A GENERALIZATION OF TAYLOR'S SERIES*

BY

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1. **Introduction.** In view of the great importance of Taylor's series in analysis, it may be regarded as extremely surprising that so few attempts at generalization have been made. The problem of the representation of an arbitrary function by means of linear combinations of prescribed functions has received no small amount of attention. It is well known that one phase of this problem leads directly to Taylor's series, the prescribed functions in this case being polynomials. It is the purpose of the present paper to discuss this same phase of the problem when the prescribed functions are of a more general nature.

Denote the prescribed functions by

$$u_0(x), \quad u_1(x), \quad u_2(x), \cdots, \tag{1}$$

real functions of the real variable all defined in a common interval $a \leqq x \leqq b$. Set

$$s_n(x) = c_0u_0(x) + c_1u_1(x) + \cdots + c_nu_n(x).$$

It is required to determine the constants c_i in such a way that $s_n(x)$ shall be the best approximation to a given function $f(x)$ that can be obtained by a linear combination of $u_0, u_1, \cdots, u_n$. Of course this problem becomes definite only after a precise definition of the phrase "best approximation" has been given. Various methods have been used, of which we mention the following:

(A) The method of least squares;
(B) The method of Tchebycheff;
(C) The method of Taylor.

In each of these cases the functions (1) may be so restricted that the constants c_i are uniquely determined. The function $s_n(x)$ thereby determined is called a *function of approximation*. Having determined the functions of approximation, one is led directly to an expansion problem. Under what conditions will $s_n(x)$ approach $f(x)$ as n becomes infinite? Or, when will the series

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$$s_0(x) + [s_1(x) - s_0(x)] + [s_2(x) - s_1(x)] + \cdots \tag{2}$$

converge and represent $f(x)$ in (a, b) or in any part of (a, b)?

There are two special sequences (1) that have received particular attention:

$$1, x, x^2, \cdots, \tag{1'}$$

$$1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots. \tag{1''}$$

The following scheme will serve as a partial reference list to this field, and to put into evidence the gap in the general theory which it is hoped the present paper will in some measure fill.

	(A)*	(B)*	(C)
(1′)	A. M. Legendre	P. L. Tchebycheff	B. Taylor
(1″)	J. J. Fourier	M. Fréchet	G. Teixeira†
(1)	E. Schmidt	A. Haar	

The entry in the upper left-hand corner, for example, means that the series (2) becomes for the method (A) and for the special sequence (1′) the expansion of $f(x)$ in a series of Legendre polynomials. It should be pointed out that the series studied by Teixeira were considered by him in another connection, and that no mention of their relation to Taylor's series was made.

It is found that if certain restrictions are imposed on the sequence (1), and if the functions of approximation are determined according to the method (C), then the general term of the series (2) may be factored, just as in Taylor's series, into two parts $c_n g_n(x)$, the second of which depends in no way on the function $f(x)$ represented, the constant c_n alone being altered when $f(x)$ is altered. As in the case of Taylor's series the constant c_n is determined by means of a linear differential operator of order n. If further restrictions, Conditions A of §6, are imposed on the sequence (1), it is found that series (2) possesses many of the formal properties of a power series. If t is a point at which $s_n(x)$ has closest contact with $f(x)$, then the interval of convergence of (2) extends equal distances on either side of t (provided that the interval of definition (a, b) permits). The familiar process of analytic extension also applies to this generalized power series.

A necessary and sufficient condition for the representation of a function $f(x)$ is obtained by generalizing a theorem of S. Bernstein. Then imposing

* For references, see Encyklopädie der Mathematischen Wissenschaften, IIC9c (Fréchet-Rosenthal), §51.

† *Extrait d'une lettre de M. Gomes Teixeira à M. Hermite*, Bulletin des Sciences Mathématiques et Astronomiques, vol. 25 (1890), p. 200.

further conditions, Conditions B of §10, it is found possible to represent an arbitrary analytic function in a series (2). It is shown that the conditions are not so strong as to exclude the case of Taylor's series, and that sequences (1) exist, satisfying the conditions, and leading to series quite different from Taylor's series. Finally the relation of the general series to Teixeira's series is shown.

2. **The Taylor method of approximation and the existence of the functions of approximation.** The Taylor method of approximation consists in determining the constants c_i of $s_n(x)$ in such a way that the approximation to $f(x)$ shall be as close as possible in the immediate neighborhood of a point $x=t$ of (a, b), irrespective of the magnitude of the error $|f(x)-s_n(x)|$ at points x remote from t. More precisely, the constants c_i are determined so that the curves $y=f(x)$ and $y=s_n(x)$ shall have closest contact at a point t. If the functions $f(x)$ and $s_n(x)$ are of class C^{m+1}(possess continuous derivatives of order $m+1$) in the neighborhood of $x=t$, then the curves $y=f(x)$ and $y=s_n(x)$ (or the functions $f(x)$ and $s_n(x)$ themselves) are said to have contact of order m at $x=t$ if and only if

$$f^{(k)}(t) = s_n^{(k)}(t), \quad k = 0,1, \cdots ,m \,; \quad f^{(m+1)}(t) \neq s_n^{(m+1)}(t).$$

We now make the following

DEFINITION. *The function*

$$s_n(x) = \sum_{i=0}^{n} c_i u_i(x)$$

is a function of approximation of order n for the point $x=t$ if the functions $u_i(x)$ are of class C^n in the neighborhood of $x=t$, and if $s_n(x)$ has contact of order n at least with $f(x)$ at $x=t$.

We shall have frequent occasion to use Wronskians, so that it will be convenient to introduce a notation. The functions $v_0(x), v_1(x), \cdots, v_n(x)$ being of class C^n, we set

$$W[v_0(x),v_1(x), \cdots ,v_n(x)] = \begin{vmatrix} v_0(x) & v_1(x) & \cdots v_n(x) \\ v_0'(x) & v_1'(x) & \cdots v_n'(x) \\ \cdot & \cdot & \cdots \quad \cdot \\ \cdot & \cdot & \cdots \quad \cdot \\ v_0^{(n)}(x) & v_1^{(n)}(x) & \cdots v_n^{(n)}(x) \end{vmatrix}.$$

In particular, for the functions of the sequence (1) we set

$$W_n(x) = W[u_0(x),u_1(x), \cdots ,u_n(x)].$$

We may now state

THEOREM I. *If the functions $f(x)$, $u_0(x)$, $u_1(x)$, $\cdots$, $u_n(x)$ are of class C^n in the neighborhood of $x=t$, and if $W_n(t)\neq 0$, then there exists a unique function of approximation*

$$s_n(x) = \sum_{i=0}^{n} c_i u_i(x) = -\left(\frac{1}{W_n(t)}\right)\begin{vmatrix} 0 & u_0(x) & u_1(x) & \cdots & u_n(x) \\ f(t) & u_0(t) & u_1(t) & \cdots & u_n(t) \\ f'(t) & u_0'(t) & u_1'(t) & \cdots & u_n'(t) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ f^{(n)}(t) & u_0^{(n)}(t) & u_1^{(n)}(t) & \cdots & u_n^{(n)}(t) \end{vmatrix}$$

of order n for $x=t$.

The proof of this theorem consists in noting that the determinant of the system of equations

$$f^{(k)}(t) = c_0 u_0^{(k)}(t) + c_1 u_1^{(k)}(t) + \cdots + c_n u_n^{(k)}(t) \qquad (k = 0,1,\cdots,n)$$

is $W_n(t)$, which is different from zero by hypothesis, and in solving the system for the constants c_i. The values of the c_i thus obtained give the above expression for $s_n(x)$.

3. **Determination of the form of the series.** In order to form the series (2) we need to know the existence of the functions of approximation of all orders. We shall assume then that $f(x)$ and $u_i(x)$, $i=0, 1, 2, \cdots$, are of class C^∞ in the interval $a\leqq x\leqq b$. Moreover we shall assume* that $W_i(x)>0$ in the same interval. This insures the existence of the functions of approximation of all orders for an arbitrary point of the interval. We are thus led naturally to a set of functions (1) possessing what G. Pólya† has called the Property W.

DEFINITION. *The sequence* (1) *is said to possess the Property W in (a, b) if each function of the sequence is of class C^∞ in $a\leqq x\leqq b$, and if $W_i(x)>0$, $i=0, 1, 2, \cdots$, in the same interval.*

We shall now be able to show that the series (2) has the form

$$c_0 h_0(x) + c_1 h_1(x) + c_2 h_2(x) + \cdots,$$

* No gain in generality would be obtained by allowing some or all of the functions $W_i(x)$ to be negative.

