than one point; H^* does not contain O. Hence H, the point set of which H^* is the image under this inversion, is a bounded point set. Let I^* denote the interior of J^*, and let J and I denote the point sets of which J^* and I^* respectively are

† See Janiszewski, loc. cit.

the images. Let G_1^* denote a set of maximal connected subsets of M^* each of which has a point within J^* and is of diameter less than 1 and such that H^* is a part of the limiting set of G_1^*; let G_2^* denote a corresponding set having H^* as a part of its limiting set and such that each element of G_2^* has a point in I^* and is of diameter less than $\frac{1}{2}$; let G_3^*, G_4^*, $\cdots$ denote corresponding sets for the numbers $\frac{1}{3}$, $\frac{1}{4}$, $\cdots$. Let G_1, G_2, G_3, $\cdots$ denote the point sets of which G_1^*, G_2^*, G_3^*, $\cdots$ are the images. Then H is a part of the limiting set of each of the sets G_1, G_2, G_3, $\cdots$. But since M satisfies Condition L, there exists a positive number ϵ_H such that H is not the limiting set of any set of maximal connected subsets of M each of diameter less than ϵ_H. Then for every positive integer n, G_n must contain at least one element g_n which is of diameter $\geqq \epsilon_H$. From each set G_i, select one such element g_i, and let B denote the sequence of sets g_1, g_2, g_3, $\cdots$ thus obtained. Since every element of B contains at least one point in the bounded point set I, it follows that the sequence B contains some subsequence A which has a sequential limiting set $\bar{g}$ which is of diameter $\geqq \epsilon_H$. But A^*, the image of A, has the property that for every positive number ϵ, there are not more than a finite number of elements of A^* of diameter greater than ϵ. Hence, the limiting set $\bar{g}^*$ of A^* must consist of only a single point; but $\bar{g}^*$ is the image of $\bar{g}$, a point set of diameter $\geqq \epsilon_H$. Thus the supposition that M^* does not satisfy Condition L leads to a contradiction. Then since M^* satisfies Condition L and lacks only the point O of being closed, it follows by Theorem 4 that there exists a simple continuous arc t which contains at least one point of every maximal connected subset of M^*. Let X and Y denote the extremities of t. There exists an arc t_0 from X to Y which has only the points X and Y in common with t. Let l^* denote the simple closed curve $t+t_0$. Now if l^* contains the point O, it can readily be shown that the point set of which l^*-O is the image is an open curve which contains at least one point of every maximal connected subset of M. In case l^* does not contain O, then M must satisfy all the conditions of Theorem 2, and with the aid of that theorem, Theorem 6 can easily be established in this case.

Proof II (depending on **Theorem 5**). I will indicate how the condition may be proved sufficient using methods very similar to those used in the proof of Theorem 1. For any definite positive integer n, let the whole plane be divided by a countable infinity of horizontal and vertical straight lines into a countable number of squares plus their interiors in such a way that the interiors of no two of these squares have a point in common and so that the diameter of each of them is less than $1/n$. Let G denote this countable set of squares (not including their interiors) and let T denote the point set obtained by adding together all the point sets of the set G. Let K denote the point set common to T and M. From each maximal connected subset Y of K, select exactly one point X. Let P denote the set of all such points (X) thus selected. It can readily be shown, then, that $\overline{P}$ is a closed and totally disconnected point set. Then from Theorem 5 it follows that there exists an open curve l_n which contains $\overline{P}$; l_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$. Hence, there exists a countable number of open curves $l_1, l_2, l_3, \cdots$, such that for every positive integer n, l_n contains at least one point of every maximal connected subset of M which is of diameter greater than $1/n$. Then by an argument almost identical with that used in the proof of Theorem 1, using this countable set of open curves instead of a countable set of arcs as in that case, it follows that M contains a closed and totally disconnected point set R which contains at least one point of every maximal connected subset of M. Then, by Theorem 5, there exists an open curve l which contains R; l satisfies all the conditions of the open curve required in the statement of Theorem 6.

THEOREM 7. *If M is any continuum whatever, and K is a closed and totally disconnected subset of M, then the point set $M-K$ satisfies Condition* L.

Suppose $M-K$ does not satisfy Condition L. Then there exists a continuum H consisting of more than one point such that for every positive number ϵ, H is a part of the limiting set of a set of maximal connected subsets of $M-K$ each of diameter less than ϵ. Since K is totally disconnected, there exists a point P of H which does not belong to K. Let C be a circle having P as center and not enclosing or containing any point of K. Let r denote the radius of C. Let N denote a set of maximal connected subsets of $M-K$ each of diameter less than $\frac{1}{4}r$, which has H as a part of its limiting set. Then some element g of N contains a point Q whose distance from P is less than $\frac{1}{4}r$; and since g is of diameter less than $\frac{1}{4}r$, $\bar{g}$ must lie wholly within C. But $\bar{g}$ has a point in K,* and C encloses no point of K. Thus the supposition that $M-K$ does not satisfy Condition L leads to a contradiction.

THEOREM 8. *If M is any continuum whatever, and K is a bounded [unbounded], closed, and totally disconnected subset of M, then there exists an arc [open curve] which contains at least one point of every maximal connected subset of the point set $M-K$.*

* By virtue of the following theorem: *If K is any closed subset of a continuum M and g is any bounded maximal connected subset of $M-K$, then K contains at least one point of g.*

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