SIMPLY TRANSITIVE PRIMITIVE GROUPS*

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1. The transitive constituents of the subgroup that leaves fixed one letter of a transitive group occur in pairs of equal degree. Transitive constituents on one letter are to be taken into account in the above statement. The two members of a pair sometimes coincide. This important property of transitive groups was proved by means of a certain quadratic invariant by Burnside in 1900.† It can be more easily demonstrated as follows:

Let G be a non-regular transitive group and let the subgroup G1 of order g1 that fixes one letter a oî G have the transitive constituents B on the letters b, b1, · · · , br−1; C on the letters c, c1, · · · , cm−1; · · · . Consider a permutation S = (cab · · · ) · · · . Every one of the g1 permutations G1S replaces a by b; and because c, c1, · · · are the letters of a transitive constituent of G1, every permutation G1S is of the form (c'ab · · · ) · · · , where c' is some one of the letters c, c1, · · · . Similarly every permutation S Gi is of the form (cab' · · · ) · · · . Then the array G1SG1 includes every permutation of G in which a is preceded by one of the s c's or is followed by one of the r b's. Now the number of distinct permutations in the array G1SG1 is g12 divided by the number of permutations common to d and S GiS−1, or common to S−1G1S and G1;‡ that is, by the number of permutations of G1 that fix c or b. These numbers are g1/s and g1/r. Therefore r = s. If, as often happens, every permutation (ab · · · ) · · · is of the form (ab) · · · or (b1ab · · · ) · · · , the transitive constituent B is paired with itself. Since the product

(a)(bb · · · ) · · · (b1ab · · · ) · · · = (ab) · · · ,

G, whenever a transitive constituent of G1 is paired with itself, is of even order. A properly chosen odd power of this product is a permutation (ab) · · · of order a power of 2.

2. As in §1, G is a transitive group of order g and G1 is a subgroup that fixes one of the n letters of G. Let H be a subgroup of G1 of degree n − m (0 < m < n) and let I be the largest subgroup of G in which H is invariant.§ If H is one of r conjugate subgroups in G1, the largest subgroup of G1 in which

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* Presented to the Society, §§1–7, San Francisco Section, October 30, 1926; §§8–9, December 31, 1926. Received by the editors December 13, 1926.


‡ Miller, these Transactions, vol. 12 (1911), p. 326.

§ Manning, these Transactions, vol. 19 (1918), p. 129.
$H$ is invariant is of order $g/\mathfrak{n}r$ and the order of $I$ is $gm/\mathfrak{n}s$, where $s$ is the total number of conjugates of $H$ under $G$ found in $G_1$. Then $I$ has a transitive constituent of degree $mr/\mathfrak{s}$ in letters fixed by $H$. The letters of this constituent are $a$, $a_1$, \ldots and $a$ is the letter fixed by $G_1$. Every permutation $(aa_1, \ldots)$ \ldots of $G$ transforms $H$ into one of its $r$ conjugates $H$, $H'$, \ldots under $G_1$. A second set of $r_1$ conjugate subgroups of $G_1$ is $H_1$, $H'_1$, \ldots. Each of the $g_1$ permutations $(ab \ldots)$ \ldots of $G$ transforms one of the $r_1$ subgroups $H_1$, $H'_1$, \ldots into $H$, so that in $G_2$, the subgroup of $G$ that fixes $b$, $H$ is one of $r_1$ conjugates and $I$ has a transitive constituent of degree $mr_1/\mathfrak{s}$ on the letters $b$, $b_1$, \ldots. The two sets of letters $a$, $a_1$, \ldots and $b$, $b_1$, \ldots do not coincide. If there is a third conjugate set of $r_2$ members $H_2$, $H'_2$, \ldots in $G_1$, there is a third transitive constituent $c$, $c_1$, \ldots of degree $mr_2/\mathfrak{s}$ in $I$, and so on. It is this correspondence between the conjugate sets $H$, $H'$, \ldots; $H_1$, $H'_1$, \ldots; \ldots of $G_1$ and the transitive constituents $a$, $a_1$, \ldots; $b$, $b_1$, \ldots; \ldots of $I$ in the letters fixed by $H$, which is to be borne in mind in the developments of the following sections.