† G. Pólya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, these Transactions, vol. 24 (1922), p. 312. We have extended the definition to apply to an infinite set.

where the functions $h_i(x)$ depend only on the sequence (1) and on the choice of the point t, and not at all on the function $f(x)$ to be expanded. The constants c_i, on the other hand, are independent of x, but depend on the function $f(x)$ and on the choice of the point t. It is this property of the series (2) that makes all the series under the method (A) of the introduction so convenient to use. The property is lacking for the method (B), and for this reason the Tchebycheff series are less useful in spite of their theoretical advantages.

The direct factorization of $[s_n(x)-s_{n-1}(x)]$ is attended with algebraic difficulties which may be avoided by means of the following device. Set

$$\phi(x) = s_n(x) - s_{n-1}(x).$$

Then by Theorem I

$$\overset{(k)}{s_n}(t) = f^{(k)}(t) = \overset{(k)}{s_{n-1}}(t),\ \ k = 0,1,\cdots,n-1,\ \ s_n^{(n)}(t) = f^{(n)}(t),$$
$$\phi^{(k)}(t) = 0,\ \ k = 0,1,\cdots,n-1,\ \ \phi^{(n)}(t) = f^{(n)}(t) - \overset{(n)}{s_{n-1}}(t).$$

But $\phi(x)$ by its form is a linear combination of $u_0(x), u_1(x), \cdots, u_n(x)$,

$$\phi(x) = a_0u_0(x) + a_1u_1(x) + \cdots + a_nu_n(x).$$

Hence the constants a_i must satisfy the equations

$$0 = a_0u_0^{(k)}(t) + a_1u_1^{(k)}(t) + \cdots + a_nu_n^{(k)}(t) \qquad (k = 0,1,\cdots,n-1),$$
$$f^{(n)}(t) - \overset{(n)}{s_{n-1}}(t) = a_0u_0^{(n)}(t) + a_1u_1^{(n)}(t) + \cdots + a_nu_n^{(n)}(t).$$

From these equations we see that $\phi(x)$ must satisfy the equation

$$\phi(x)W_n(t) = [f^{(n)}(t) - \overset{(n)}{s_{n-1}}(t)]\begin{vmatrix} u_0(t) & u_1(t) & \cdots & u_n(t) \\ u_0'(t) & u_1'(t) & \cdots & u_n'(t) \\ \cdot & \cdot & \cdots & \cdot \\ u_0^{(n-1)}(t) & u_1^{(n-1)}(t) & \cdots & u_n^{(n-1)}(t) \\ u_0(x) & u_1(x) & \cdots & u_n(x) \end{vmatrix}.$$

The factorizaton of $\phi(x)$ which we set out to perform is thus completed. For brevity we set

$$g_n(x,t) = \left(\frac{1}{W_n(t)}\right)\begin{vmatrix} u_0(t) & u_1(t) & \cdots & u_n(t) \\ u_0'(t) & u_1'(t) & \cdots & u_n'(t) \\ \cdot & \cdot & \cdots & \cdot \\ u_0^{(n-1)}(t) & u_1^{(n-1)}(t) & \cdots & u_n^{(n-1)}(t) \\ u_0(x) & u_1(x) & \cdots & u_n(x) \end{vmatrix}, \tag{3}$$

so that $g_n(x, t)$ is the function $h_n(x)$ sought. For convenience in later work we have put into evidence the point t chosen. We see that

$$s_n(x) - s_{n-1}(x) = [f^{(n)}(t) - s_{n-1}^{(n)}(t)]g_n(x,t). \tag{4}$$

Now by reference to the explicit form of $s_{n-1}(x)$ given in Theorem I it becomes clear that

$$\begin{aligned} f^{(n)}(t) - s_{n-1}^{(n)}(t) &= f^{(n)}(t) + \left(\frac{1}{W_{n-1}(t)}\right)\begin{vmatrix} 0 & u_0^{(n)}(t) & u_1^{(n)}(t) & \cdots & u_{n-1}^{(n)}(t) \\ f(t) & u_0(t) & u_1(t) & \cdots & u_{n-1}(t) \\ f'(t) & u_0'(t) & u_1'(t) & \cdots & u_{n-1}'(t) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ f^{(n-1)}(t) & u_0^{(n-1)}(t) & u_1^{(n-1)}(t) & \cdots & u_{n-1}^{(n-1)}(t) \end{vmatrix} \\ &= \frac{W[u_0(t), u_1(t), \cdots, u_{n-1}(t), f(t)]}{W_{n-1}(t)}. \end{aligned}$$

It will now be convenient to introduce a linear differential operator defined by the relation

$$L_nf(x) = \frac{W[u_0(x), u_1(x), \cdots, u_{n-1}(x), f(x)]}{W_{n-1}(x)}.$$

By use of this notation equation (4) becomes

$$s_n(x) - s_{n-1}(x) = L_nf(t)g_n(x,t),$$

and the expansion of the function $f(x)$ has the form

$$f(x) \sim L_0f(t)g_0(x,t) + L_1f(t)g_1(x,t) + L_2f(t)g_2(x,t) + \cdots, \tag{5}$$

$$g_0(x,t) = \frac{u_0(x)}{u_0(t)}, \qquad L_0f(x) = f(x).$$

Incidentally, we have proved the following formula:

$$\begin{aligned} L_nf(t) = f^{(n)}(t) - L_0f(t)g_0^{(n)}(t,t) - L_1f(t)g_1^{(n)}(t,t) - \cdots \\ - L_{n-1}f(t)g_{n-1}^{(n)}(t,t). \end{aligned}$$

4. **The properties of the functions $g_n(x, t)$ and of the operators L_n.** From the equation (3) defining the function $g_n(x, t)$ we read off at once certain properties. Considered as a function of x, it is evidently a linear combination of $u_0(x), u_1(x), \cdots, u_n(x)$ satisfying the equations

$$\frac{\partial^k}{\partial x^k} g_n(x,t)\bigg|_{x=t} = \begin{cases} 0, & k = 0,1,\cdots,n-1, \\ 1, & k = n. \end{cases} \tag{6}$$

The operator L_n is seen to be a linear differential operator of order n which annuls the first n functions of the set (1), and which satisfies the relation

$$L_n x^n \big|_{x=0} = n!.$$

The expanded form of $L_n f$ is

$$L_n f(x) = f^{(n)}(x) + p_1(x) f^{(n-1)}(x) + \cdots + p_n(x) f(x),$$

the coefficient of $f^{(n)}(x)$ being unity.

The function $g_n(x, t)$ is the function of Cauchy* used in obtaining a particular solution of the non-homogeneous equation

$$L_{n+1} f(x) = p(x)$$

from the solutions of the corresponding homogeneous equation. The particular solution of this equation vanishing with its first n derivatives at $x=t$ is known to be

$$f(x) = \int_t^x g_n(x,t) p(t)\, dt.$$

When L_n operates on the functions $g_m(x, t)$ the result is particularly simple. Since L_n annuls the first n functions of the sequence (1), it follows that

$$L_n g_m(x,t) \equiv 0, \quad m < n.$$

Let us also compute $L_n g_m(x, t)$ for $x=t$ and $m \geqq n$. By means of the relations (6) we find that

$$L_n g_n(x,t)\big|_{x=t} = \frac{\partial^n}{\partial x^n} g_n(x,t)\big|_{x=t} + \left[p_1(x) \frac{\partial^{n-1}}{\partial x^{n-1}} g_n(x,t) \right]_{x=t} + \cdots = 1,$$

$$L_n g_{n+p}(x,t)\big|_{x=t} = \frac{\partial^n}{\partial x^n} g_{n+p}(x,t)\big|_{x=t} + \left[p_1(x) \frac{\partial^{n-1}}{\partial x^{n-1}} g_{n+p}(x,t) \right]_{x=t} + \cdots = 0$$

$$(p = 1,2,\cdots).$$

These properties† may be summed up as follows:

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* E. Goursat, *Cours d'Analyse Mathématique*, vol. 2, p. 430.

† An a priori discussion of the series in question might be made by starting with these formulas. They may evidently be used to determine the coefficients of the series formally.

$$L_n g_m(x,t)\big|_{x=t} = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases} \tag{7}$$

The relation of the series (5) to Taylor's series is brought out more clearly if the sequence (1) is replaced by the sequence (1′) in the preceding work. Simple computations show that for this case

$$W_n(x) = n!(n-1)!(n-2)!\cdots 2!,$$

$$L_n f(x) = f^{(n)}(x),$$

$$g_n(x,t) = (x-t)^n/n!.$$

The series (5) now has precisely the form of Taylor's series.

