CUBIC CURVES AND DESMIC SURFACES; SECOND PAPER*

BY

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1. **Introduction.** It is evident superficially that there is some connection between cubic curves and desmic surfaces. It is well known that the equation of every cubic curve of the sixth class can be reduced to the form

$$y^2 = 4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3), \tag{1}$$

and the coördinates of a point on this curve are given parametrically as $x=\wp(u)$, $y=\wp'(u)$, where $\wp(u)$ is Weierstrass's elliptic $\wp$-function which is a solution of the differential equation

$$[\wp'(u)]^2 = 4[\wp(u)]^3 - g_2\wp(u) - g_3. \tag{2}$$

To every cubic of genus 1 there corresponds such a $\wp$-function and to every $\wp$-function there corresponds a projective class of cubics. On the other hand, the desmic surface may be defined analytically as the locus of a point whose coördinates are

$$x_0 = \frac{\sigma(u)}{\sigma(v)}, \qquad x_1 = \frac{\sigma_1(u)}{\sigma_1(v)}, \qquad x_2 = \frac{\sigma_2(u)}{\sigma_2(v)}, \qquad x_3 = \frac{\sigma_3(u)}{\sigma_3(v)}, \tag{3}$$

where

$$\left(\frac{\sigma_i(u)}{\sigma(u)}\right) = \wp(u) - e_i. \tag{4}$$

Thus to every desmic surface corresponds a $\wp$-function and to every such function a class of projective desmic surfaces.

This superficial analytic indication of relationship between the classes of curves and surfaces sets the problem of finding intimate geometrical connections. I have shown some of these relations in a former paper† and now present others.

2. **Setting of the problem.** We recall first some known facts. (i) From an arbitrary point A on a cubic curve C_6^3 of the sixth class four tangents can

* Presented to the Society, April 2, 1926; received by the editors in January, 1927.

† These Transactions, vol. 28 (1926) pp. 502–522.

be drawn. The cross ratios of this pencil of tangents are constant as A describes the curve, and they are the mutual ratios of the roots of $z^2-3z+2/I^{1/2}=0$ where I denotes the absolute invariant $64S^3/(64S^3+T^2)$ of the cubic, or equally well of the corresponding $\wp$-function. (ii) If ABC are three collinear points on C_6^3, then the three quadrangles of the points of contact of the tangents from them form a $(12_4, 16_3)$ Hessian configuration on the curve. Any two of the quadrangles are perspective from each vertex of the third. (iii) Three tetrahedra are in *desmic* formation when their vertices form a $(12_4, 16_3)$ space configuration, any two of the tetrahedra being perspective from each vertex of the third. (iv) They form the base of a pencil of quartic surfaces, called *desmic surfaces*. The points are the 12 nodes and the lines of perspectivity are the 16 lines on each surface. (v) The pencil contains three degenerate surfaces, namely the tetrahedra. If $\lambda=0$, $\mu=0$, $\nu=0$ be the equations of these three surfaces, then $\lambda+\mu+\nu\equiv 0$. (vi) If these forms be evaluated for an arbitrary point $P(x)$ of space, not on the tetrahedra, then the equation of the desmic surface D through P may be written

$$(5)\qquad \lambda_x(y_0^2y_1^2+y_2^2y_3^2)+\mu_x(y_0^2y_2^2+y_3^2y_1^2)+\nu_x(y_0^2y_3^2+y_1^2y_2^2)=0.$$

