PRIMITIVE GROUPS WHICH CONTAIN SUBSTITUTIONS OF PRIME ORDER p AND OF DEGREE $6p$ OR $7p$*

BY

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1. In a memoir on primitive groups in the first volume of the Bulletin of the Mathematical Society of France† Jordan announced the following theorem:

Let q be a positive integer < 6, p any prime $>q$. The degree of a primitive group G that contains a substitution of order p on q cycles (without including the alternating group) cannot exceed $pq+q+1$.

He gave proofs for the cases $q=1$, and $q=2$, but no proofs for greater values of q. In the same memoir, he also found a limit for the degree of G when q is not restricted to numbers <6. His results may be stated as follows:

Let q be any positive integer, p any prime $>2q\log_2 q+q+1$. The degree of a primitive group G that contains a substitution of order p on q cycles (without including the alternating group) cannot exceed $qp+2q\log_2 2q$.

Manning has studied this problem further. He not only gave proofs for the cases $q=2$, 3, 4, 5, finding a somewhat closer limit than the one announced by Jordan, but also found a much lower limit for the degree of G in the general case. Using the theory developed in the proof of his general theorem, he investigated the case $q=6$. A brief statement of these theorems follows:

Let q be any integer greater than unity and <5, p any prime $>q+1$. Then the degree of a primitive group of class >3 which contains a substitution of order p and of degree qp cannot exceed $qp+q$. When $p=q+1$, the degree cannot exceed $qp+q+1$.

If a primitive group of class >3 contains a substitution of prime order p on 5 cycles, $p>5$, its degree cannot exceed $5p+6$. Moreover, if a primitive group of degree $5p+6$ exists, it is doubly transitive.

The degree of a primitive group G of class >3 which contains a substitution of prime order p and of degree $qp(p>2q-3, q>1)$, does not exceed $qp+4q-4$. Moreover p^2 does not divide the order of G.

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† C. Jordan, Bulletin de la Société Mathématique de France, vol. 1 (1873), pp. 175–221.

The degree of a primitive group of class >3 *which contains a substitution of prime order* p *and of degree* $6p(p>6)$ *does not exceed* $6p+10$.

He published these results in a series of 4 papers, entitled *On the order of primitive groups*.* This title draws attention to the fact that the problem of finding a limit for the degree of these primitive groups is equivalent to the problem of finding a limit for the order of a primitive group in terms of its degree. The fact that the order of these groups is limited by their degree is discussed by Manning in his second paper.

2. In the present paper, the case $q=7$ will be investigated and the limit for the degree of G for the case $q=6$ will be lowered. The following theorems will be proved:

The degree of a primitive group G *of class* >3 *which contains a substitution of prime order* $p(p>7)$ *and of degree* $6p$ *cannot exceed* $6p+6$. *If* $p=7$, *the true limit for the degree of* G *is* $6p+7$. *Moreover, if* G *of degree* $6p+7$ *exists, it is doubly transitive.*

The degree of a primitive group G *of class* >3 *which contains a substitution of prime order* p *and of degree* $7p$, $p>7$, *does not exceed* $7p+8$. *Moreover, if* G *of degree* $7p+8$ *exists, it is doubly transitive.*

Although the first theorem is not proved until §17, we shall use it in the proof of the second theorem, for the former depends in no way upon the latter. The proof of the latter theorem is based on the general method developed by Manning in his third paper *On the order of primitive groups*, of which especially §§9–20 should be carefully read before reading the following proof. All definitions and the fundamental theory will be found in these sections. The assumption (III, §12) that the degree of H_{r+1} exceeds $qp+q$ should be noted, for with the exception of §15 this hypothesis is held throughout the present paper. We now proceed to the proof of this theorem.

3. If H_{r+1} is imprimitive, H_{r+s} (III, §18) has systems of imprimitivity of 7 letters only, for the number of letters in a system must divide 7 (III, §17). Moreover, if H_{r+1} is of degree $>qp+q$, the systems of imprimitivity of H_{r+s} are permuted according to a primitive group which is not triply transitive (III, §18), thus according to a primitive group of degree $p+1$ at most. Then the degree of H_{r+s} does not exceed $7p+7$. Now since a generator introduces 7 letters or none, $s=1$. We shall now consider H_{r+1} of degree $7p+7$. Clearly the order of H_{r+1} is not divisible by p^2 when $p>7$. Then J_1

* W. A. Manning, these Transactions, vol. 10 (1909), p. 247; vol. 16 (1915), p. 139; vol. 19 (1918), p. 127; vol. 20 (1919), p. 66. These 4 papers will be referred to by the Roman numerals I. II, III, IV, respectively.

is transitive of degree 7. Now H_{r+1} may be contained in a doubly transitive group of degree $7p+8$, for if H_{r+2} exists, J_2 is multiply transitive of degree 8. Since the J group is always a transitive representation of a subgroup or quotient group of the direct product of a cyclic group of degree a divisor of $p-1$ and the symmetric group of degree 7 (see §5), J_2 must be the simple group of order 168, for it is the only primitive group of degree 8 which does not occur for the first time on 8 letters. A triply transitive group of degree $7p+9$ does not exist, for then J_3 is of degree 9. However, J_2 is not contained in any non-alternating primitive group of higher degree, and J is not alternating when its degree exceeds 7 (III, §20).

4. Let H_{r+1} be a primitive group. Its subgroup F is intransitive. If $p>2q-3$, no constituent of F is alternating, nor does an imprimitive constituent permute its systems according to an alternating group (III, §35).*

We shall now assume $p>2q-3$. The case of $p=2q-3$ will be taken up in §16. Then the degree of F does not exceed $7p+14$. The order of F is not divisible by p^2 (III, §27), nor is the order of the subgroup L of H_{r+1} that leaves one letter fixed.

5. The group I_1 that occurs in H_{r+1} has no substitutions on the letters of J_1 only, for then G would be of class $\leqq 2q-4$ (III, §21) and therefore of degree $\leqq 25$.†

We shall need the following theorems (I, Theorems 5 and 6, p. 251) in discussing the J groups.

Let P be a cyclic group of prime order p and of degree $qp(q<p)$. The largest group G on the same letters that transforms P into itself and that contains no substitution of order p with $<q$ cycles is of order $p(p-1)(q!)$.

The quotient group G/P is the direct product of a cyclic group of order $p-1$ and a group isomorphic to the symmetric group of degree q.

Then the constituent of I_1 on the letters of A_1 is the group of order $p(p-1)(q!)$ or a subgroup of it. Thus J_1 is a transitive representation of the direct product of a cyclic group of order d (d a divisor of $p-1$) and a group

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* An accurate statement of this theorem follows: If $p>2q-3$, the degree of H_{r+1} does not exceed $qp+q$, when H_{ij} has a transitive constituent which is alternating or which permutes systems of imprimitivity according to an alternating group. If $p=2q-3$, there is one exception to this limit of the degree of H_{r+1}. It occurs when E_1 (III, § 30) has a transitive constituent simply isomorphic to the alternating constituent of degree p. Then E_1 has three constituents of degrees $(p-1)p/2$, p, p. The third constituent cannot be of degree $p+k$, for such a constituent is of too small a degree to be simply isomorphic to the alternating constituent of degree p and (III, §33) E_1 cannot have a primitive constituent multiply isomorphic to the alternating constituent of degree p. Then H_{r+1} is of degree $\leqq qp+2q$.