For many purposes it will be convenient to use another form of the differential operator L_n. It is known* that if the Property W holds for the set $u_0(x), u_1(x), \cdots, u_{n-1}(x)$ in (a, b), then $L_n f(x)$ may be written as

$$L_n f(x) = \phi_0(x)\phi_1(x)\cdots\phi_{n-1}(x)\frac{d}{dx}\frac{1}{\phi_{n-1}(x)}\frac{d}{dx}\frac{1}{\phi_{n-2}(x)}\frac{d}{dx}\cdots\frac{d}{dx}\frac{1}{\phi_1(x)}\frac{d}{dx}\frac{f(x)}{\phi_0(x)}, \tag{8}$$

where

$$\phi_0(x) = W_0(x), \quad \phi_1(x) = \frac{W_1(x)}{[W_0(x)]^2}, \quad \phi_k(x) = \frac{W_k(x)W_{k-2}(x)}{[W_{k-1}(x)]^2}$$
$$(k = 2, 3, \cdots, n-1).$$

The functions $\phi_i(x)$ will all be positive for $a \leqq x \leqq b$ since we are assuming that the Property W holds in that interval. The differential expression adjoint to $L_n f(x)$ may then be written†

$$M_n f(t) = (-1)^n \frac{1}{\phi_0(t)}\frac{d}{dt}\frac{1}{\phi_1(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{\phi_{n-1}(t)}\frac{d}{dt}\phi_0(t)\phi_1(t)\cdots\phi_{n-1}(t)f(t). \tag{9}$$

In formulas (8) and (9) the operation of differentiation applies to all that follows.‡

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* For a simple proof of this fact see G. Pólya, loc. cit., p. 316.

† L. Schlesinger, *Lineare Differential-Gleichungen*, vol. 1, p. 58.

‡ Throughout this paper the independent variable for the operator L_n is x; for M_n, t. The expression $L_n f(t)$ means $L_n f(x)\big|_{x=t}$.

The functions $g_n(x, t)$ can be expressed in terms of the functions $\phi_i(x)$. For $g_n(x, t)$, considered as a function of x, satisfies the differential system

$$L_{n+1}u(x) = 0,$$

$$L_m u(t) = \begin{cases} 0, & m = 0,1,\cdots,n-1, \\ 1, & m = n. \end{cases}$$

The system has a unique solution since the boundary conditions are equivalent to

$$u^{(m)}(t) = \begin{cases} 0, & m = 0,1,\cdots,n-1, \\ 1, & m = n. \end{cases}$$

But by virtue of formula (8) the solution takes the form

(10)
$$u(x) = g_n(x,t) = \frac{\phi_0(x)}{\phi_0(t)\cdots\phi_n(t)}\int_t^x \phi_1(x_1)\int_t^{x_1} \cdots \int_t^{x_{n-2}} \phi_{n-1}(x_{n-1})\int_t^{x_{n-1}} \phi_n(x_n)dx_1dx_2\cdots dx_n,$$

a formula which we shall also write as follows:

$$g_n(x,t) = \frac{\phi_0(x)}{\phi_0(t)\cdots\phi_n(t)}\int_t^x \phi_1(x)\int_t^x \phi_2(x)\int_t^x\cdots\int_t^x \phi_{n-1}(x)\int_t^x \phi_n(x)(dx)^n.$$

That this function satisfies the differential equation is obvious. That it satisfies the boundary conditions may be seen by forming the functions

$$L_m g_n(x,t) = \frac{\phi_0(x)\cdots\phi_m(x)}{\phi_0(t)\cdots\phi_n(t)}\int_t^x \phi_{m+1}(x)\int_t^x\cdots\int_t^x \phi_n(x)(dx)^{n-m}, \quad m < n,$$

$$L_n g_n(x,t) = \frac{\phi_0(x)\cdots\phi_n(x)}{\phi_0(t)\cdots\phi_n(t)},$$

and substituting $x=t$.

It is a familiar fact, and one that may be directly verified by use of formulas (6) and (9), that $g_n(x, t)$ considered as a function of t satisfies the adjoint differential system

(11)
$$M_{n+1}v(t) = 0,$$

(12)
$$v^{(m)}(x) = \begin{cases} 0, & m = 0,1,2,\cdots,n-1, \\ (-1)^n, & m = n. \end{cases}$$

But an argument similar to that given above shows that the solution of this system has the form

$$
\begin{aligned}
(13) \qquad v(t) &= g_n(x,t) \\
&= (-1)^n \frac{\phi_0(x)}{\phi_0(t)\cdots\phi_n(t)} \int_x^t \phi_n(t) \int_x^t \phi_{n-1}(t) \int_x^t \cdots \int_x^t \phi_1(t)(dt)^n.
\end{aligned}
$$

This formula has the advantage over (10) that it enables one to express $g_n(x, t)$ in terms of $g_{n-1}(x, t)$:

$$g_n(x,t) = \frac{1}{\phi_0(t)\cdots\phi_n(t)} \int_t^x \phi_0(t)\cdots\phi_n(t) g_{n-1}(x,t)dt.$$

By use of this formula the functions $g_n(x, t)$ may be computed step by step from the functions $\phi_i(x)$, the computations involving only one new integration for each new function $g_n(x, t)$.

It should be pointed out that for many purposes it is convenient to consider the functions $\phi_i(x)$ as the given functions instead of the $u_i(x)$. For if the $\phi_i(x)$ are given positive functions in (a, b), then a set of functions $u_i(x)$ possessing the property W in that interval is

$$u_i(x) = g_i(x,t) \qquad (i = 0,1,2,\cdots; a \leqq t \leqq b).$$

Evidently any function $\phi_i(x)$ may be multiplied by an arbitrary constant not zero without affecting the form of the series; for a glance at formulas (8) and (10) will show that neither the operators L_n nor the functions $g_n(x, t)$ will be thereby affected. For the special sequence (1′) we have

$$
\begin{aligned}
\phi_k(x) &= k \qquad (k = 1,2,3,\cdots), \\
\phi_0(x) &= 1.
\end{aligned}
$$

However, one is led equally well to Taylor's series by taking

$$\phi_k(x) = 1 \qquad (k = 0,1,2,\cdots).$$

5. **Remainder formulas.** Let us begin by deriving an exact remainder formula, the analogue of a well known formula for Taylor's series.* Set

$$R_n(x) = f(x) - L_0f(t)g_0(x,t) - L_1f(t)g_1(x,t) - \cdots - L_nf(t)g_n(x,t).$$

By Theorem I this function has a zero of order $(n+1)$ at least at $x=t$. Furthermore it satisfies the differential equation

$$L_{n+1}R_n(x) = L_{n+1}f(x).$$

* See for example E. Goursat, loc. cit., vol. 1, p. 209.

But it is known that the only solution of this equation vanishing with its first n derivatives at $x=t$ is

$$R_n(x) = \int_t^x g_n(x,t)L_{n+1}f(t)dt. \tag{14}$$

This gives the remainder formula desired:

$$\begin{aligned} f(x) = L_0f(x)g_0(x,t) + L_1f(t)g_1(x,t) + \cdots + L_nf(t)g_n(x,t) \\ + \int_t^x g_n(x,t)L_{n+1}f(t)dt. \end{aligned} \tag{15}$$

For the special sequence (1′) this becomes

$$f(x) = f(t) + f'(t)(x-t) + \cdots + f^{(n)}(t)\frac{(x-t)^n}{n!} + \int_t^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)dt.$$

In the previous section we assumed that the functions $f(x)$ and $u_i(x)$ were of class C^∞. For the validity of the remainder formula (15) it is clearly sufficient to assume that $f(x)$, $u_0(x)$, $u_1(x)$, $\cdots$, $u_n(x)$ are of class C^{n+1} and that the Wronskians $W_0(x)$, $W_1(x)$, $\cdots$, $W_n(x)$ are positive in (a, b).