3. Examples of transitive groups in illustration of the preceding theory may be helpful.

\begin{align*}
G_{10}^5 &= \{(bb_1)(cc_1), (ab)(b_1c)\}.
G_{21}^7 &= \{(bb_1b_2)(cc_1c_2), (abc)(b_1c_2b_2)\}.
G_{72}^9 &= \{(bb_1)(bb_2)(cc_1), (bb_2)(cc_2)(c_1c_2), (ab)(b_1c_2)(b_2c_1)\}.
G_{122}^{10} &= \{(bb_1)(cc_4)(c_1c_6), (bb_2)(cc_2c_4)(c_3c_6), (cc_1)(c_2c_4)(c_3c_6),
\quad (ab)(b_1c_4)(b_2c_4)(c_1c_2)\}.
\end{align*}

These four groups are primitive. In the one of degree 10 the subgroup that leaves $a$ fixed is the following:

\begin{align*}
1, \quad & (cc_1)(c_2c_3)(c_4c_6), \\
(bb_1b_2)(cc_4c_6c_1c_6), \quad & (bb_1b_2)(cc_4c_6)(c_1c_6c_4), \\
(bb_2b_1)(cc_1c_3c_4c_6), \quad & (bb_2b_1)(cc_1c_4c_6)(c_1c_3c_4), \\
(bb_1)(cc_4)(c_1c_6), \quad & (bb_1)(cc_4)(c_1c_6)(c_3c_6), \\
(bb_2)(cc_2c_4)(c_3c_6), \quad & (bb_2)(cc_3)(c_2c_4)(c_3c_4), \\
(bb_1b_2)(cc_2)(c_1c_6), \quad & (bb_1b_2)(cc_1c_4c_6)(c_1c_2c_6). 
\end{align*}

4. In all that follows $G$ is a simply transitive primitive group. $G_1$ is intransitive of degree $n - 1$ and is a maximal subgroup of $G$. Let $H$ be invariant in $G_1$ and of degree $<n - 1$. Then $I$ and $G_1$ coincide and $I$ may be said to have one transitive constituent on the one letter $a$ fixed by $G_1$. Hence the factor $m/s$ of \S2 is unity. Since $H$ is invariant in $G_1$, it must fix all the letters of
one or more transitive constituents of $G_i$. Because $G_i$ does not fix two letters of $G$, of all the conjugates of $H$ under $G$, only $H$ is invariant in $G_i$. Therefore $H$, like the letter $a$, is characteristic of $G_i$. It is not, however, to be inferred from this statement that $H$ is a “characteristic” subgroup of $G_i$ in the strong sense of being invariant in the holomorph of $G_i$. What is meant is that the $n$ conjugate subgroups $H, \cdots$ are in one-to-one correspondence to the $n$ subgroups $G_i, \cdots$ and therefore are in one-to-one correspondence to the $n$ letters $a, \cdots$ of $G$. In $G_i$ there are exactly $m - 1$ non-invariant subgroups $H_i, \cdots$, and “they are transformed by $G_i$ in the same manner as the letters of one of $G_i$’s constituent groups of degree $m - 1$.”* The constituent group may be transitive or intransitive. This well known conclusion leaves open the question as to whether or not this constituent of $G_i$ according to which the $m - 1$ subgroups $H_i, \cdots$ are permuted contains letters displaced by $H$.† This is an unsolved problem of fundamental importance.

If the transitive constituent $B$ (on letters fixed by $H$) is paired with itself in the sense of §1, the permutation $S = (ab) \cdots$, known to exist in $G$, which transforms $H_i$ into $H$, has an inverse $S^{-1} = (ab) \cdots$ which transforms $H$ into $H_1$ and which transforms some member $H'_1$ of the conjugate set $H_1, H'_1, \cdots$ of $G_i$ into $H$ (§2). Now $H_1$ is the invariant subgroup of the subgroup $G_i$ that fixes $b$ and in which $H$ is included. The $r_i$ conjugate subgroups $H_1, H'_1, \cdots$ are therefore permuted according to the permutations of the transitive constituent $B$ of $G_i$. Conversely, if in the permutation $S = (ab) \cdots$ of §2 which transforms $H_1$ into $H$, the letter $b$ is one of the transitive set according to which $H_1, H'_1, \cdots$ are permuted by $G_i$, $B$ is paired with itself.

If two transitive constituents $B$ and $C$, both in letters not displaced by $H$, are paired, there is in $G$ a permutation $S = (cab \cdots) \cdots$ such that

$$S^{-1}H_1S = H, \quad SHS^{-1} = H;$$

and hence

$$SHS^{-1} = H_1, \quad S^{-1}HS = H_2.$$

Then $H_1, H'_1, \cdots$ are permuted according to the transitive constituent $C$, and $H_2, H'_2, \cdots$ are permuted according to the transitive constituent $B$. Here also the converse is true.