† W. A. Manning, American Journal of Mathematics, vol. 28 (1906), p. 226.

K which occurs as a quotient group among the groups of degree $\leqq 7$. Let the group K of order kk' be multiplied into a cyclic group of order d. The direct product of order $kk'd$ can be represented as a transitive group on dk letters if and only if the group K of order kk' has a subgroup K' of order k' which contains no invariant subgroup of K. Call the subgroup of J_1 that leaves one letter fixed J_1'. It should be noted that J_1' is not the identity if H_{r+1} is of degree $>qp+q$. If the subgroup K' is invariant in a group of order mk', J_1', isomorphic to K', is invariant in a group of order mk' and therefore fixes exactly m letters.

6. The following theorems on simply transitive primitive groups by Manning will be used repeatedly:

THEOREM 1. *Let G_1, the subgroup that fixes one letter of a simply transitive primitive group G of degree n and order g, have a multiply transitive constituent of degree m. If G_1 has no transitive constituent whose degree $(>m)$ is a divisor of $m(m-1)$, all the transitive constituents of G_1 are simply isomorphic multiply transitive groups of degree m and order g/n.**

THEOREM 2. *If only one transitive constituent of G_1 is an imprimitive group (of order f), G_1 is of order f.*

THEOREM 3. *If G_1 has an intransitive constituent of order f, and if all the transitive constituents on the remaining letters of G_1 are primitive groups, G_1 is of order f.*†

THEOREM 4. *Let G_1 have a primitive constituent M of degree m, in which the subgroup M_1 that fixes one letter is primitive. Let M be paired with itself in G_1 and let the order of M be $<g/n$. Then G_1 contains an imprimitive constituent in which there is an invariant intransitive subgroup with m transitive constituents of $m-1$ letters each, permuted according to the permutations of the primitive group M.*‡

7. We wish to see how far the reasoning used in the proof of Theorem 1 may be applied to the subgroup F of H_{r+1} when H_{r+1} is a primitive group of degree $>qp+q$ $(q>5,\ p>2q-3)$. Let $L(x)$ $(=L$, III, §21) be the subgroup of H_{r+1} that fixes the letter x. Then $L(x)$ has an invariant subgroup $F(x)=F$ generated by all of its substitutions of order p. Let $F(x)$ have a transitive constituent of degree $p+2$ on the letters $a_1, a_2, \cdots, a_{p+2}$. If the order of $F(x)$ is t, the order of $F(x)(a_1)$ is $t/(p+2)$ and the order of $F(x)(a_1)(a_2)$

* W. A. Manning, *Primitive Groups*, 1921, p. 83.

† W. A. Manning, Proceedings of the National Academy of Sciences, vol. 12 (1926), p. 755.

‡ W. A. Manning, these Transactions, vol. 29 (1927) pp. 815–825.

is $t/[(p+2)(p+1)]$. In $F(a_1)$, x belongs to a transitive constituent of degree $p+2$. Since $F(a_1)(x)$ has a transitive constituent on the letters $a_2, a_3, \cdots, a_{p+2}$, the transitive constituent to which x belongs in $F(a_1)$ contains either $p+1$ a's or none. Suppose that $F(a_1)$ has a transitive constituent on the letters $x, a_2, a_3, \cdots, a_{p+2}$. Then the group $\{F(a_1), F(x)\}$ has a transitive constituent of degree $p+3$ on the letters $x, a_1, a_2, \cdots, a_{p+2}$. Now a group of degree $p+3$ that contains a substitution of degree and order p is alternating. The subgroup of $\{F(a_1), F(x)\}$ that fixes the letter x contains an invariant subgroup, $F(x)$, generated by all of its substitutions of order p and has an alternating constituent on the letters $a_1, a_2, \cdots, a_{p+2}$. Since an alternating group of degree >4 is simple, $F(x)$ has an alternating constituent on the letters $a_1, a_2, \cdots, a_{p+2}$. However, if $p>2q-3$, F has no alternating constituents. If $F(a_1)$ has a transitive constituent on the letters $a_2, a_3, \cdots, a_{p+2}$, $F(x)$ also has a transitive constituent of degree $p+1$. But it can be shown that F cannot have constituents of degree $p+2$ and $p+1$ at the same time. The constituent groups of F are positive groups. In order that the constituent of degree $p+2$ contain no negative substitutions, the substitution from its I group that is associated* with the substitution from its J group must be negative. This negative substitution is from the metacyclic group and consequently it is negative in the I group of the constituent of degree $p+1$. Substitutions from the metacyclic group of I_1 have cycles on letters of each cycle of A_1. Then the letters $a_2, a_3, \cdots, a_{p+2}$ belong in $F(a_1)$ to a transitive constituent of degree $\mu \geqq p+2$. Thus the order of $F(a_1)(a_2)$ is t/μ, and if x belongs to a transitive constituent of degree δ in $F(a_1)(a_2)$, the order of $F(a_1)(a_2)(x)$ is $t/(\mu\delta)$. Then $t/[(p+2)(p+1)]=t/(\mu\delta)$. If $F(x)$ contains no constituent whose degree $(>p+2)$ divides $(p+2)(p+1)$, $\mu=p+2$.

The next difficulty in applying the proof of Theorem 1 arises when we consider the constituents of degree $p+2$ which contain $p+1$ a's in the groups $F(a_1), F(a_2), \cdots, F(a_{p+2})$. However, if $c_1=c_2$, the group $\{F(a_1), F(a_2)\}$ contains a transitive constituent on the letters $c_1, a_1, a_2, \cdots, a_{p+2}$. Thus as above, $F(a_1)$ has an alternating constituent on the letters $c_1, a_2, a_3, \cdots, a_{p+2}$.

* An intransitive group may be regarded as formed from its transitive constituents by establishing an isomorphism between one transitive constituent and the constituent (transitive or intransitive) on the remaining letters, and then multiplying corresponding substitutions. Thus any substitution of an intransitive group is the product of substitutions from all transitive constituents (taking the identity into account). These substitutions from different transitive constituents which occur as factors in a given substitution are said to be *associated*. For example, in the intransitive octic group written out in §13, the substitution (47) is said to be associated with the substitution (58) (69).

We may now follow the proof of Theorem 1 until the statement "if B and C coincide." If B and C coincide, the group $\{F(a_1), F(x)\}$ has a transitive constituent of degree $2p+5$. Such a constituent is alternating. Now the proof (III, §28) that no alternating constituent of H_{ij} involves letters of more than one cycle of A_1 without causing the presence of a substitution of order p and of degree $<qp$ in G applies to any intransitive group generated by substitutions of order p and of degree qp, thus also to the group $\{F(a_1), F(x)\}$. Since transitive groups generated by substitutions of order p and of degrees $3p+7$, $4p+9$, $5p+11$, $6p+13$ are alternating, we see that if $F(x)$ has one transitive constituent of degree $p+2$ and no transitive constituent whose degree $(>p+2)$ divides $(p+2)(p+1)$, it has at least six constituents of degree $p+2$.