Let us now obtain remainder formulas analogous to certain other of the classical remainder formulas for Taylor's series. Let $F(s)$ be a function of class C' in the interval (a, b), and such that $F'(s)$ is not zero in the interval (t, x) except perhaps at the point t. Then formula (14) may evidently be written as

$$R_n(x) = \int_t^x \frac{g_n(x,s)}{F'(s)}L_{n+1}f(s)F'(s)ds.$$

We may now apply the first mean-value theorem for integrals,* and obtain

$$R_n(x) = \frac{g_n(x,\xi)}{F'(\xi)}L_{n+1}f(\xi)[F(x) - F(t)] \qquad (t < \xi < x,\ x < \xi < t). \tag{16}$$

This is the analogue of the remainder given by Schömilch,†

$$R_n(x) = \frac{f^{(n+1)}(\xi)(x-\xi)^n}{F'(\xi)n!}[F(x) - F(t)],$$

to which it reduces for the special sequence (1′).

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* E. Goursat, loc. cit., vol. 1, p. 181. It is to be noted that $[g_n(x,s)\ L_{n+1}f(s)]/(F'(s))$ may be discontinuous at $s=t$. The ordinary treatments of the theorem do not admit this possibility, but it may be shown that the theorem is still applicable to this case; cf. G. D. Birkhoff, these Transactions, vol. 7 (1906), p. 115.

† For references see Encyklopädie der Mathematischen Wissenschaften, IIA2 (Pringsheim), § 11.

By specializing the function $F(s)$ a variety of remainders may be obtained. Let us take

$$F(s) = \int_t^s g_m(x,s)ds, \quad m \leqq n.$$

Then $F(s)$ obviously possesses the continuity properties imposed above. That it is a function of one sign in the open interval (t, x) may be seen by direct inspection of formula (10) or by the general theory of G. Pólya.* For, by formulas (11) and (12) we see that $g_m(x, s)$ considered as a function of s has a zero of order m at the point x and satisfies the differential equation

$$M_{m+1}v(s) = 0.$$

But no solution of this equation not identically zero can vanish more than m times in any interval in which the Property W holds. Consequently $g_m(x, s)$ is different from zero in (a, b) except at x. With this special choice of $F(s)$, (16) becomes

$$R_n(x) = \frac{g_n(x,\xi)}{g_m(x,\xi)} L_{n+1}f(\xi) \int_t^x g_m(x,s)ds. \tag{17}$$

This is the analogue of a remainder of Roche,†

$$R_n(x) = \frac{(x-\xi)^{n-m}(x-t)^{n+1}}{n!(m+1)} f^{(n+1)}(\xi),$$

to which it reduces for the sequence (1′).

By taking $m = n$, (17) becomes

$$R_n(x) = L_{n+1}f(\xi) \int_t^x g_n(x,s)ds,$$

and this is the analogue of the familiar Lagrange† remainder. Finally by taking $m = 0$ we obtain

$$R_n(x) = \frac{g_n(x,\xi)}{g_0(x,\xi)} L_{n+1}f(\xi) \int_t^x g_0(x,s)ds,$$

as the analogue of Cauchy's† remainder,

$$R_n(x) = \frac{(x-\xi)^n(x-t)}{n!} f^{(n+1)}(\xi).$$

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* G. Pólya, loc. cit., p. 317.

† Encyklopädie, II A2, loc. cit.

A simpler remainder which also reduces to that of Cauchy for the sequence (1′) is

$$R_n(x) = g_n(x,\xi)L_{n+1}f(\xi)(x-t).$$

Let us sum up the results in

THEOREM II. *Let the functions $f(x)$, $u_0(x)$, $u_1(x)$, $\cdots$, $u_n(x)$ be of class C^{n+1}, and let the Wronskians $W_0(x)$, $W_1(x)$, $\cdots$, $W_n(x)$ be positive in the interval $a\leqq x\leqq b$. Then if t is a point of this interval,*

$$f(x) = L_0f(t)g_0(x,t) + L_1f(t)g_1(x,t) + \cdots + L_nf(t)g_n(x,t) + R_n(x), \tag{18}$$

where

$$g_k(x,t) = \left(\frac{1}{W_k(t)}\right)\begin{vmatrix} u_0(t) & u_1(t) & \cdots & u_k(t) \\ u_0'(t) & u_1'(t) & \cdots & u_k'(t) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ u_0^{(k-1)}(t) & u_1^{(k-1)}(t) & \cdots & u_k^{(k-1)}(t) \\ u_0(x) & u_1(x) & \cdots & u_k(x)\end{vmatrix},$$

$$L_kf(x) = \frac{W[u_0(x),\cdots,u_{k-1}(x),f(x)]}{W_{k-1}(x)}$$

$$(k = 1,2,\cdots,n+1\,;\ L_0f(x) = f(x)),$$

and where $R_n(x)$ has one of the forms

$$R_n(x) = \int_t^x g_n(x,t)L_{n+1}f(t)dt,$$

$$R_n(x) = \frac{g_n(x,\xi)}{g_m(x,\xi)}L_{n+1}f(\xi)\int_t^x g_m(x,s)ds$$

$$(m\leqq n\,;\ t<\xi<x\,;\ x<\xi<t).$$

The function

$$N_m(x,t) = \int_t^x g_m(x,s)ds$$

that appears in the remainder may be expressed in a different form, which will be useful in what is to follow. From the form of the function it is seen to satisfy the following differential system when considered as a function of x:

$$L_{m+1}u(x) = 1,$$

$$u^{(k)}(t) = 0 \qquad (k = 0,1,2,\cdots,m).$$

But the unique solution of this system may also be written in the form

$$N_m(x,t) = \phi_0(x) \int_t^x \phi_1(x) \int_t^x \cdots \int_t^x \phi_m(x) \int_t^x \frac{(dx)^{m+1}}{\phi_0(x) \cdots \phi_m(x)}. \tag{19}$$

For the special sequence (1′) this is equal to $(x-t)^{m+1}/(m+1)!$.

6. **Generalized power series.** If in formula (18) n is allowed to become infinite, a series of the form

$$a_0 g_0(x,t) + a_1 g_1(x,t) + \cdots \tag{20}$$

results. Before discussing the behavior of the remainder as n becomes infinite, we discuss the general properties of a series of this type, a series which evidently reduces to a power series for the sequence (1′). In particular if $t=0$ is a point of (a, b), we shall set

$$g_n(x) = g_n(x,0) \qquad (n = 0,1,2, \cdots).$$

As has already been observed, no change is made in the series if any function $\phi_i(x)$ is multiplied by a non-vanishing constant. Consequently, no essential restriction will be introduced by the assumption, which will be made in the remainder of this paper, that $\phi_i(0)=1$. With this assumpton we may write

$$g_n(x) = \phi_0(x) \int_0^x \phi_1(x) \int_0^x \cdots \int_0^x \phi_n(x)(dx)^n. \tag{21}$$

In order that the series (20) may retain many of the formal properties of a power series we introduce

CONDITIONS A: (a) *The functions $\phi_i(x)$ are of class C^∞ in the interval $a \leqq x \leqq b$;*

(b) $$\phi_i(x) > 0 \qquad (i = 0,1,2, \cdots, a \leqq x \leqq b),$$

(c) $$\lim_{n=a} \frac{M_n}{m_n} = 1,$$

where

$$M_n = \text{maximum } \phi_n(x), \quad m_n = \text{minimum } \phi_n(x) \textit{ in } a \leqq x \leqq b.$$

In the case of the sequence (1′), $\phi_i(x)$ is constant, and the Conditions A are surely satisfied. It is a simple matter to construct other sequences of functions satisfying the conditions. For example, take

$$\phi_n(x) = e^{-x/n}.$$

Then

$$M_n = e^{-a/n}, \quad m_n = e^{-b/n},$$

and the conditions are evidently satisfied in any interval (a, b) however large.