**5. Theorem I.** If all the transitive constituents of $H$ are of the same degree, or if no two (not of maximum degree in $H$) belong to the same transitive constituent of $G_i$, every subgroup of $G_i$ similar to $H$ is transformed into itself by $H$.

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The above conclusion is equivalent to the statement that the constituent of $G_1$ according to which the $m-1$ subgroups $H_1, H'_1, \cdots$ are permuted displaces no letter of $H$. For if $H_1, H'_1, \cdots$ are permuted according to the constituent $B$ of $G_1$ on the $r_1$ letters $b, b_1, \cdots$ fixed by $H$, the subgroup of $G_1$ that fixes $b$, say, is the largest subgroup of $G_1$ in which $H_1$ is invariant and includes $H$. Conversely, if each of the $r_1$ subgroups $H_1, H'_1, \cdots$ is transformed into itself by every permutation of $H$, the transitive constituent of degree $r_1$ of $G_1$ according to which they are permuted displaces no letter of $H$.

The letter of $G$ fixed by $G_i$ is $a$. Let $b, b_1, \cdots$ be certain letters fixed by $H$ but permuted transitively by $G_1$. Let $\alpha, \alpha_1, \cdots, \beta, \beta_1, \cdots, \cdots$ be the letters of some transitive constituent of $G_1$ and such that $\alpha, \alpha_1, \cdots$ is one transitive constituent of $H, \beta, \beta_1, \cdots$ is another, and so on. Of course if $H$ displaces one letter of a transitive constituent of $G_1$ it displaces every letter of that constituent.

Now transform $G_1$ into $G_2$ by means of a permutation $(ab \cdots) \cdots$ of the primitive group $G$. At the same time $H$, an invariant subgroup of $G_1$, is transformed into an invariant subgroup of $G_2$. Call the latter subgroup $H_b$. We wish to show that $H_b$ is necessarily a subgroup of $G_1$. Suppose it is not a subgroup of $G_1$. Then since a primitive group is generated by a subgroup leaving one letter fixed and any permutation of the group not in that subgroup, $\{G_1, H_b\} = G$. But if $H_b$ fails to connect transitively letters of $H$ and letters fixed by $H$, $\{G_1, H_b\}$ is intransitive. Hence, if $H_b$ is not a subgroup of $G_1$, at least one of its permutations unites letters of $H$ and letters fixed by $H$. Let us now impose upon the transitive constituent $\alpha, \alpha_1, \cdots$ of $H$ the condition that no transitive constituent of $H$ is of higher degree. The set $\beta, \beta_1, \cdots$, being in the same transitive constituent of $G_1$, will have exactly the same number of letters as the set $\alpha, \alpha_1, \cdots$. If $\alpha$ and $x$ (let $x$ be one of the $m$ letters $a, b, b_1, \cdots$ fixed by $H$) are in the same transitive constituent of $H_b$, so also are all the other letters $\alpha_1, \alpha_2, \cdots$ of that transitive constituent of $H$. For since $H$ fixes $b$, it is a subgroup of $G_2$, and in consequence every permutation of $H$ transforms $H_b$ into itself. Then $H_b$ has a transitive constituent $x, \alpha, \alpha_1, \cdots$ of higher degree than any transitive constituent of $H$, to which $H_b$ is conjugate under $G$; — an absurd result. Similarly the constituent $\beta, \beta_1, \cdots$ of $H_b$ displaces no letter fixed by $H$. If $H_b$ fixes all the letters $\alpha, \beta, \cdots$ of one transitive constituent of $G_1$, the group $\{G_1, H_b\}$ is intransitive. Then $H_b$ connects the letters $\alpha, \beta, \cdots$ only with letters of $H$. Thus if all the transitive constituents of $H$ are of the same degree the theorem is proved.

The letters $\lambda, \lambda_1, \cdots$ of a transitive constituent of $H$ of lower degree are by hypothesis the letters of a transitive constituent of $G_1$. The transitive
constituent $x, \lambda, \lambda_1, \cdots$ of $H_b$ contains all the letters of the transitive constituent $\lambda, \lambda_1, \cdots$ or $G_1$. Then the transitive constituents $\alpha, \cdots$ and $x, \lambda, \cdots$ of $H_b$ are not united by $G_1$.

**Corollary I.** If $G_1$ has only two transitive constituents and contains an invariant subgroup $H$ of degree $<n-1$, every subgroup of $G_1$ similar to $H$ is transformed into itself by $H$, and $G$ is of even order.

In this case $H$ is an invariant intransitive subgroup of an imprimitive group and all its transitive constituents are of the same degree. It is of even order because each transitive constituent of $G_1$ is paired with itself.