The results of the above discussion may be summarized in

THEOREM 5. *Let H_{r+1} be a primitive group of degree $>qp+q$ $(p>2q-3, q>5)$. If F has a transitive constituent of degree $p+2$ and no transitive constituent whose degree $(>p+2)$ divides $(p+2)(p+1)$, it has at least six transitive constituents of degree $p+2$.*

The following theorem will also be useful.

THEOREM 6. *Let G be a simply transitive primitive group. If the subgroup that fixes one letter of G has a transitive constituent of degree m, it must also have another transitive constituent whose degree divides mk_i, where k_i $(\geqq 1)$ is the degree of a transitive constituent of the subgroup that fixes one letter of the constituent of degree m.*

Let $G(x)$ be the subgroup of G that fixes the letter x. Let $G(x)$ have a transitive constituent of degree m on the letters $a_1, a_2, \cdots, a_m$. Note that the theorem is proved if $G(x)$ has a second transitive constituent of degree m. It may thus be assumed that $G(x)$ has only one transitive constituent of degree m. Now let $G(x)(a_1)$ have transitive constituents of degrees k_1, $k_2, \cdots, k_v$ on the letters $a_2, a_3, \cdots, a_m$. The order of $G(x)$ is g/n, if n is the degree of G. Then the order of $G(x)(a_1)$ is $g/(nm)$, and the order of $G(x)(a_1)(a_2)$ is $g/(nmk_1)$, $g/(nmk_2)$, $\cdots$, or $g(nmk_v)$, according as a_2 belongs to a transitive constituent of degree $k_1, k_2, \cdots$, or k_v in $G(x)(a_1)$. Now x belongs to a transitive constituent of degree m in $G(a_1)$. We know that at least one of the transitive constituents of degree k_i, $i=1, 2, \cdots, v$, on the letters $a_2, a_3, \cdots, a_m$ in $G(x)(a_1)$ belongs in $G(a_1)$ to a transitive constituent of degree $r>k_i$, which does not include the letter x.* Then the order of

* W. A. Manning, these Transactions, vol. 29 (1927), p. 815.

$G(a_1)(a_2)$ is $g/(nr)$, if a_2 is the letter that occurs in the transitive constituent of degree k_i in $G(x)(a_1)$, and if x belongs to a transitive constituent of degree s in $G(a_1)(a_2)$, the order of $G(a_1)(a_2)(x)$ is $g/(nrs)$. Then at least one of the following equations is true:

$$r = mk_i/s \qquad (i = 1, 2, \cdots, v).$$

If $k=1$, a condition that may arise when the constituent of degree m is imprimitive, we conclude that r is a divisor of m.

8. We are now prepared to study the subgroup F of a primitive H_{r+1}. Let F be of degree $7p+14$ (§4). Since F includes H_3, it has at most 5 constituents. The partitions of the degree of F are the following:

$$\begin{array}{lllll}
6p+12, & p+2 & & & \\
5p+10, & 2p+4 & & & \\
4p+8, & 3p+6 & & & \\
5p+10, & p+2, & p+2 & & \\
4p+8, & 2p+4, & p+2 & & \\
3p+6, & 3p+6, & p+2 & & \\
3p+6, & 2p+4, & 2p+4 & & \\
4p+8, & p+2, & p+2, & p+2 & \\
3p+6, & 2p+4, & p+2, & p+2 & \\
2p+4, & 2p+4, & 2p+4, & p+2 & \\
3p+6, & p+2, & p+2, & p+2, & p+2 \\
2p+4, & 2p+4, & p+2, & p+2, & p+2.
\end{array}$$

The following partitions are impossible: $6p+12$, $p+2$; $4p+8$, $2p+4$, $p+2$; $3p+6$, $3p+6$, $p+2$; $2p+4$, $2p+4$, $2p+4$, $p+2$, for in each case L has a transitive constituent of degree $p+2$, paired with itself, whose order is less than that of L. By Theorems 2 and 3, the imprimitive constituents of L determine the order of L. The multiple isomorphism between the constituent of degree $p+2$ and the constituent whose order determines that of L follows from the multiple isomorphism between the J groups of these constituents. Then all the conditions of Theorem 4 are satisfied and consequently L should have a transitive constituent of degree $(p+2)(p+1)$.

Now J_1 is a transitive group of degree 15. Then $dk=15$, and $d=1$, 3, or 5. When $d=3$ or 5, J_1' fixes 3 and 5 letters respectively. However, the partitions of the degree of F show that J_1' fixes one letter only and therefore $d=1$ and $k=15$. Now the partitions also show that k' is even. No orders will be listed when there are no groups or quotient groups of these orders on <8 letters. Then the possible values of $15k'$ are 60, 120, 240, 360, 720, 2520,

5040. The order 5040 is impossible, for the symmetric group of degree 7 contains no subgroup of order 336.

If $15k'=60$, J_1' is of order 4. Since the J group of a constituent of degree $2p+4$ is octic (IV, §3), the partitions of the degree of F exclude this representation.

Let $15k'=120$. J_1' is of order 8. Since the least order of the J group of a constituent of degree $4p+8$ is 16 (IV, §3), we need consider for this representation only the following partition of the degree of F: $2p+4$, $2p+4$, $p+2$, $p+2$. From the groups of order 120 on <8 letters, only one distinct representation is obtained. This representation is given by the symmetric group of degree 5 with respect to its octic subgroup. J_1' has three transitive constituents of degrees 2, 4, and 8. It is generated by

$$\{23\cdot 4567\cdot 8yux\cdot 9zvo,\ \ 23\cdot 46\cdot 8z\cdot ou\cdot xv\cdot 9y\}.$$

Then in L this partition of the degree of F becomes $4p+8$, $2p+4$, $p+2$. Thus the constituent of degree $p+2$ in L satisfies all the conditions of Theorem 4, but L has no transitive constituent of degree $(p+2)(p+1)$.

If $15k'=240$, J_1' is of order 16. However the only group of order 240 on <8 letters contains no non-invariant subgroup of order 16.

If $15k'=360$, J_1' is of order 24. There is only one representation of the alternating group of degree 6 on 15 letters and J_1' has two transitive constituents one of degree 6 and one of degree 8. The group J_1' is generated by

$$\{25\cdot 34\cdot 68\cdot 79\cdot xy\cdot zu,\ \ 28\cdot 36\cdot 45\cdot 79\cdot ox\cdot vu,\ \ 29\cdot 37\cdot 45\cdot 68\cdot oz\cdot yv\}.$$

Now consider the possible partitions of the degree of F for this J_1'. A constituent of degree $3p+6$ is impossible, for its J group is of order 18, 36, or 72 (IV, §3). A constituent of degree $4p+8$ is likewise impossible, for its J group is of order 16, 32, 128 or greater. Then there is only the partition $2p+4$, $2p+4$, $p+2$, $p+2$, $p+2$ to be considered. It calls for an invariant intransitive subgroup of degree and order 8 in J_1'. However all the subgroups of order 8 are conjugate in J_1'.