We are now in a position to prove

THEOREM III. *If the functions $\phi_i(x)$ satisfy the Conditions* A *in (a, b), and if the series*

$$\sum_{n=0}^{\infty} c_n g_n(x,t), \qquad a \leqq t \leqq b, \tag{22}$$

converges for a value $x=x_0\neq t$ of that interval, then it converges absolutely in the interval $|x-t|<|x_0-t|$, $a\leqq x\leqq b$, and uniformly in any closed interval included therein. If the sum of the series is denoted by $f(x)$, then

$$L_k f(x) = \sum_{n=0}^{\infty} c_n L_k g_n(x,t) \tag{23}$$

$$(k = 0,1,2,\cdots;\ |x-t| < |x_0-t|;\ a \leqq x \leqq b).$$

Since the series (22) converges for $x=x_0$, it follows that there exists a constant M independent of n for which

$$|c_n g_n(x_0,t)| < M.$$

We are thus led immediately to a dominant series for (22),

$$\sum_{n=0}^{\infty} c_n g_n(x,t) \ll M \sum_{n=0}^{\infty} \frac{|g_n(x,t)|}{|g_n(x_0,t)|}.$$

We now obtain a more convenient form for $g_n(x, t)$ by successive applications of the mean-value theorem for integrals:

$$g_n(x,t) = \frac{\phi_0(x)\phi_1(\xi_1)\phi_2(\xi_2)\cdots\phi_n(\xi_n)}{\phi_0(t)\phi_1(t)\phi_2(t)\cdots\phi_n(t)}\,\frac{(x-t)^n}{n!},$$

$$t < \xi_n < \xi_{n-1} < \cdots < \xi_1 < x,$$

$$t > \xi_n > \xi_{n-1} > \cdots > \xi_1 > x.$$

Here the first line of inequalities holds if $t<x$; the second if $t>x$. Now making use of the upper and lower bounds M_n and m_n of ϕ_n in (a, b), we see that

$$|g_n(x,t)| < \frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\,\frac{|x-t|^n}{n!},$$

$$|g_n(x_0,t)| > \frac{m_0m_1\cdots m_n}{M_0M_1\cdots M_n}\,\frac{|x_0-t|^n}{n!},$$

$$\sum_{n=0}^{\infty} c_n g_n(x,t) \ll M \sum_{n=0}^{\infty} \left(\frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\right)^2 \frac{|x-t|^n}{|x_0-t|^n}.$$

The test ratio

$$\left(\frac{M_{n+1}}{m_{n+1}}\right)^2 \left|\frac{x-t}{x_0-t}\right|$$

of the dominant series has the limit $|x-t|/|x_0-t|$ as n becomes infinite by Condition A (c). The first part of the theorem is thus established. It remains to show that the operation term by term by L_k is permissible. Now

$$(24)\quad \begin{aligned} L_k g_n(x,t) &= \frac{\phi_0(x)\cdots\phi_k(x)}{\phi_0(t)\cdots\phi_n(t)}\int_t^x \phi_{k+1}(x)\int_t^x\cdots\int_t^x \phi_n(x)(dx)^{n-k}, \quad n \geqq k,\\ L_k g_n(x,t) &= \frac{\phi_0(x)\cdots\phi_k(x)\phi_{k+1}(\xi_{k+1})\cdots\phi_n(\xi_n)}{\phi_0(t)\cdots\phi_n(t)}\,\frac{(x-t)^{n-k}}{(n-k)!},\end{aligned}$$

$$t < \xi_n < \xi_{n-1} < \cdots < \xi_{k+1} < x,$$

$$t > \xi_n > \xi_{n-1} > \cdots > \xi_{k+1} > x.$$

Hence

$$\begin{aligned}\sum_{n=k}^{\infty} c_n L_k g_n(x,t) &\ll M\sum_{n=k}^{\infty}\frac{|L_k g_n(x,t)|}{|g_n(x_0,t)|}\\ &\ll M\sum_{n=k}^{\infty}\left(\frac{M_0\cdots M_n}{m_0\cdots m_n}\right)^2\frac{|x-t|^{n-k}}{|x_0-t|^n}\,\frac{n!}{(n-k)!}.\end{aligned}$$

Consequently the series (23) is uniformly convergent for $|x-t|\leqq r$, $a\leqq x\leqq b$, where $r<|x_0|$. This is sufficient to establish the result stated.

As a result of this theorem it follows that there exists an interval of convergence for the series extending equal distances on either side of t (provided the length of the interval of definition (a, b) permits). In particular, the interval may reduce to a single point, or it may be the entire interval (a, b) (which in turn may, in special cases, be the entire x-axis). The following examples will show that all of these cases are possible. Take $\phi_n(x)=e^{-x/n}$. Then

$$\sum_{n=0}^{\infty}(n!)^2 g_n(x) \textit{ diverges except for } x = 0\,;$$

$$\begin{aligned}\sum_{n=0}^{\infty} n!\,g_n(x) \quad &\textit{converges for } |x| < 1,\\ &\textit{diverges for } |x| > 1\,;\end{aligned}$$

$$\sum_{n=0}^{\infty} g_n(x) \quad \textit{converges for all } x.$$

Theorem III has a further important consequence. If in equations (23) we set $x=t$, we see that

$$c_k = L_k f(t).$$

Since the coefficients c_k are uniquely determined by the values of $f(x)$ and its derivatives at $x=t$, it follows that the development of a function $f(x)$ in a series (22) is unique.

7. **A generalization of Abel's theorem.** If a series (22) has an interval of convergence $(-r, r)$,* then by Theorem III it has a continuous sum in the interval $-r<x<r$. As in the case of power series the series may or may not converge at the extremities of the interval. We shall show that if (22) converges at r(or $-r$), then the sum of the series is continuous in the interval $-r<x\leqq r$ (or $-r\leqq x<r$) by use of the following

LEMMA. *If the functions $\phi_n(x)$ satisfy the conditions* A (a), (b), *then the determinant*

$$\Delta = \begin{vmatrix} g_{n-1}(x) & g_n(x) \\ g_{n-1}(y) & g_n(y) \end{vmatrix}$$

is positive or negative according as $0<x<y$ *or* $0>x>y$.

First it will be shown that $\Delta\neq 0$. If Δ were equal to zero for two values x_0 and y_0 distinct from each other and from the origin, it would be possible to determine constants c_1 and c_2 not both zero such that the function

$$\phi(x) = c_1 g_{n-1}(x) + c_2 g_n(x) \tag{25}$$

would vanish at x_0 and y_0. But $g_{n-1}(x)$ and $g_n(x)$ both vanish $(n-1)$ times at the origin so that $\phi(x)$ would have at least $(n+1)$ zeros in $(-a, a)$. This however is impossible. For, according to the general results of Pólya already cited, no linear combination of $g_0(x), g_1(x), \cdots, g_n(x)$ not identically zero can vanish $(n+1)$ times in an interval in which the Property W holds. Hence $\Delta\neq 0$.

It remains to discuss the sign of Δ. Regard y as fixed, so that Δ becomes a function of x alone. Evidently

$$\lim_{x\to y} \frac{\Delta}{y-x} = W(y) = \begin{vmatrix} g_{n-1}(y) & g_n(y) \\ g'_{n-1}(y) & g'_n(y) \end{vmatrix}.$$

We shall show presently that $W(y)>0$ for all values of y different from zero in $(-a, a)$. This will be sufficient to establish the Lemma.

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* Throughout this section we assume that $a<-r<r<b$; t is taken equal to zero for simplicity.

For, if x is allowed to approach a positive value of y through values less than y, then $\Delta/(y-x)$ remains a function of one sign (with the same sign as Δ), and approaches a positive value. The variable Δ must therefore have been positive. By allowing x to approach a negative y through values between y and zero, we see that

$$\Delta < 0, \quad y < x < 0.$$

To prove that $W(y) > 0$ throughout $(-a, a)$ except at the origin, first note that

$$W^{(k)}(0) = 0 \qquad (k = 0, 1, \cdots, 2n-3),$$

$$W^{(2n-2)}(0) = \begin{vmatrix} g_{n-1}^{(n-1)}(0) & g_n^{(n-1)}(0) \\ g_{n-1}^{(n)}(0) & g_n^{(n)}(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ g_{n-1}^{(n)}(0) & 1 \end{vmatrix} = 1$$

by virtue of relations (6). Hence

$$W(y) = \frac{y^{2n-2}}{(2n-2)!} + \frac{y^{2n-1}}{(2n-1)!} W^{(2n-1)}(\xi), \qquad 0 < \xi < y, \ 0 > \xi > y.$$

This shows that $W(y) > 0$ for values of y sufficiently near the origin. But the same argument used above to show that Δ is different from zero may be used to show that $W(y)$ is different from zero away from the origin. The Lemma is thus completely established.

By use of this Lemma it is possible to prove

THEOREM IV. *Let the function $\phi_i(x)$ satisfy Conditions* A *in $(-a, a)$, and let the interval of convergence of the series*

$$\sum_{n=0}^{\infty} c_n g_n(x)$$

be $(-r, r)$. Then if the series converges for $x = r$ (or $x = -r$), its sum is continuous in the interval $-r < x \leqq r$ (or $-r \leqq x < r$).