It was proved by Rietz* that if $G$ is of odd order and if $G_1$ has only two transitive constituents, $G_1$ is a simple isomorphism between its two constituents.

**Corollary II.** If $G$ is of even order and $G_1$ has only two transitive constituents, each transitive constituent of $G_1$ is paired with itself.

For $G$ certainly contains a permutation $(ab) \cdots$ of order 2 which pairs one of the transitive constituents $(B)$ with itself.

**Corollary III.** If $G_1$ has three and only three transitive constituents and contains an invariant subgroup $H$ of degree $<n-1$, every subgroup of $G_1$, similar to $H$, is transformed into itself by $H$.

This is true except perhaps when $H$ has transitive constituents $\alpha, \alpha_1, \cdots, \beta, \beta_1, \cdots, \cdots$ of degree $t$; and transitive constituents $\lambda, \lambda_1, \cdots, \mu, \mu_1, \cdots, \cdots$ of degree $v (\leq t)$. The letters fixed by $H$ are $b, b_1, \cdots$ of the transitive constituent $B$.

Assume as before that $H_b$ displaces $a$, that is, $H_b$ is not a subgroup of $G_1$. If the only letters in the transitive constituent of $H_b$ with $a$ are letters of $B$, $G$ contains a permutation $(b'ab_1 \cdots) \cdots$, where $b'$ is a letter of $B$, and the constituent $B$ is paired with itself. This would prove the Corollary, so that $H_b$ has a transitive constituent $a, \lambda, \cdots$ of degree $t$. Then $G_1$ is transformed into $G_2$ by a permutation $S = (aab \cdots) \cdots$ which pairs the constituents $b, b_1, \cdots$ and $a, a_1, \cdots$ of $G_1$. If the transitive constituent $a, \lambda, \cdots$ of $H_b$ contains a letter of $B$, $H_b$ contains a permutation $(a_1b_1 \cdots) \cdots$ in which $a$ is preceded by a letter of $B$ or by one of the letters $\lambda, \cdots, \mu, \cdots$. But every permutation of $G$ which replaces $a$ by $a'$ belongs to the array $G_1SG_1$, in which only $a$'s precede $a$. Now the transitive constituent $a, \lambda, \lambda_1, \cdots$ of $H_b$ is of degree $t = kv + 1$ if it contains letters of $k$ transitive constituents of $H$. In one of the transitive constituents of $H_b$ are found letters $\alpha$ and letters $\mu$.

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* Rietz, loc. cit., p. 11, Theorem 10.
for only thus can $H_b$ unite these two transitive constituents of $G_i$. This transitive constituent of degree $v$ in $H_b$ cannot displace all the $t$ letters $\alpha, \alpha_1, \ldots, \alpha_{t-1}$. But $H_b$, being transformed into itself by $H$, displaces all the $t+v$ letters $\alpha, \alpha_1, \ldots, \mu, \mu_1, \ldots$. $H$ permutes $q$ (say) transitive constituents $\alpha, \mu, \ldots$ of $H_b$. This means that the transitive constituent $\alpha$ of $H$ has $q$ systems of imprimitivity of $t/q$ letters and that the transitive constituent $\mu$ of $H$ has $q$ systems of imprimitivity of $v/q$ letters. That $t$ and $v$ should have a common factor $q$ is inconsistent with the preceding result, $t = kv + 1$. Therefore $H_b$ is a subgroup of $G_i$.

6. **Theorem II.** Let $G$ be a simply transitive primitive group in which each of the $m$ subgroups $H, H_1, \ldots$ of $G_i$ is transformed into itself by every permutation of $H$, the invariant subgroup of $G_i$. If the degree of the group generated by the complete set of conjugates $H_1, H_1, \ldots$ of $G_i$ is less than $n - 1$, the letters of $G_i$ left fixed by it are the letters of one or more of the transitive constituents whose letters are already fixed by $H$.