If $15k'=720$, J_1' is of order 48. There is one representation of the symmetric group of degree 6 with respect to its subgroup $\{abc, ad, ef\}$. J_1' has two constituents of degrees 8 and 6, respectively, in a two-to-one isomorphism. It is generated by

$$\{246\cdot 573\cdot ozv\cdot xuy,\ \ 28\cdot 39\cdot xz\cdot yv,\ \ 23\cdot 45\cdot 67\cdot 89\}.$$

Now the J group of a constituent of degree $3p+6$ in F is incompatible with a J_1' of order 48. Then the only possible partitions are $4p+8$, $p+2$, $p+2$,

$p+2$, and $2p+4$, $2p+4$, $p+2$, $p+2$, $p+2$. Neither partition, however, allows a two-to-one isomorphism between the constituents of J_1'.

If $15k'=2520$, there is one representation of the alternating group of degree 7 with respect to its subgroup of order 168. Now this subgroup contains substitutions of order 7 and consequently J_1' contains substitutions of the same order. Then J_1' has two constituents of degree 7 or it is transitive of degree 14. The partitions of the degree of F show that either case is impossible. Hence H_{r+1} is not of degree $7p+15$.

9. Let F be of degree $7p+13$. The partitions of the degree of F are the following:

$$\begin{array}{llllll}
6p+12, & p+1 & & & & \\
5p+10, & p+2, & p+1 & & & \\
4p+8, & 2p+4, & p+1 & & & \\
3p+6, & 3p+6, & p+1 & & & \\
4p+8; & p+2, & p+2, & p+1 & & \\
3p+6, & 2p+4, & p+2, & p+1 & & \\
2p+4, & 2p+4, & 2p+4, & p+1 & & \\
3p+6, & p+2, & p+2, & p+2, & p+1 & \\
2p+4, & 2p+4, & p+2, & p+2, & p+1 & \\
2p+4, & p+2, & p+2, & p+2, & p+2, & p+1.
\end{array}$$

All these partitions have one and only one constituent of degree $p+1$. Then by Theorem 1, L should have a transitive constituent whose degree divides $(p+1)p$. Thus H_{r+1} is not of degree $7p+14$

10. H_{r+1} cannot be of degree $7p+13$. If $p>13$, J_1 is transitive of degree 13, but no subgroup of the direct product of a cyclic group of order a divisor of $p-1$ and the symmetric group of degree 7 can be written as a group of degree 13. Then $p=13$. However, a primitive group of degree $qp+p$ does not exist unless $p<2q-2$ (III, §22).

Similarly if H_{r+1} is of degree $7p+11$, $p=11$, but by hypothesis $p>11$.

11. Let F be of degree $7p+11$. The partitions of the degree of F are the following:

$$\begin{array}{llll}
5p+10, & 2p+1 & & \\
4p+8, & 3p+3 & & \\
*5p+10, & p+1, & p & \\
*4p+8, & 2p+2, & p+1 & \\
\dagger 4p+8, & 2p+1, & p+2 & \\
\dagger 3p+6, & 3p+3, & p+2 & \\
3p+6, & 2p+4, & 2p+1 & \\
3p+3, & 2p+4, & 2p+4 & \\
*4p+8, & p+2, & p+1, & p
\end{array}$$

$4p+8$,	$p+1$,	$p+1$,	$p+1$		
$*3p+6$,	$2p+4$,	$p+1$,	p		
$*3p+6$,	$2p+2$,	$p+2$,	$p+1$		
$3p+6$,	$2p+1$,	$p+2$,	$p+2$		
$3p+3$,	$2p+4$,	$p+2$,	$p+2$		
$*2p+4$,	$2p+4$,	$2p+2$,	$p+1$		
$\dagger 2p+4$,	$2p+4$,	$2p+1$,	$p+2$		
$*3p+6$,	$p+2$,	$p+2$,	$p+1$,	p	
$\dagger 3p+6$,	$p+2$,	$p+1$,	$p+1$,	$p+1$	
$3p+3$,	$p+2$,	$p+2$,	$p+2$,	$p+2$	
$*2p+4$,	$2p+4$,	$p+2$,	$p+1$,	p	
$2p+4$,	$2p+4$,	$p+1$,	$p+1$,	$p+1$	
$*2p+4$,	$2p+2$,	$p+2$,	$p+2$,	$p+1$	
$2p+4$,	$2p+1$,	$p+2$,	$p+2$,	$p+2$	
$*2p+4$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	p
$\ddagger 2p+4$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$
$*2p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$
$**2p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+2$.

In this and the remaining sections all the partitions of the degree of F which are impossible by Theorem 1 are prefixed by the asterisk *. Likewise all partitions which are impossible by Theorem 4 are prefixed by the dagger † and those which contradict Theorem 5 by the two asterisks **.

In the present case, among the partitions which remain after eliminating those impossible by Theorems 1, 4, and 5, the partition $2p+4$, $p+2$, $p+2$, $p+1$, $p+1$, $p+1$ is also impossible, for F cannot have constituents of degrees $p+2$ and $p+1$ at the same time without containing a negative substitution (see §7). In the remaining cases, partitions of the degree of F which are impossible for this reason will be prefixed by the double dagger ‡.

Now consider the possible J groups. The 10 partitions of the degree of F that remain to be considered show that the order of J_1' is even. Then if $d=1$, $k=12$, and $12k'=24, 48, 72, 120, 144, 168, 240, 360, 720, 2520, 5040$. The orders 360, 2520, 5040 are impossible, for there are no groups of orders 30, 210, or 420 on <8 letters. Since there are no partitions which allow J_1' to be of order 2, $12k'\neq 24$. Neither are there any partitions which allow J_1' to be of order 4, for the J group of a constituent of degree $2p+4$ is octic.

If $12k'=72$, the only possible partition is $3p+3$, $p+2$, $p+2$, $p+2$, $p+2$, for the J group of a constituent of degree $3p+6$ is at least of order 18. This partition brings a substitution of degree and order 3 into J_1'. The group J_1 is imprimitive, for a primitive J_1 would be alternating, but if J_1 is of degree

$>q$, it is not alternating (III, §20). Then J_1 has systems of imprimitivity of 3, 4, or 6 letters and its order is 324, 648, or greater.

If $12k'=120$, no partition of the degree of F is possible, for the least order of the J group of a constituent of degree $5p+10$ is 50 (IV, §3). If $12k'=144$ or 720, the only possible partition is again $3p+3$, $p+2$, $p+2$, $p+2$, $p+2$, but as has been seen this partition is incompatible with the order of J_1'. When $12k'=168$ or 240, no partition of the degree of F allows J_1' to be of order 14 or 20.

If $d=2$, $k=6$, and $6k'=12$, 24, 36, 48, 60, 72, 120, 360, 720. As we have seen, no partition of the degree of F allows J_1' to be of orders 2, 4, 6, 10, 12, 20, or 60.