Since the series converges for $x = r$, then to an arbitrary positive ϵ there corresponds a number m such that

$$\left| c_{m+1} g_{m+1}(r) + \cdots + c_{m+p} g_{m+p}(r) \right| < \epsilon \qquad (p = 1, 2, 3, \cdots).$$

Now by the Lemma the set of values

$$\frac{g_0(x)}{g_0(r)}, \quad \frac{g_1(x)}{g_1(r)}, \quad \frac{g_2(x)}{g_2(r)}, \cdots, \quad 0 < x < r,$$

forms a decreasing set. Hence by Abel's lemma*

* E. Goursat, *Cours d'Analyse Mathématique*, vol. 1, p. 182.

$$\left| c_{m+1}g_{m+1}(r)\,\frac{g_{m+1}(x)}{g_{m+1}(r)} + \cdots + c_{m+p}g_{m+p}(r)\,\frac{g_{m+p}(x)}{g_{m+p}(r)} \right| < \epsilon\frac{g_{m+1}(x)}{g_{m+1}(r)},$$

$$\leqq \epsilon\frac{g_0(x)}{g_0(r)} \leqq \epsilon M, \quad 0 \leqq x \leqq r,$$

where M is the maximum of $g_0(x)/g_0(r)$ in $0\leqq x\leqq r$. Consequently the series converges uniformly in $0\leqq x\leqq r$, and represents a continuous function there. That the sum is continuous in $-r<x\leqq 0$ follows from Theorem III. A similar proof shows that if the series converges at $-r$, then the sum is continuous in $-r\leqq x<r$.

8. **Generalization of the process of analytic continuation.** Let us first obtain formulas analogous to the binomial formulas

$$(x-t)^n = x^n - nx^{n-1}t + \frac{n(n-1)}{2!}x^{n-2}t^2 - \cdots + (-1)^n t^n, \tag{26}$$

$$x^n = t^n + nt^{n-1}(x-t) + \frac{n(n-1)}{2!}t^{n-2}(x-t)^2 + \cdots + (x-t)^n. \tag{27}$$

Let t and u be two distinct points of the interval (a, b). Since $g_n(x, t)$ and $g_n(x, u)$ are both linear combinations of $u_0(x), u_1(x), \cdots, u_n(x)$,

$$g_n(x,u) = c_0g_0(x,t) + c_1g_1(x,t) + \cdots + c_ng_n(x,t).$$

The constants c_i may be determined by use of formulas (7). The result is

$$g_n(x,u) = L_0g_n(t,u)g_0(x,t) + L_1g_n(t,u)g_1(x,t) + \cdots + L_ng_n(t,u)g_n(x,t). \tag{28}$$

For the particular sequence (1′) this reduces to (26) with $t=0$, $u=t$, and to (27) with $u=0$, $t=t$.

In order to generalize the process of analytic continuation we begin with a consideration of the double series

$$\begin{aligned} &c_0L_0g_0(t,u)g_0(x,t) \\ &+ c_1L_0g_1(t,u)g_0(x,t) + c_1L_1g_1(t,u)g_1(x,t) \\ &+ c_2L_0g_2(t,u)g_0(x,t) + c_2L_1g_2(t,u)g_1(x,t) + c_2L_2g_2(t,u)g_2(x,t) \\ &+ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &+ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\,. \end{aligned} \tag{29}$$

Let us suppose that the interval of convergence of the series

$$\sum_{n=0}^{\infty} c_ng_n(x,u) \tag{30}$$

is $|x-u|<r, a\leqq x\leqq b$. It will now be possible to show the double series (29) absolutely convergent in a certain interval. The general term of that series is

$$c_nL_kg_n(t,u)g_k(x,t), \quad 0\leqq k\leqq n.$$

Assuming Conditions A, we may obtain an upper bound for this term as follows:

$$|g_k(x,t)|\leqq \frac{M_0M_1\cdots M_k}{m_0m_1\cdots m_k}\,\frac{|x-t|^k}{k!},$$

$$|L_kg_n(t,u)|\leqq \frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\,\frac{|t-u|^{n-k}}{(n-k)!},\qquad n\geqq k.$$

Let x_0 be a point in the interval of convergence of the series (30). Then there exists a constant M independent of n for which

$$|c_ng_n(x_0,u)|<M.$$

Hence we have

$$|g_n(x_0,u)|>\frac{m_0m_1\cdots m_n}{M_0M_1\cdots M_n}\,\frac{|x_0-u|^n}{n!},$$

$$|c_n|<\frac{MM_0\cdots M_n}{m_0\cdots m_n}\,\frac{n!}{|x_0-u|^n}.$$

Consequently, observing that $M_k/m_k\geqq 1$,

$$|c_nL_kg_n(t,u)g_k(x,t)|<M\left(\frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\right)^3\frac{n!}{k!(n-k)!}\,\frac{|x-t|^k|t-u|^{n-k}}{|x_0-u|^n}.$$

We are thus led to a dominant double series, which will now be shown convergent under certain conditions. First form the sum of the nth row of this series:

$$M\left(\frac{M_0\cdots M_n}{m_0\cdots m_n}\right)^3\frac{1}{|x_0-u|^n}\sum_{k=0}^{n}\frac{n!|x-t|^k|t-u|^{n-k}}{k!(n-k)!}$$
$$=M\left(\frac{M_0\cdots M_n}{m_0\cdots m_n}\right)^3\frac{1}{|x_0-u|^n}(|t-u|+|x-t|)^n.$$

Then form the sum of the row values

$$M\sum_{n=0}^{\infty}\left(\frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\right)^3\left(\frac{|t-u|+|x-t|}{|x_0-u|}\right)^n.$$

The test ratio of this series is

$$\left(\frac{M_n}{m_n}\right)^s \frac{|t-u|+|x-t|}{|x_0-u|},$$

and by Condition A (c) this has the limit

$$\frac{|t-u|+|x-t|}{|x_0-u|}$$

as n becomes infinite. Consequently the series (29) is absolutely convergent if

$$|t-u|+|x-t|<|x_0-u|, \quad a \leqq x \leqq b.$$

The sum of the series may be obtained by summing by rows or by columns. In the one case, using formula (28), we find the sum to be the convergent series

$$\sum_{n=0}^{\infty} c_n g_n(x,u). \tag{31}$$

In the other case, the sum is found to be

$$\sum_{n=0}^{\infty} L_n f(t) g_n(x,t), \tag{32}$$

where $f(x)$ is defined as the sum of the series (31). That

$$L_k f(t) = \sum_{n=0}^{\infty} c_n L_k g_n(t,u)$$

follows from Theorem III. We thus have two representations for $f(x)$, the first of which, (31), holds in $|x-u|<r$, $a\leqq x\leqq b$, and the second of which, (32), holds in $|x-t|<r-|t-u|$, $a\leqq x\leqq b$. Conceivably, series (32) may converge in a larger interval, in which case an extension or prolongation of $f(x)$ would be at hand. We sum up the results in

THEOREM V. *If the functions $\phi_i(x)$ satisfy Conditions* A *in* (a, b), *and if*

$$f(x) = \sum_{n=0}^{\infty} c_n g_n(x,u), \qquad |x-u|<r, \quad a\leqq x\leqq b, \ a\leqq u\leqq b,$$

then

$$f(x) = \sum_{n=0}^{\infty} L_n f(t) g_n(x,t)$$

for all x and t satisfying the relation

$$|x-t|<r-|t-u|, \quad a\leqq x\leqq b, \quad a\leqq t\leqq b.$$

9. **A generalization of a theorem of S. Bernstein.** We shall now obtain a necessary and sufficient condition for the representation of a function $f(x)$ in a series of the type in question. The method consists in generalizing a familiar theorem of S. Bernstein.* The results to be proved are stated in

THEOREM VI. *Let the functions* ϕ_i *satisfy Conditions* A *in* (a, b). *Then a necessary and sufficient condition that a function* $f(x)$, *defined in the interval* $a \leqq x < b$, *can be represented by a series*

$$f(x) = \sum_{n=0}^{\infty} L_n f(a) g_n(x, a), \quad a \leqq x < b, \tag{33}$$

is that $f(x)$ *be the difference of two functions of class* C^∞ *in* $a \leqq x < b$,

$$f(x) = \phi(x) - \psi(x),$$

such that

$$L_n\phi(x) > 0 \text{ or } \phi(x) \equiv 0\,; \quad L_n\psi(x) > 0, \text{ or } \psi(x) \equiv 0, \quad a < x < b$$
$$(n = 0, 1, 2, \cdots).$$

We begin by proving the necessity of the condition. We suppose that

$$f(x) = \sum_{n=0}^{\infty} L_n f(a) g_n(x, a), \quad a \leqq x < b.$$

By Theorem III this series is absolutely convergent in $a \leqq x < b$, and hence we may set

$$\phi(x) = \sum_{n=0}^{\infty} |L_n f(a)| g_n(x, a),$$

$$\psi(x) = \sum_{n=0}^{\infty} \{ |L_n f(a)| - L_n f(a) \} g_n(x, a),$$

$$f(x) = \phi(x) - \psi(x), \qquad a \leqq x < b.$$

Again using the results of Theorem III, we have

$$L_k\phi(x) = \sum_{n=0}^{\infty} |L_n f(a)| L_k g_n(x, a) \qquad (k = 0, 1, 2, \cdots),$$

$$L_k\psi(x) = \sum_{n=0}^{\infty} \{ |L_n f(a)| - L_n f(a) \} L_k g_n(x, a).$$

* S. Bernstein, *Sur la définition et les propriétés des fonctions analytiques d'une variable réelle*, Mathematische Annalen, vol. 75 (1914), p. 449.