The group $K$ generated by the $r_1$ conjugates $H_1, H'_1, \ldots$ is an invariant subgroup of $G_i$. By hypothesis $K$ is of degree $< n - 1$. There is therefore in $G_i$ a complete set of conjugate subgroups $K_1, K'_1, \ldots$, similar to $K$, permuted according to a transitive constituent $X$ of $G_i$. Now $K_1$ is conjugate to $K$ under transformation by some permutation of $G$, so that $K_1$ is generated by some of the $n$ subgroups $H, H_1, \ldots$. Because $K_1$ fixes $a$, its $r_1$ generating subgroups are subgroups of $G_i$. All the permutations of $X$ except the identity actually permute two or more of the subgroups $K_1, K'_1, \ldots$, conjugate under $G_i$. Then the identity is the only permutation of $X$ that can occur in a constituent of $H$, because by hypothesis every permutation of $H$ transforms each of the $m$ subgroups $H, H_1, \ldots$ of $G_i$ into itself. Then $X$ is one of the transitive constituents $B, C, \ldots$ of $G_i$ that displace no letter of $H$. What is true of $K_1, K'_1, \ldots$ is true of all such conjugate sets of non-invariant subgroups of $G_i$ similar to $K$. Hence the only letters of $G_i$ fixed by $K$ are letters already fixed by $H$.

7. **Theorem III.** If only one transitive constituent of $G_i$ is an imprimitive group (of order $f$), $G_i$ is of order $f$.

Let $B$ be the imprimitive constituent of $G_i$. Suppose that a subgroup $H$ of $G_i$ corresponds to the identity of $B$. All the $n - m$ letters of $G_i$ primitive constituents of $G_i$ are displaced by $H$. The $m - 1$ other letters of $G_i$ are distributed among $w$ transitive constituents $B, C, \ldots$. Since an invariant subgroup of a primitive group is transitive, no two transitive constituents of $H$ belong to the same transitive constituent of $G_i$. Then by Theorem I, each of the $m - 1$ non-invariant subgroups $H_1, H'_1, \ldots$ similar to $H$ of $G_i$ is
transformed into itself by $H$. By Theorem II, the group $K$ generated by the conjugate set $H_1, H'_1, \cdot \cdot \cdot$ of $G_1$ displaces all the letters of $H$. Since $K$ is not $H$ or a subgroup of $H$, $K$ also displaces the letters of the imprimitive constituent $B$. If one of the generators $H_1, H'_1, \cdot \cdot \cdot$ of $K$ fixes all the letters of a transitive constituent of $G_1$, $K$ fixes all the letters of that constituent. Hence $H_1$ has $v+1$ or more transitive constituents. Under $G$, $H$ is conjugate to $H_1$ and therefore also has $v+1$ or more transitive constituents. But $H$ displaces the letters of $v$ primitive constituents of $G_1$ and has exactly $v$ transitive constituents. Hence it is impossible that the order of $G_1$ exceeds that of the imprimitive constituent $B$.

**Corollary I.** If all the transitive constituents of $G_1$ are primitive groups, $G_1$ is a simple isomorphism between its transitive constituents.

Each of the primitive constituents of $G_1$ may be put in turn in the place of the imprimitive constituent of Theorem III.

**Corollary II.** If $G_1$ has an intransitive constituent of order $f$, and if all the other transitive constituents of $G_1$ are primitive groups, $G_1$ is of order $f$.

8. It has been known since 1921 that if one of the transitive constituents of $G_1$ of maximum degree is doubly transitive, $G_1$ is a simple isomorphism between its transitive constituents. Moreover the transitive constituents are similar groups whose corresponding permutations are multiplied together. For this is an immediate consequence of the following theorem:*

**Theorem IV.** Let $G_1$ have a $t$-ply ($t \geq 2$) transitive constituent of degree $m$. If $G_1$ has no transitive constituent whose degree is a divisor ($> m$) of $m(m-1)$, all the transitive constituents of $G_1$ are similar groups of order $g/n$.

To guard against misunderstanding, we recall that two groups $G$ and $G'$ are equivalent when there exists a permutation by which one can be transformed into the other; and that two equivalent groups are similar when, if $S_i$ and $S'_i$ are corresponding permutations in some isomorphism of $G$ to $G'$, a permutation $T$ exists such that $T^{-1}S_iT = S'_i$ ($i = 1, 2, \cdot \cdot \cdot, g$).† In the group $G^8_{10}$ of §3, the two octic constituents of $G_1$ are isomorphic and equivalent but are not similar. In $G^8_{19}$ and $G^7_{21}$ of §3, the two constituents of $G_1$ are similar.

A useful set of theorems having to do with simply transitive primitive groups was given by Dr. E. R. Bennett in 1912.‡ In particular, Corollary II to Theorem V, page 6, reads:

"If the transitive constituent of degree \( m \) in \( G_1 \) is a \( t \)-times transitive group \((t \geq 3)\), \( G_1 \) always contains an imprimitive group of degree \( m(m-1) \)."