Now when $d=2$, J_1' is invariant in a subgroup of twice its order and therefore fixes two letters of J_1. Then when $d>1$, J_1 may be constructed by first writing down the transitive representation of K on k letters with respect to its subgroup of order k' and then making it simply isomorphic to itself in d different sets of letters and in such a way that the subgroup of order d permutes these d transitive constituents cyclically and is commutative with each substitution of K. With the above in mind consider the case when J_1' is of order 8. There are only two possible partitions of the degree of F: $2p+4$, $2p+4$, $p+1$, $p+1$, $p+1$, and $2p+4$, $2p+1$, $p+2$, $p+2$, $p+2$. In the second partition since the constituent of degree $2p+4$ gives J_1' a constituent of degree 4, J_1' must have two constituents of degree 4. Thus J_1' is of degree 8 and this partition is then incompatible with such a J_1'. The first partition is also impossible because all the constituents of degree $p+1$ cannot unite in L if J_1' fixes two letters. Then L has a multiply transitive constituent of degree $p+1$ and is impossible by Theorem 1.

If J_1' is of order 120, it has two constituents of degree 5. However no partition of the degree of F is possible.

If $d=3$, $k=4$, and J_1' is invariant in a group of three times its order and therefore it fixes three letters. The only possible partitions of the degree of F, $4p+8, p+1, p+1, p+1$, and $2p+4, 2p+4, p+1, p+1, p+1$, have a multiply transitive constituent of degree $p+1$ in L, for the constituents of degree $p+1$ cannot unite in L if J_1' fixes three letters. These partitions are then impossible by Theorem 1. If $d=4$, and $k=3$, the only possible partitions are the above and again they are impossible. Since no partition of the degree of F fixes so many as 6 letters, $d\neq 6$.

12. Let F be of degree $7p+9$. The partitions of the degree of F are the following:

$$\begin{array}{ll} 5p+5, & 2p+4 \\ 4p+8, & 3p+1 \end{array}$$

$4p+3$,	$3p+6$			
$*5p+6$,	$p+2$,	$p+1$		
$5p+5$,	$p+2$,	$p+2$		
$4p+8$,	$2p+1$,	p		
$4p+8$,	$2p$,	$p+1$		
$*4p+4$,	$2p+4$,	$p+1$		
$3p+6$,	$3p+3$,	p		
$*3p+6$,	$3p+2$,	$p+1$		
$\dagger 3p+6$,	$3p+1$,	$p+2$		
$\dagger\dagger 3p+6$,	$2p+2$,	$2p+1$		
$\dagger\dagger 3p+3$,	$2p+4$,	$2p+2$		
$3p+1$,	$2p+4$,	$2p+4$		
$4p+8$,	$p+1$,	p,	p	
$*4p+4$,	$p+2$,	$p+2$,	$p+1$	
$**4p+3$,	$p+2$,	$p+2$,	$p+2$	
$*3p+6$,	$2p+2$,	$p+1$,	p	
$\dagger 3p+6$,	$2p+1$,	$p+2$,	p	
$\dagger\dagger 3p+6$,	$2p+1$,	$p+1$,	$p+1$	
$\dagger 3p+6$,	$2p$,	$p+2$,	$p+1$	
$\dagger 3p+3$,	$2p+4$,	$p+2$,	p	
$\dagger\dagger 3p+3$,	$2p+4$,	$p+1$,	$p+1$	
$\dagger\dagger 3p+3$,	$2p+2$,	$p+2$,	$p+2$	
$*3p+2$,	$2p+4$,	$p+2$,	$p+1$	
$3p+1$,	$2p+4$,	$p+2$,	$p+2$	
$2p+4$,	$2p+4$,	$2p+1$,	p	
$2p+4$,	$2p+4$,	$2p$,	$p+1$	
$*2p+4$,	$2p+2$,	$2p+2$,	$p+1$	
$\dagger 2p+4$,	$2p+2$,	$2p+1$,	$p+2$	
$\dagger 3p+6$,	$p+2$,	$p+1$,	p,	p
$\dagger\dagger 3p+6$,	$p+1$,	$p+1$,	$p+1$,	p
$\ddagger 3p+3$,	$p+2$,	$p+2$,	$p+2$,	p
$\ddagger 3p+3$,	$p+2$,	$p+2$,	$p+1$,	$p+1$
$*3p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$
$**3p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+2$
$2p+4$,	$2p+4$,	$p+1$,	p,	p
$*2p+4$,	$2p+2$,	$p+2$,	$p+1$,	p
$2p+4$,	$2p+2$,	$p+1$,	$p+1$,	$p+1$
$\ddagger 2p+4$,	$2p+1$,	$p+2$,	$p+2$,	p
$\dagger 2p+4$,	$2p+1$,	$p+2$,	$p+1$,	$p+1$
$\ddagger 2p+4$,	$2p$,	$p+2$,	$p+2$,	$p+1$

$^*2p+2$,	$2p+2$,	$p+2$,	$p+2$,	$p+1$	
$^{**}2p+2$,	$2p+1$,	$p+2$,	$p+2$,	$p+2$	
$\ddagger 2p+4$,	$p+2$,	$p+2$,	$p+1$,	p,	p
$\dagger 2p+4$,	$p+2$,	$p+1$,	$p+1$,	$p+1$,	p
$2p+4$,	$p+1$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
$^*2p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	p
$^{**}2p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$
$^{**}2p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	p
$^{**}2p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$
$^{**}2p$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$.

We now delete all those partitions of the degree of F which contradict Theorems 1, 4, and 5, and those which cause a constituent group of F to have a negative substitution. In the partitions preceded by the two daggers $\dagger\dagger$, J_1' has a substitution (IV, §3) of degree and order 3. If J_1 contains such a substitution it is imprimitive, for a primitive J_1 is alternating, and we know that J_1 is not alternating when its degree $>q$. The group J_1 has systems of imprimitivity of 5 letters only and its least possible order is 7200. Since a J_1 of this or greater order cannot be written on 7 or fewer letters, these partitions are impossible.

Now consider the possible J groups. If $d=1$, $k=10$, and $10k'=20$, 40, 60, 120, 240, 360, 720, 2520, 5040. Odd values of k' need not be considered, for the partitions of the degree of F show that the order of J_1' is even. Since there are no groups of order 252 or 504 on <8 letters, $10k'\neq 2520$, or 5040. The orders 20, 40, 60, and 120 are also impossible, for the partitions of the degree of F do not allow J_1' to be of order 2, 4, 6, or 12.

If $10k'=240$, the group $\{abcde, ab, fg\}$ may be represented on 10 letters by means of its symmetric subgroup of degree 4. This representation gives a J_1' with two constituents of degree 4 in a simple isomorphism. The only possible partitions of the degree of F are the following: $3p+1$, $2p+4$, $2p+4$; $2p+4$, $2p+4$, $2p+1$, p; $2p+4$, $2p+4$, $2p$, $p+1$; $2p+4$, $2p+4$, $p+1$, p, p. However these partitions are all impossible, for they all contain a constituent of degree $2p+4$ which demands that an octic group be invariant in the symmetric group of degree 4.

If $10k'=360$, the alternating group of degree 6 with respect to its subgroup $\{abc, def, aebd\cdot cf\}$ gives a doubly transitive J_1. Since there are no partitions which allow J_1 to be doubly transitive, $10k'\neq 360$. If $10k'=720$, the symmetric group of degree 6 with respect to its subgroup $\{ab, ac, de, df, ad\cdot be\cdot cf\}$ also gives a doubly transitive J_1.