(vii) The tetrahedra are also the base of a net of quadrics and the generators of these quadrics constitute a desmic cubic complex of lines. (viii) Now, a point $P(x)$ determines a desmic surface of the pencil and is the vertex of a cubic cone of the complex. As shown in the former paper, this cone passes through the vertices of the tetrahedra and so cuts an arbitrary transversal plane in a cubic curve with the desmic points projecting into a Hessian configuration on it. Moreover, the tangent plane to D at P cuts the cone in the three generators which give on C_6^3 the three collinear points ABC proper to the Hessian configuration. Conversely, it was shown in the first paper (p. 509) how to construct a set of tetrahedra and a desmic surface to correspond to a given C_6^3. When one of the desmic tetrahedra is taken for reference and a vertex of a second for unit point, then the equation of the cubic curve on $y_3=0$ is

$$(6)\qquad \begin{aligned}(x_0^2-x_3^2)y_1y_2(x_2y_1-x_1y_2)+(x_1^2-x_3^2)y_2y_0(x_0y_2-x_2y_0)&\\ +(x_2^2-x_3^2)y_0y_1(x_1y_0-x_0y_1)&=0.^*\end{aligned}$$

3. **Developments for the general case.** We seek some of the consequences of these properties. We compute the absolute invariant of the cubic curve by Salmon's formulas, and find

* Loc. cit. equation 11.

$$I = \frac{4(\mu\nu + \nu\lambda + \lambda\mu)^3}{4(\mu\nu + \nu\lambda + \lambda\mu)^3 + (\mu - \nu)^2(\nu - \lambda)^2(\lambda - \mu)^2}. \tag{7}$$

Thus the invariant of the cubic curve and of the cubic cone is expressed in terms of the coefficients of the surface. The cross ratios of the four tangent planes through a generator of a cone, or of the four tangent lines which they cut on the transversal plane, are of the type form $-\lambda_x:\mu_x$; and as these ratios must be the same for all points of D (cf. equation (5)) it follows that

All cones of a desmic cubic complex whose vertices lie on one desmic surface have the same absolute invariant and are projective.

4. The assignment of $\sigma u/\sigma v$ as x_0, etc. is arbitrary. For a given $P(x_i)$, 23 other points may be obtained by permuting the coördinates, and the 24 correspond to the symmetric group G^4. This group contains as subgroup the symmetric group G_6^3. We find that the 24 points lie by fours on six distinct desmic surfaces of the pencil and the absolute invariant is the same for all six surfaces, for λ, μ, ν are merely permuted by the permutations of the G_6. Conversely, the absolute invariant, when equated to an arbitrary number, gives an equation of the 24th degree which factors into six of the fourth degree, and these give the six conjugate desmic surfaces of the pencil. Hence

The locus of the vertices of the projective cubic cones in a desmic cubic complex consists of six conjugate desmic surfaces.

5. The cross ratios $-\lambda:\mu$ have a still closer relation to the features of the surface. Those generators of the cone which lie in the tangent plane at P are three of the bitangents which can be drawn from P to D, and their second points of contact can be determined as follows. If the vertices of two of the tetrahedra be taken as the eight invariant points of a cubic involution $x' = 1/x$, then D transforms into itself and P interchanges with P', the second point of contact on one of the bitangents. If tetrahedra II and III be the invariant base of the transformation, we shall say that the bitangents signalized in this manner are *of the first system*. Now the four planes determined on $PP'A$ by the vertices of tetrahedron I have cross ratios of type $-\lambda:\mu$. Hence

The bitangents of the same system on a desmic surface subtend at the vertices of the proper tetrahedron four planes of constant cross ratio.

6. By a theorem of Steiner's these lines also cut the faces of the tetrahedron in a range of the same cross ratio. Thus

Bitangents of the same system on a desmic surface belong to a complex of Reye (tetrahedral complex) for which the corresponding tetrahedron of nodes is fundamental.

Under a cubic involution whose eight invariant points are the vertices of two desmic tetrahedra, a desmic surface transforms into itself and the lines which join corresponding points belong to a tetrahedral complex on the third tetrahedron.

7. **Special surfaces.** When the value of I is 1, 0, or ∞, the corresponding cubic curve is *harmonic*, *equianharmonic*, or *nodal*. We determine the corresponding desmic surfaces.