This result, in common with Dr. Bennett's Theorems I to VI, is subject to the following conditions upon the simply transitive primitive group \( G \) (of degree \( n \)) and its maximal subgroup \( G_x \):

1. The constituent \( M \) (of degree \( m \)) of \( G_1 \) is a non-regular transitive group.
2. \( M \) is the only transitive constituent of \( G_1 \) whose degree divides \( m \).
3. Corresponding to the identity of \( M \) there is a subgroup \( H \) in \( G_1 \) of degree \( n-m-1 \).

This corollary raises interesting questions as to possible extensions of our Theorem IV, the proof of which is based merely upon the hypothesis that one constituent of \( G_1 \) is (at least) doubly transitive. Conditions (2) and (3) may be replaced by the weaker conditions of the following theorem:

**Theorem V.** Let \( G_1 \), the subgroup that leaves fixed one letter of the simply transitive primitive group \( G \) of degree \( n \) and order \( g \), have a primitive constituent \( M \) of degree \( m \), in which the subgroup \( M_x \) that fixes one letter is primitive. Let \( M \) be paired with itself in \( G_1 \) and let the order of \( M \) be \(<g/n\). Then \( G_1 \) contains an imprimitive constituent in which there is an invariant intransitive subgroup with \( m \) transitive constituents of \( m-1 \) letters each, permuted according to the permutations of the primitive group \( M \).

The letter of \( G \) fixed by \( G_1 \) is \( x \), and the letters of \( M \) are \( a, a_1, \ldots, a_{m-1} \). The subgroup of \( G \) that fixes both \( x \) and \( a \) is \( F \). In \( F \), \( a_1, a_2, \ldots, a_{m-1} \) are the letters of a primitive constituent group. Because \( M \) is paired with itself in \( G_1 \) (§1), \( G \) contains a permutation \( S = (xa) \ldots \) which transforms \( F \) into itself, \( G_1 \) into \( G_2 \) (fixing \( a \)), \( M \) of \( G_1 \) into a transitive constituent of \( G_2 \) on the letters \( x, b_1, \ldots, b_{m-1} \), and the transitive constituent \( a_1, a_2, \ldots, a_{m-1} \) of \( F \) into a transitive constituent \( b_1, b_2, \ldots, b_{m-1} \) of \( F \). These two transitive constituents of \( F \) are distinct because if they have one letter in common they have every letter in common and therefore \( \{G_1, G_2\} \) would permute the \( m+1 \) letters \( x, a, a_1, \ldots, a_{m-1} \) only among themselves, making \( G \) either intransitive or doubly transitive, contrary to hypothesis. Nor can the letters \( b_1, b_2, \ldots, b_{m-1} \) be the letters of a transitive constituent of \( G_1 \). For if so, \( \{G_1, G_2\} \) has a transitive constituent of degree \( m \) in the letters \( x, b_1, \ldots, b_{m-1} \). The letters \( b_1, b_2, \ldots \) belong to a transitive constituent \((P)\) of \( G_1 \) of degree \( \geq m \).

There is an invariant subgroup \( H \) in \( G_1 \) corresponding to the identity of \( M \). Because \( M \) is paired with itself in \( G_1 \), there is a complete set of \( m \) conjugate subgroups \( H_1, H'_1, \ldots \), similar to \( H \), in \( G_1 \) which are permuted
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according to the transitive constituent $M$ ($\S 4$). Let $H_1$ correspond to the
text below:

letter $a$ of $M$. The largest subgroup of $G_1$ in which $H_1$ is invariant is $F$;
$H$ and $H_1$ are the invariant subgroups of $G_1$ and $G_2$ respectively. Since $H_1$
is a subgroup of $G_1$ and is not a subgroup of $H$, it displaces at least one letter
of $M$, and since it is invariant in $F$ it displaces the $m-1$ letters $a_1, a_2, \ldots, a_{m-1}$. Since $S^{-1}H_1S = H$, $H$ displaces the $m-1$ letters $b_1, b_2, \ldots, b_{m-1}$.

Now $P$ (of degree $\geq m$) has an invariant intransitive subgroup in $H$ with one
transitive constituent of degree $m-1$. It is therefore an imprimitive group
with systems of $m-1$ letters each. The only permutations of $G_1$ that permute
these $m-1$ letters $b_1, b_2, \ldots, b_{m-1}$ among themselves are the permutations
of $F$, the subgroup of $G_1$ that fixes $a$. Because $M$ is primitive, $F$ is a maximal
subgroup of $G_1$ and is one of $m$ conjugates under $G_1$. Then there are $m$ systems
of $m-1$ letters each in $P$ and they are permuted according to a primitive
group of degree $m$. That this primitive group is exactly $M$ is evident from
a consideration of the $m$ conjugate subgroups $F, F_1, \ldots$. For $F$ fixes $a$
and fixes the constituent $b_1, b_2, \ldots, b_{m-1}$ of $P, F_1$ fixes $a_1$ and the constituent
$b_1', b_2', \ldots, b_{m-1}'$ of $P$, and so on.