If $d=2$, $k=5$, $5k'=10$, 20, 60, 120. We have seen that J_1' cannot be of order 2, 4, or 12. If $5k'=120$, we have a J_1' with two constituents of degree

4. Such a J_1' group has been seen to be impossible. (See this section, paragraph 4.)

13. Let F be of degree $7p+8$. The partitions of the degree of F are the following:

$*6p+6$,	$p+2$		
$*5p+6$,	$2p+2$		
$5p+4$,	$2p+4$		
$4p+8$,	$3p$		
$4p+2$,	$3p+6$		
$*5p+6$,	$p+2$,	p	
$*5p+6$,	$p+1$,	$p+1$	
$*5p+5$,	$p+2$,	$p+1$	
$**5p+4$,	$p+2$,	$p+2$	
$4p+8$,	$2p$,	p	
$4p+4$,	$2p+4$,	p	
$*4p+4$,	$2p+2$,	$p+2$	
$*4p+3$,	$2p+4$,	$p+1$	
$\dagger 4p+2$,	$2p+4$,	$p+2$	
$3p+6$,	$3p+2$,	p	
$*3p+6$,	$3p+1$,	$p+1$	
$\dagger 3p+6$,	$3p$,	$p+2$	
$3p+3$,	$3p+3$,	$p+2$	
$3p+6$,	$2p+2$,	$2p$	
$3p+6$,	$2p+1$,	$2p+1$	
$3p+3$,	$2p+4$,	$2p+1$	
$3p+2$,	$2p+4$,	$2p+2$	
$3p$,	$2p+4$,	$2p+4$	
$4p+8$,	p,	p,	p
$**4p+4$,	$p+2$,	$p+2$,	p
$*4p+4$,	$p+2$,	$p+1$,	$p+1$
$*4p+3$,	$p+2$,	$p+2$,	$p+1$
$**4p+2$,	$p+2$,	$p+2$,	$p+2$
$3p+6$,	$2p+2$,	p,	p
$*3p+6$,	$2p+1$,	$p+1$,	p
$\dagger 3p+6$,	$2p$,	$p+2$,	p
$3p+6$,	$2p$,	$p+1$,	$p+1$
$*3p+3$,	$2p+4$,	$p+1$,	p
$*3p+3$,	$2p+2$,	$p+2$,	$p+1$
$3p+3$,	$2p+1$,	$p+2$,	$p+2$
$\dagger 3p+2$,	$2p+4$,	$p+2$,	p

$3p+2$,	$2p+4$,	$p+1$,	$p+1$		
$^{**}3p+2$,	$2p+2$,	$p+2$,	$p+2$		
$^{*}3p+1$,	$2p+4$,	$p+2$,	$p+1$		
$3p$,	$2p+4$,	$p+2$,	$p+2$		
$2p+4$,	$2p+4$,	$2p$,	p		
$2p+4$,	$2p+2$,	$2p+2$,	p		
$^{*}2p+4$,	$2p+2$,	$2p+1$,	$p+1$		
$^{\dagger}2p+4$,	$2p+2$,	$2p$,	$p+2$		
$^{*}2p+2$,	$2p+2$,	$2p+2$,	$p+2$		
$^{\dagger}3p+6$,	$p+2$,	p,	p,	p	
$3p+6$,	$p+1$,	$p+1$,	p,	p	
$^{*}3p+3$,	$p+2$,	$p+2$,	$p+1$,	p	
$^{\ddagger}3p+3$,	$p+2$,	$p+1$,	$p+1$,	$p+1$	
$^{**}3p+2$,	$p+2$,	$p+2$,	$p+2$,	p	
$^{**}3p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$	
$^{*}3p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+1$	
$^{**}3p$,	$p+2$,	$p+2$,	$p+2$,	$p+2$	
$2p+4$,	$2p+4$,	p,	p,	p	
$^{\dagger}2p+4$,	$2p+2$,	$p+2$,	p,	p	
$2p+4$,	$2p+2$,	$p+1$,	$p+1$,	p	
$^{*}2p+4$,	$2p+1$,	$p+2$,	$p+1$,	p	
$2p+4$,	$2p+1$,	$p+1$,	$p+1$,	$p+1$	
$^{\ddagger}2p+4$,	$2p$,	$p+2$,	$p+2$,	p	
$^{\dagger}2p+4$,	$2p$,	$p+2$,	$p+1$,	$p+1$	
$^{**}2p+2$,	$2p+2$,	$p+2$,	$p+2$,	p	
$^{*}2p+2$,	$2p+2$,	$p+2$,	$p+1$,	$p+1$	
$^{*}2p+2$,	$2p+1$,	$p+2$,	$p+2$,	$p+1$	
$^{**}2p+2$,	$2p$,	$p+2$,	$p+2$,	$p+2$	
$^{**}2p+1$,	$2p+1$,	$p+2$,	$p+2$,	$p+2$	
$^{\ddagger}2p+4$,	$p+2$,	$p+2$,	p,	p,	p
$^{\dagger}2p+4$,	$p+2$,	$p+1$,	$p+1$,	p,	p
$2p+4$,	$p+1$,	$p+1$,	$p+1$,	$p+1$,	p
$^{**}2p+2$,	$p+2$,	$p+2$,	$p+2$,	p,	p
$^{**}2p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	p
$^{**}2p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
$^{*}2p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	p
$^{**}2p+1$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$
$^{**}2p$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	p
$^{**}2p$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$.

First strike out all the partitions of the degree of F which are impossible

by Theorem 1. A partition containing a constituent of degree $5p+6$ is included in this category, for such a constituent is doubly transitive (II, p. 147). Likewise eliminate all those partitions which contradict Theorems 4 and 5 and those which cause a constituent group of F to contain a negative substitution. Then of the original 75 partitions of the degree of F, only 25 remain to be considered.

Now consider the possible J groups. If $d=3$, $k=3$, and J_1' can be of order 2 only, but none of the partitions that remain allow J_1' to be of order 2. Then $d=1$, $k=9$, and $9k'=18, 36, 72, 144, 360, 720, 2520, 5040$. The orders 720, 2520, and 5040 are impossible, for there are no groups of orders 80, 280, or 560 on <8 letters. Odd values of k' have not been considered, for the partitions of the degree of F show that J_1' is of even order. Moreover, no partition allows J_1' to be of order 2 or 4.

If $9k'=72$, there is only one group of order 72 on <8 letters which contains no invariant subgroup in its subgroup of order 8. The group $\{ab, ac, de, df, ad\cdot be\cdot cf\}$ with respect to one of its octic subgroups gives the J_1'

$$\begin{array}{c} 1 \\ 2437\cdot 5698 \\ 2734\cdot 5896 \\ 23\cdot 47\cdot 59\cdot 68 \\ 47\cdot 58\cdot 69 \\ 27\cdot 34\cdot 59 \\ 24\cdot 37\cdot 68 \\ 23\cdot 56\cdot 89. \end{array}$$

The possible partitions of the degree of F for such a J_1' are the following: $5p+4$, $2p+4$; $4p+4$, $2p+4$, p; $3p$, $2p+4$, $2p+4$; $3p$, $2p+4$, $p+2$, $p+2$; $2p+4$, $2p+4$, $2p$, p; $2p+4$, $2p+2$, $2p+2$, p; $2p+4$, $2p+4$, p, p, p. The J groups of the partitions $3p$, $2p+4$, $p+2$, $p+2$, and $2p+4$, $2p+2$, $2p+2$, p, are in multiple isomorphism while the constituents of J_1' are in simple isomorphism.