By reference to (24) it is seen that every non-vanishing term of each of these series is positive throughout the interval $a<x<b$. The necessity of the condition is thus established.

Conversely, suppose that $f(x)=\phi(x)-\psi(x)$, where $\phi(x)$ and $\psi(x)$ satisfy the conditions of the theorem. It will be enough to show that $\phi(x)$ can be represented in a series (33), for a similar proof will apply to $\psi(x)$; and, since the operators L_n are linear, we will then obtain a representation of the form desired for $f(x)$ by subtracting the series for $\phi(x)$ and $\psi(x)$.

We suppose that $\phi(x)$ is not identically zero, for otherwise the result is obvious. Choose a point x_0 of the interval $a<x<b$, and consider the following exact remainder formula:

$$\phi(x_0) = L_0\phi(t)g_0(x_0,t) + L_1\phi(t)g_1(x_0,t) + \cdots + L_n\phi(t)g_n(x_0,t)$$
$$+ \int_t^{x_0} g_n(x_0,t)L_{n+1}\phi(t)dt,$$

where $a<t<x_0$. Since the functions $\phi_n(x)$ are all positive, the functions $g_i(x_0, t)$ are all positive. By hypothesis $L_{n+1}\phi(t)$ is positive. Consequently the above integral is surely positive, as is each term on the right-hand side of the equation. Hence

$$\phi(x_0) > L_n\phi(t)g_n(x_0,t),$$

$$L_n\phi(t) < \frac{\phi(x_0)}{g_n(x_0,t)} < \phi(x_0)\frac{M_0M_1\cdots M_n}{m_0m_1\cdots m_n}\frac{n!}{(x_0-t)^n}. \tag{34}$$

Now referring to Theorem II and to formula (19), we see that

$$\phi(x) = L_0\phi(t)g_0(x,t) + L_1\phi(t)g_1(x,t) + \cdots + L_n\phi(t)g_n(x,t) + R_n,$$

$$R_n = L_{n+1}\phi(\xi)\phi_0(x)\int_t^x\phi_1(x)\int_t^x\phi_2(x)\int_t^x\cdots\int_t^x\phi_n(x)\int_t^x\frac{(dx)^{n+1}}{\phi_0(x)\cdots\phi_n(x)},$$
$$t<\xi<x,$$
$$t>\xi>x,$$

$$R_n = L_{n+1}\phi(\xi)\frac{\phi_0(x)\phi_1(\xi_1)\cdots\phi_n(\xi_n)}{\phi_0(\xi_{n+1})\phi_1(\xi_{n+1})\cdots\phi_n(\xi_{n+1})}\frac{(x-t)^{n+1}}{(n+1)!},$$
$$t<\xi_{n+1}<\xi_n<\cdots<\xi_1<x,$$
$$t>\xi_{n+1}>\xi_n>\cdots>\xi_1>x.$$

Setting $t=\xi$ in (34), we have

$$|R_n| < \phi(x_0)\left(\frac{M_0\cdots M_{n+1}}{m_0\cdots m_{n+1}}\right)^2\frac{|x-t|^{n+1}}{(x_0-\xi)^{n+1}}.$$

Evidently the remainder approaches zero as n becomes infinite if

$$|x-t| < \frac{x_0 - t}{2}, \qquad a \leqq x.$$

If now t is allowed to approach a, the following expansion results:

$$\phi(x) = \sum_{n=0}^{\infty} L_n\phi(a)g_n(x,a), \qquad 0 \leqq x-a < \frac{x_0-a}{2}. \tag{35}$$

But the series (35) converges in a larger interval. For

$$\sum_{n=0}^{\infty} L_n\phi(a)g_n(x,a) \ll \phi(x_0)\sum_{n=0}^{\infty}\left(\frac{M_0\cdots M_n}{m_0\cdots m_n}\right)^2 \frac{|x-a|^n}{(x_0-a)^n}, \tag{36}$$

and the dominant series converges for $|x-a|<x_0-a$. It remains only to show that the sum of the series is $\phi(x)$ throughout the interval $a\leqq x<b$.

Denote the sum of the series (36) by $H(x)$. Then $H(x)=\phi(x)$ for $a\leqq x<(a+x_0)/2$. Choose a point t in this interval near to $(x_0+a)/2$. We have seen above that

$$\phi(x) = \sum_{n=0}^{\infty} L_nH(t)g_n(x,t) = \sum_{n=0}^{\infty} L_n\phi(t)g_n(x,t), \quad |x-t| < \frac{x_0-t}{2}, \quad x \geqq a.$$

But by Theorem V

$$H(x) = \sum_{n=0}^{\infty} L_nH(t)g_n(x,t), \quad |x-t| < x_0 - t, \quad x \geqq a.$$

Consequently $H(x)=\phi(x)$ for $a\leqq x<(x_0+t)/2$. Now choose a point t' in this interval near to $(x_0+t)/2$, and proceed as before to show that $H(x)=\phi(x)$ in $a\leqq x<(x_0+t')/2$. By continuing the process we see that $H(x)$ and $\phi(x)$ coincide in the entire interval $a\leqq x<x_0$. But x_0 was an arbitrary point of $a<x<b$. Consequently equation (33) holds in this interval, and the proof is complete.

10. **The expansion of an arbitrary analytic function.** After imposing further conditions on the functions $\phi_i(x)$ it will be found possible to represent an arbitrary analytic function in a generalized power series. We define

CONDITIONS B. (a) *Conditions* A *are satisfied in* (a, b);

(b) $$\frac{d^k}{dx^k}\left(\frac{1}{\phi_n(x)}\right) \geqq 0 \qquad (k = 1,2,3,\cdots; n = 0,1,2,3,\cdots; a \leqq x \leqq b).$$

We now state a very simple lemma, the proof of which follows immediately from Leibniz's rule for the differentiation of a product.

LEMMA. *If $f(x)$ is positive with positive derivatives of all orders, and if $\phi(x)$ is positive with derivatives of all orders that are positive or zero, then $(d/dx)\cdot(f(x)\cdot\phi(x))$ is positive with all its derivatives.*

We are now in a position to prove

THEOREM VII. *If Conditions* B *are satisfied in (a, b), and if $f(x)$ is analytic in $a<x<b$, then*

$$f(x) = \sum_{n=0}^{\infty} L_n f(t) g_n(x,t), \qquad a < t < b,$$

the series being convergent in some neighborhood of t.

Since $f(x)$ is analytic at t, it can be represented as a power series

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(t)\frac{(x-t)^n}{n!}, \qquad |x-t| < r.$$

Then it follows that the expansion

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}\left(t - \frac{r}{3}\right)\left(x - t + \frac{r}{3}\right)^n$$

is certainly valid in the interval $t-r<x<t+r/3$.