9. It is worth while to extend Dr. Bennett's Theorems I to VI, replacing
the three given conditions by the single condition that $M$ is a transitive
constituent of $G_1$ "paired with itself," and adding other limitations only as
needed. We shall use the notation of the preceding section ($\S 8$).

Suppose $G_1$ has a transitive constituent $Q$ of degree $q$. This constituent
$Q$ is transformed by $S$ into a transitive constituent of $G_2$ which must include
at least one letter new to $Q$.

If $F$ permutes the letters of $Q$ transitively, $S$ transforms $F$ into itself and
therefore transforms this transitive constituent $(A)$ of $F$ on the letters of $Q$
into a second transitive constituent $(B)$ of $F$. There is no letter of $Q$ in $B$.
Since $G_1$ and $G_2$ cannot have transitive constituents on the same letters,
$G_1$ has a transitive constituent of degree $>q$ in which all the letters of $B$
 occur.

If $F$ does not permute the letters of $Q$ transitively, the letters of at
least one transitive constituent of degree $l (\geq 1)$ of a subgroup of $Q$ found in
$F$ is replaced by $S$ by letters new to $Q$.

Instead of $Q$, consider now $M$ and its subgroup $M_1$ that fixes $a$. The
permutation $S$ transforms the constituent $M$ of $G_1$ into a transitive constituent
of $G_2$ on the letters $x, b_1, b_2, \ldots, b_{m-1}$. The letters $b_1, b_2, \ldots, b_{m-1}$
do not coincide with the letters $a_1, a_2, \ldots, a_{m-1}$, for $\{G_1, G_2\}$ is simply
transitive. If $M_1$ is transitive, $\{F, S\}$ has an imprimitive constituent with
the two distinct systems $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$. If $M_1$ is intransitive,
or if \( M \) is regular, at least one transitive constituent \( b_1, b_2, \ldots, b_l \) of \( S^{-1}M_1S \) (\( l \geq 1 \)) contains none of the letters \( a_1, a_2, \ldots, a_{m-1} \).

We now impose the condition that the order of \( M \) is \(< g/n \).

The invariant subgroup \( H \) of \( G_1 \), corresponding to the identity of \( M \), is conjugate under \( G \) to \( m \) subgroups \( H_1, H_1', \ldots \) which \( G_1 \) permutes according to the transitive constituent \( M \). The subgroup \( H_1 \), say, is an invariant subgroup of \( G_2 \), and must displace some \( j(>1) \) letters of \( M: a_1, a_2, \ldots, a_j \).

Thus if \( M \) is a regular group, its order is \( g/n \).* The transform of \( H_1 \) by \( S \) is \( H \). Let \( b_1, b_2, \ldots, b_l \) be the letters by which \( S \) replaces \( a_1, a_2, \ldots, a_j \).

By definition \( H \) fixes all the letters of \( M \). All the letters of the transitive constituents of \( G_1 \) to which \( b_1, b_2, \ldots, b_l \) belong are displaced by \( H \).

Suppose the letters \( b_1, b_2, \ldots, b_k \) \((k \geq j)\), new to \( M \) in the constituent of \( G_2 \) by which \( S \) replaces \( M \), are permuted only among themselves by \( G_1 \). Clearly \( k < m - 1 \), for if \( k = m - 1 \), \( \{G_1, G_2\} \) has a transitive constituent on the letters of \( M \). Then \( \{G_1, G_2\} \) is of degree \( m + k + 1 < 2m \). Of the letters \( b_1, b_2, \ldots, b_k, H \) displaces only \( b_1, b_2, \ldots, b_l \) and therefore \( H_1 \) displaces only \( a_1, a_2, \ldots, a_j \). The subgroup \( \{H_1, H_1', \ldots, H_1^{m-1}\} \), invariant in \( G_1 \), displaces only letters of \( M \). Being a subgroup of \( G \) of degree \( < n \), it must be intransitive and therefore \( M \), in which it is invariant, is imprimitive. If \( k = j \), all the non-invariant subgroups of \( G_1 \), similar to \( H \), are in a single conjugate set and are permuted according to the permutations of \( M \). Each of these subgroups \( H_1, H_1', \ldots, H_1^{m-1} \) is transformed into itself by \( H \), and Theorem II is applicable to the group \( \{H_1, H_1', \ldots\} \), which however is seen to fix the wrong letters. Then \( k > j \).