Now consider the partition $5p+4$, $2p+4$. The subgroup L of H_{r+1} has the same transitive constituents. We shall now apply Theorem 6 to the constituent of degree $2p+4$. We know (IV, §3) that the subgroup that fixes one letter of the constituent of degree $2p+4$ has a transitive constituent of degree $2p+2$. Thus $k_1=1$, $k_2=2p+2$, and $r=5p+4$. Thus $s=(2p+4)/(5p+4)$ or $(2p+4)(2p+2)/(5p+4)$. A moment's calculation shows that these are impossible equations, for s is an integer.

In an imprimitive constituent of degree $2p+4$, generated by substitutions of order p and of degree $2p$, the invariant substitution in its J group fixes the $2p$ letters of A_1 (IV, §3). Consequently the substitution from the I

group of another constituent, associated with it, cannot be a substitution from the metacyclic group, for the substitutions from the metacyclic group have cycles on letters of each cycle of A_1. Now the I group of a constituent of degree p in F is the metacyclic group or one of its subgroups. Then in the partition $2p+4$, $2p+4$, p, p, p, F has a substitution of order 2 and degree 8. There may be two kinds of substitutions in the I group of a constituent of degree $2p$ or $3p$ in F: substitutions from the metacyclic group which do not permute cycles of A_1 and substitutions that permute cycles of A_1. The latter may again be of two kinds: substitutions which are commutative with each substitution in A_1 and substitutions which are the product of these and substitutions from the metacyclic group. The latter substitutions have cycles on letters of each cycle of A_1. Thus in the partition $2p+4$, $2p+4$, $2p$, p, the invariant substitution of order 2 and degree 8 in J_1' either fixes the $2p$ letters of the constituent of degree $2p$ or is associated with a substitution of order 2 and degree $2p$ from it. The latter is a negative substitution, while the constituent groups of F are positive groups. In the partition $2p+4$, $2p+4$, $3p$, there is likewise a substitution of order 2 and degree 8 or $8+2p$, for the invariant substitution of degree 8 may fix the $3p$ letters of A_1 in the constituent of degree $3p$, be associated with a substitution of order 2 and degree $2p$, or be associated with a substitution of order 3 and of degree $3p$ from it. Thus this partition, $2p+4$, $2p+4$, $3p$, is also impossible.

The systems of imprimitivity of a constituent of degree $2p+4$ can be chosen in only one way and the choice is determined by the transpositions in its octic J group. Then the non-invariant substitutions of order 2 and of degree 4 in the octic group permute systems of imprimitivity. Such substitutions cannot fix the $2p$ letters of A_1, for then the primitive group according to which the constituent of degree $2p+4$ permutes its systems contains a transposition and is consequently symmetric, but since the constituent groups of F are positive groups, the group of the systems is also positive. Again, since the constituent groups of F are positive groups, a positive permutation must be associated with the non-invariant substitutions of the axial subgroup of the J group. The only positive substitutions in the I group of a constituent of degree $2p+4$, on the letters of A_1 only, are substitutions from the metacyclic group. Then there must be substitutions from the metacyclic group in the I group of a constituent of degree $2p+4$. Now consider the partition $4p+4$, $2p+4$, p. The group L has transitive constituents of the same degrees. In it the constituent of degree p is simply transitive, for it cannot be doubly transitive by Theorem 1. Consequently it is a subgroup of the metacyclic group.* Since the metacyclic group has only one

* W. Burnside, Proceedings of the London Mathematical Society, vol. 33, pp. 162–185

subgroup of order p, the constituent of degree p in F is cyclic. Therefore there are no substitutions from the metacyclic group in F, for as we have seen, a permutation from the metacyclic group in I has cycles on letters of each cycle of A_1.

Let $9k'=144$. There is only one group of order 144 on <8 letters, the group $\{abc, ad, ef, eg\}$. However, it contains no subgroup of order 16 which does not contain the axial group, an invariant subgroup of the group of order 144.

If $9k'=360$, there is no representation, for the alternating group of degree 6 contains no subgroup of order 40.

14. Let F be of degree $7p+7$. The partitions of the degree of F are the following:

$*6p+6$,	$p+1$		
$*6p+5$,	$p+2$		
$*5p+6$,	$2p+1$		
$5p+5$,	$2p+2$		
$\dagger\dagger 5p+3$,	$2p+4$		
$4p+4$,	$3p+3$		
$\dagger\dagger 4p+1$,	$3p+6$		
$*5p+6$,	$p+1$,	p	
$\dagger 5p+5$,	$p+2$,	p	
$5p+5$,	$p+1$,	$p+1$	
$*5p+4$,	$p+2$,	$p+1$	
$**5p+3$,	$p+2$,	$p+2$	
$*4p+4$,	$2p+2$,	$p+1$	
$*4p+4$,	$2p+1$,	$p+2$	
$\dagger\dagger 4p+3$,	$2p+4$,	p	
$*4p+3$,	$2p+2$,	$p+2$	
$*4p+2$,	$2p+4$,	$p+1$	
$\dagger 4p+1$,	$2p+4$,	$p+2$	
$\dagger\dagger 3p+6$,	$3p+1$,	p	
$*3p+3$,	$3p+3$,	$p+1$	
$\dagger\dagger 3p+3$,	$3p+2$,	$p+2$	
$\dagger\dagger 3p+6$,	$2p+1$,	$2p$	
$\dagger\dagger 3p+3$,	$2p+4$,	$2p$	
$\dagger\dagger 3p+3$,	$2p+2$,	$2p+2$	
$3p+2$,	$2p+4$,	$2p+1$	
$3p+1$,	$2p+4$,	$2p+2$	
$*4p+4$,	$p+2$,	$p+1$,	p
$4p+4$,	$p+1$,	$p+1$,	$p+1$