Now set

$$g(x) = \sum_{n=0}^{\infty} \left|f^{(n)}\left(t - \frac{r}{3}\right)\right|\left(x - t + \frac{r}{3}\right)^n,$$

$$h(x) = \sum_{n=0}^{\infty} \left\{\left|f^{(n)}\left(t - \frac{r}{3}\right)\right| - f^{(n)}\left(t - \frac{r}{3}\right)\right\}\left(x - t + \frac{r}{3}\right)^n,$$

so that

$$f(x) = g(x) - h(x),$$

$g(x)$ and $h(x)$ being functions that are either identically zero or positive with all their derivatives in $t-r/3<x<t+r/3$. The trivial case in which $g(x)$ or $h(x)$ is identically zero may be discarded. Now by making successive applications of the Lemma it is seen that

$$L_n g(x) > 0, \ L_n h(x) > 0 \qquad \left(n = 0,1,2, \cdots ; t - \frac{r}{3} < x < t + \frac{r}{3}\right).$$

Consequently Theorem VI may be applied to give

$$g(x) = \sum_{n=0}^{\infty} L_n g\left(t - \frac{r}{3}\right) g_n\left(x, t - \frac{r}{3}\right), \quad t - \frac{r}{3} \leqq x < t + \frac{r}{3},$$

$$h(x) = \sum_{n=0}^{\infty} L_n h\left(t - \frac{r}{3}\right) g_n\left(x, t - \frac{r}{3}\right).$$

Finally, we make use of Theorem V, and see that

$$g(x) = \sum_{n=0}^{\infty} L_n g(t) g_n(x,t),$$

$$h(x) = \sum_{n=0}^{\infty} L_n h(t) g_n(x,t),$$

$$f(x) = \sum_{n=0}^{\infty} L_n f(t) g_n(x,t), \qquad |x - t| < r.$$

The theorem is thus established.

It should be pointed out that Conditions B are not so strong as to exclude the case of Taylor's development. For, they are surely satisfied for $\phi_n(x) = 1$. Moreover, other sets of functions $\phi_n(x)$ exist satisfying the conditions. Witness the set

$$\phi_n(x) = e^{-x/n}.$$

11. **Teixeira's series.** In the introduction reference was made to certain series studied by Teixeira. We wish to show by a consideration of the sequence (1″) how these series arise naturally as a generalization of Taylor's series. In order that the Wronskians $W_n(x)$ may all be positive we change the sign of certain of the functions of the sequence (1″), an alteration that will not affect the form of the series. Consider then the sequence

$$(37) \qquad 1, \sin x, -\cos x, -\sin 2x, \cos 2x, \cdots, (-1)^{n-1}\sin nx, \\ (-1)^n \cos nx, \cdots.$$

The operators L_{2n+1} corresponding to this sequence have a particularly simple form:

$$L_{2n+1} = D(D^2 + 1^2)(D^2 + 2^2) \cdots (D^2 + n^2) \qquad (n = 0, 1, 2, \cdots),$$

where D indicates the operation of differentiation. The operators L_{2n} are more complicated. Direct computations show that

$$W_{2n} = n! [(2n-1)!]^2 [(2n-3)!]^2 \cdots [3!]^2.$$

By definition of the operator L_{2n+1} we have

$$L_{2n+1}(-1)^n \sin (n+1)x$$
$$= \frac{W(1, \sin x, \cos x, \cdots, \sin nx, \cos nx, (-1)^n \sin (n+1)x)}{W_{2n}} = \frac{W_{2n+1}}{W_{2n}},$$

whence

$$W_{2n+1} = W_{2n}D(D^2+1^2)\cdots(D^2+n^2)(-1)^n \sin (n+1)x$$
$$= W_{2n}(2n+1)! \cos (n+1)x.$$

Hence the Wronskians $W_n(x)$ are all positive at the origin, and the functions of approximation, $g_n(x)$, all exist. We shall show that

$$g_{2n}(x) = \frac{2^n}{(2n)!}[1 - \cos x]^n, \quad g_{2n+1}(x) = \frac{2^n}{(2n+1)!}[1 - \cos x]^n \sin x.$$

By a familiar formula of trigonometry we have

$$\frac{2^n}{(2n)!}[1 - \cos x]^n = \sum_{k=-n}^{n} \frac{(-1)^k \cos kx}{(n-k)!(n+k)!}.$$

This function clearly satisfies the differential equation

$$L_{2n+1}u(x) = 0. \tag{38}$$

Moreover, it satisfies the boundary conditions

$$u^{(k)}(0) = 0 \quad (k = 0,1,2,\cdots,2n-1); \quad u^{(2n)}(0) = 1. \tag{39}$$

But the differential system (38) (39) has only one solution, the function of approximation $g_{2n}(x)$.

By noting that

$$\frac{2^n}{(2n+1)!}[1 - \cos x]^n \sin x = \frac{d}{dx}\frac{2^{n+1}}{(2n+2)!}[1 - \cos x]^{n+1}$$
$$= \sum_{k=-n-1}^{n+1} \frac{(-1)^{k+1} k \sin kx}{(n+1-k)!(n+1+k)!}$$

it is seen that this function satisfies the system

$$L_{2n+2}v(x) = 0,$$
$$v^{(k)}(0) = 0 \quad (k = 0,1,2,\cdots,2n); \quad v^{(2n+1)}(0) = 1,$$

and consequently must be $g_{2n+1}(x)$.

In the expansion of the function $f(x)$, the coefficients of the terms $g_{2n}(x)$ will involve the complicated differential operator L_{2n}. We may, however, express this coefficient in terms of a simpler operator of order $2n$. In doing this use will be made of the functions $\phi_n(x)$ which will now be computed:

$$\phi_0(x) = 1, \quad \phi_1(x) = \cos x,$$

$$\phi_{2n}(x) = \frac{n}{\cos^2 nx}, \quad \phi_{2n+1}(x) = 2(2n+1)\cos(n+1)x\cos nx \qquad (n = 1,2,3,\cdots).$$

Evidently,

$$L_{2n+2}f(x) = \phi_0(x)\cdots\phi_{2n+1}(x)\frac{d}{dx}\frac{L_{2n+1}f(x)}{\phi_0(x)\cdots\phi_{2n+1}(x)}$$

$$= \frac{(\cos(n+1)x)DL_{2n+1}f(x) + (n+1)(\sin(n+1)x)L_{2n+1}f(x)}{\cos(n+1)x}.$$

Consequently it follows that,

$$L_{2n+2}f(0) = D^2(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)f(0).$$

The expansion of $f(x)$ now takes the form

$$f(x) \sim \sum_{n=0}^{\infty} A_n\frac{2^n}{(2n)!}[1-\cos x]^n + B_n\frac{2^n}{(2n+1)!}[1-\cos x]^n\sin x,$$

$$A_n = D^2(D^2+1^2)(D^2+2^2)\cdots(D^2+(n-1)^2)f(0),$$

$$B_n = D(D^2+1^2)(D^2+2^2)\cdots(D^2+n^2)f(0).$$

Although the sequence (37) does not satisfy the Conditions A directly, a simple substitution reduces the series (40) to one for which these conditions are satisfied. Indeed we shall see that the substitution $y = \sin(x/2)$ reduces the series to the sum of two Taylor's series, so that the convergence can be easily discussed.

An alternative form of the series is obviously

$$f(x) \sim \sum_{n=0}^{\infty} A_n\frac{2^{2n}}{(2n)!}\left(\sin\left(\frac{x}{2}\right)\right)^{2n} + B_n\frac{2^{2n+1}}{(2n+1)!}\left(\sin\left(\frac{x}{2}\right)\right)^{2n+1}\cos\left(\frac{x}{2}\right).$$

If the change of variable $x/2 = y$ is made, the form of the series employed by Teixeira* is obtained.

* For reference see § 1.

Now any function $f(x)$ analytic in the neighborhood of $x=0$ can be expanded in a series of this type for a sufficiently small neighborhood of $x=0$. For, if

$$\phi(x) = \frac{f(x) + f(-x)}{2}, \quad \psi(x) = \frac{f(x) - f(-x)}{2}, \quad \phi(x) + \psi(x) = f(x),$$

then the functions

$$\phi(2 \sin^{-1} y) \quad \text{and} \quad \frac{\psi(2 \sin^{-1} y)}{\cos \sin^{-1} y}$$

are both analytic in some neighborhood $|y|<\delta$ of $y=0$. Hence they can be expanded in powers of y:

$$\phi(2 \sin^{-1} y) = \sum_{n=0}^{\infty} a_{2n} y^{2n}, \quad |y| < \delta,$$

$$\frac{\psi(2 \sin^{-1} y)}{\cos \sin^{-1} y} = \sum_{n=0}^{\infty} b_{2n+1} y^{2n+1}, \quad |y| < \delta.$$

We have now only to make the substitution $y=\sin(x/2)$ in these series and to add in order to be assured that $f(x)$ can be expanded in a series (40) in some neighborhood of the origin. For simplicity expansion have been considered in the neighborhood of the origin, but the results clearly hold for an arbitrary point.

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