Let it be assumed that \( M \), if imprimitive, is of degree \( \leq n/2 \). Another assumption that might be made is that \( k = j \). This last is a weaker form of Dr. Bennett's condition upon the degree of \( H \): that it is \( n - m - 1 \). It follows that there is in \( F \) at least one transitive constituent \((B_l)\) on \( l \) of the \( k \) letters \( b_1, b_2, \ldots \) which is a part of a transitive constituent \( P \) of \( G_1 \) in which occur letters \( c_1, \ldots \) new to \( S^{-1}MS \). Finally put upon \( G_1 \) the condition that these \( l \) letters of \( B_1 \) are permuted transitively by \( H \). They may now be called \( b_1, b_2, \ldots, b_l \) \((1 < l \leq j)\). This transitive constituent \( P \) of \( G_1 \) is imprimitive because of its invariant intransitive subgroup in \( H \).

If \( M \) is primitive, \( F \) is a maximal subgroup of \( G_1 \), and therefore \( F \) is the largest subgroup of \( G_1 \) by which the letters \( b_1, b_2, \ldots, b_l \) are permuted only among themselves. There are \( m \) conjugate subgroups \( F, F_1, \ldots \) in \( G_1 \). Hence \( P \) permutes \( m \) systems of imprimitivity, of which \( b_1, b_2, \ldots, b_l \) is one, according to the primitive group \( M \).

* Rietz, loc. cit., p. 9, Theorem 7.
If $M$ is imprimitive, $F$ is not maximal, and the letters $b_1, b_2, \ldots, b_i$ may be transformed among themselves by a subgroup of $G_i$ of which $F$ is a subgroup. Hence our transitive constituent has $m'$ (a divisor of $m$) systems of imprimitivity of $l$ letters each. These last results may be formulated as follows. The notation of this section is used.

**Theorem VI.** Let $G_1$ have a transitive constituent $M$, of order $<g/n$, paired with itself, and of degree $\leq n/2$ if $M$ is imprimitive. There is a transitive constituent $B_i$ in $S_1^{-1}M_1S$ on $l$ letters new to $M$ which is a part of a transitive constituent $P$ of $G_i$ in which are letters new to $M$ and to $S_1^{-1}MS$. If the letters of $B_i$ are permuted transitively by $H$, $P$ has $m$ systems of imprimitivity of $l$ letters each if $M$ is primitive, or $m'$ ($m' > 1$ and a divisor of $m$) systems of $l$ letters each if $M$ is imprimitive.

For example, if all the transitive constituents of $M_1$ are primitive groups and if $H_1$ displaces $m - 1$ or $m - 2$ letters of $M$, then certainly the letters of $B_i$ are permuted transitively by $H$. By putting on the restriction that $n \geq 2m$ when $M$ is imprimitive, and with no corresponding condition when $M$ is primitive, the condition "if no transitive constituent of degree $l$ occurs in $G_i$, where $l$ represents the degree of any of the transitive constituents of the subgroup of $M$ composed of all the substitutions leaving one letter of $M$ fixed" of Dr. Bennett's Theorems V and VI, has been avoided. In the following theorem this condition is restored in a modified form.

**Theorem VII.** Let $G_1$ have a transitive constituent $M$ of order $<g/n$, paired with itself. Let those transitive constituents of $M_1$ whose $j$ letters are displaced by $H_1$ be primitive groups. Let $G_i$ have no constituent of degree $j$. Then $G_1$ has an imprimitive constituent of degree $m'$, where $l$ is the degree of one of the transitive constituents of $M_1$ whose letters are displaced by $H_1$ and where $m'$ divides $m$ if $M$ is imprimitive and is equal to $m$ if $M$ is primitive.

It was seen that $H_1$ displaces $j$ letters $a_1, a_2, \ldots, a_j$ of $M_1$ and that $H$ displaces $b_1, b_2, \ldots, b_i$ and no other letters of $S_1^{-1}M_1S$. Since $H_1$ is an invariant subgroup of $F$, $H_1$ displaces all the letters of a transitive constituent of $M_1$ if it displaces one of the letters of that constituent. If $G_i$ does not permute $b_1, b_2, \ldots, b_i$ in one or several transitive constituents, one primitive constituent $b_1, b_2, \ldots, b_i$ of $F$ has a transitive subgroup in $H$ and is part with letters $c, \ldots$, new to $M$ and to $S_1^{-1}MS$, of an imprimitive constituent $P$ of degree $m'$. Our theorem follows as before.

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