Let $I=1$, then $T^2=0$; thus the desmic surfaces on which the vertices of the cubic cones lie when the cubics are harmonic are

$$\mu_y - \nu_y = 0\,, \quad \nu_y - \lambda_y = 0\,, \quad \lambda_y - \mu_y = 0\,.$$

These are obtained by setting each factor of T equal to zero and considering x as a variable y. As $T^2=0$, each factor counts twice and the set of six desmic surfaces reduces to three.

8. If $I=0$, then $S^3=0$. The solution of the system

$$S = \rho(\mu\nu + \nu\lambda + \lambda\mu) = 0,$$

$$\lambda + \mu + \nu \equiv 0,$$

gives

$$\lambda = \lambda, \quad \mu = \omega\lambda, \quad \nu = \omega^2\lambda,$$

where ω is a complex cube root of unity. Hence the desmic surfaces for equianharmonic cubics are

$$y_0^2y_1^2 + y_2^2y_3^2 + \omega(y_1^2y_3^2 + y_2^2y_0^2) + \omega^2(y_2^2y_1^2 + y_0^2y_3^2) = 0$$

and

$$y_0^2y_1^2 + y_2^2y_3^2 + \omega^2(y_1^2y_3^2 + y_2^2y_0^2) + \omega(y_2^2y_1^2 + y_0^2y_3^2) = 0.$$

These are the only two distinct desmic surfaces in the set of six, for all the others obtained by permutation of the coefficients can also be obtained in this instance by multiplying each of the equations by ω and ω^2.

9. If $I=\infty$, then $S^3+64T^3=0$. This corresponds to $g_2^3-27g_3^2=0$ for the elliptic $\wp$-function. If the four numbers of the set (x^2) be taken as the roots of the quartic

$$f(z) \equiv a_0z^4 + 4a_1z^3 + 6a_2z^2 + 4a_3z + a_4 = 0,$$

then $g_2^3-27g_3^2$ is the discriminant of that equation and when equated to zero implies that at least two of the roots are equal. Therefore point (x) is on one of the six pairs of planes $y_i^2-y_j^2=0$. These are just the degenerate pairs of planes which taken again in twos make the three degenerate desmic surfaces of the pencil. Thus the locus of (x) for a nodal cubic consists of the three degenerate desmic surfaces of the pencil:

$$\lambda = 0, \quad \mu = 0, \quad \nu = 0.$$

Each of the corresponding nodal cubics degenerates to a line and a conic.

10. For cuspidal cubics T and S are zero simultaneously. We find that the corresponding cubics degenerate to three concurrent lines.

11. From these last two results we see that there is no correspondence between proper desmic surfaces and proper singular cubics; or otherwise, *no cone in a desmic complex has a single double line.*

12. **Associated surfaces of order eight.** It happens that the eighteen edges of the desmic tetrahedra are also the edges, in a different grouping, of a counter-set of desmic tetrahedra, and there is another pencil of desmic surfaces D' on these (First paper, p. 507). Through an arbitrary point P_x there passes a surface of each pencil. As any one surface D is cut by every surface of the other pencil, the values of the absolute invariant for the two surfaces through a point are different, in general. If we seek the locus of points for which they are equal, that is, form the combinations of type $\lambda_x\mu_x' - \lambda_x'\mu_x = 0$, we find six surfaces of the eighth order whose coefficients are *numerical.* A typical equation is

$$(x_0^2 - x_1^2)(x_2^2 - x_3^2)(\sum x_0^4 - 2\sum x_0^2 x_1^2 + 8x_0x_1x_2x_3) \\ + 16x_0x_1x_2x_3(x_0^2 - x_2^2)(x_3^2 - x_1^2) = 0.$$

Thus we have a set of surfaces intimately connected with a set of desmic tetrahedra and its counter-set. Each surface passes through the 16 lines of the pencil $\{D\}$ and the 16 of pencil $\{D'\}$. Moreover, each passes through six of the eighteen edges of the tetrahedra.

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