*$4p+3$,	$p+2$,	$p+1$,	$p+1$	
*$4p+2$,	$p+2$,	$p+2$,	$p+1$	
**$4p+1$,	$p+2$,	$p+2$,	$p+2$	
††$3p+6$,	$2p+1$,	p,	p	
††$3p+6$,	$2p$,	$p+1$,	p	
††$3p+3$,	$2p+4$,	p,	p	
††$3p+3$,	$2p+2$,	$p+2$,	p	
††$3p+3$,	$2p+2$,	$p+1$,	$p+1$	
*$3p+3$,	$2p+1$,	$p+2$,	$p+1$	
††$3p+3$,	$2p$,	$p+2$,	$p+2$	
*$3p+2$;	$2p+4$,	$p+1$,	p	
*$3p+2$,	$2p+2$,	$p+2$,	$p+1$	
**$3p+2$,	$2p+1$,	$p+2$,	$p+2$	
†$3p+1$,	$2p+4$,	$p+2$,	p	
$3p+1$,	$2p+4$,	$p+1$,	$p+1$	
**$3p+1$,	$2p+2$,	$p+2$,	$p+2$	
†$3p$,	$2p+4$,	$p+2$,	$p+1$	
$2p+4$,	$2p+2$,	$2p+1$,	p	
$2p+4$,	$2p+2$,	$2p$,	$p+1$	
*$2p+4$,	$2p+1$,	$2p+1$,	$p+1$	
†$2p+4$,	$2p+1$,	$2p$,	$p+2$	
*$2p+2$,	$2p+2$,	$2p+2$,	$p+1$	
*$2p+2$,	$2p+2$,	$2p+1$,	$p+2$	
††$3p+6$,	$p+1$,	p,	p,	p
††$3p+3$,	$p+2$,	$p+2$,	p,	p
††$3p+3$,	$p+2$,	$p+1$,	$p+1$,	p
††$3p+3$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
*$3p+2$,	$p+2$,	$p+2$,	$p+1$,	p
**$3p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$
**$3p+1$,	$p+2$,	$p+2$,	$p+2$,	p
**$3p+1$,	$p+2$,	$p+2$,	$p+1$,	$p+1$
**$3p$,	$p+2$,	$p+2$,	$p+2$,	$p+1$
$2p+4$,	$2p+2$,	$p+1$,	p,	p
†$2p+4$,	$2p+1$,	$p+2$,	p,	p
$2p+4$,	$2p+1$,	$p+1$,	$p+1$,	p
†$2p+4$,	$2p$,	$p+2$,	$p+1$,	p
$2p+4$,	$2p$,	$p+1$,	$p+1$,	$p+1$
*$2p+2$,	$2p+2$,	$p+2$,	$p+1$,	p
$2p+2$,	$2p+2$,	$p+1$,	$p+1$,	$p+1$
**$2p+2$,	$2p+1$,	$p+2$,	$p+2$,	p

$^*2p+2$,	$2p+1$,	$p+2$,	$p+1$,	$p+1$	
$^{**}2p+2$,	$2p$,	$p+2$,	$p+2$,	$p+1$	
$^*2p+1$,	$2p+1$,	$p+2$,	$p+2$,	$p+1$	
$^{**}2p+1$,	$2p$,	$p+2$,	$p+2$,	$p+2$	
$^\dagger 2p+4$,	$p+2$,	$p+1$,	p,	p,	p
$2p+4$,	$p+1$,	$p+1$,	$p+1$,	p,	p
$^{**}2p+2$,	$p+2$,	$p+2$,	$p+1$,	p,	p
$^{**}2p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$,	p
$2p+2$,	$p+1$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
$^{**}2p+1$,	$p+2$,	$p+2$,	$p+2$,	p,	p
$^{**}2p+1$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	p
$^{**}2p+1$,	$p+2$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
$^{**}2p$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	p
$^{**}2p$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	$p+1$.

In addition to striking out all those partitions of the degree of F which are impossible by Theorems 1, 4, and 5, we shall also exclude all those which bring a circular substitution of degree 3 into J_1'. For, if J_1 contains such a substitution, it is imprimitive of order 288, 576, or 1152. All groups of these orders occur for the first time on 8 letters. These partitions will be preceded by the two daggers ††. The partitions $5p+5$, $2p+2$, and $5p+5$, $p+1$, $p+1$ are also impossible, for if J_1 contains a substitution of degree and order 5 it is alternating, but J_1 is not alternating when its degree exceeds q.

The 13 partitions of the degree of F that need still to be considered show that the order of J_1' is even. Then if $d=1$, $k=8$, and $8k'=16$, 48, 144, 240, 720, 5040. The only possible orders are 16, 48, and 144, for there are no groups of orders 30, 90, and 630 on <8 letters. Moreover, the order 16 is also impossible, for all of the partitions which allow J_1' to be of order 2 have a multiply transitive constituent in L.

There are three groups of order 48 on <8 letters, namely, $\{ad, ab\cdot de, ac\cdot df\}$, $\{abc, ad, ef\}$, and $\{ab, ac\cdot bd, ef, eg\}$. The last group contains no non-invariant subgroup of order 6, while the first two give a J_1' with two transitive constituents of degree 3 each. Such constituents are incompatible with any of the partitions that remain to be considered.

The only group of order 144 on <8 letters, $\{abc, ad, ef, eg\}$, contains no subgroup of order 18 which includes no non-invariant subgroup.

If $d=2$, $k=4$, and J_1' can be of order 2 only. If $d=4$, $k=2$, and $k'=1$, but the partitions of the degree of F show that $k'\neq 1$.

15. If H_{r+1} is primitive of degree $7p+7$, it may lead to a doubly transitive group of degree $7p+8$. Then J_1 is transitive of degree 7 and J_2 is multiply transitive of degree 8. As we have seen (§3), J_2 must be the simple group of

order 168. Then if J_2 exists, J_1' is intransitive with two cyclic constituents of degree 3 each. Now F is of degree $7p+6$. The only partitions of its degree compatible with J_1' are the following: $4p+3$, $3p+3$; $3p+3$, $3p+3$, p; $p+1$, $p+1$, $p+1$, $p+1$, $p+1$, $p+1$, p. Any partition containing a constituent of degree $2p+3$ or $p+3$ is impossible, for the J groups of such constituents are of order 6.

Now consider the partition $4p+3$, $3p+3$. The constituent of degree $4p+3$ is a simply transitive primitive group (Theorem 1). Its subgroup that fixes one letter has the following partitions of its degree: $3p+2$, p; $3p+1$, $p+1$; $3p$, $p+2$; $2p+2$, $2p$; $2p+1$, $2p+1$; $2p+2$, p, p; $2p+1$, $p+1$, p; $2p$, $p+2$, p; $2p$, $p+1$, $p+1$; $p+2$, p, p, p; $p+1$, $p+1$, p, p. Since the J group of the constituent of degree $4p+3$ must be cyclic, all the partitions which bring a transposition into it are impossible. Theorem 1 excludes the following partitions: $3p+1$, $p+1$; $2p+1$, $p+1$, p; $p+1$, $p+1$, p, p. This leaves the two partitions $2p+1$, $2p+1$, and $2p$, $p+1$, $p+1$. Now apply Theorem 6 to the partition $4p+3$, $3p+3$. Let the constituent of degree $4p+3$ be the constituent of degree m. Then $k_i=2p+1$, $2p$, or $p+1$, and $r=3p+3$. Thus $s=(4p+3)(2p+1)/(3p+3)$, $(4p+3)(2p)/(3p+3)$, or $(4p+3)(p+1)/(3p+3)$. Since s is an integer all of these equations are impossible.

Theorem 6 will also be used to eliminate the partitions $3p+3, 3p+3, p$, and $p+1$, $p+1$, $p+1$, $p+1$, $p+1$, $p+1$, p. The group J_1' demands that for these partitions L have constituents of degrees $3p+3$, $3p+3$, p. Now we know that a single constituent of degree p in L is a subgroup of the metacyclic group (§13, paragraph 7). Let the constituent of degree p be the constituent of degree m of Theorem 6. Then $k_i=b$, say, is a divisor of $p-1$, and $r=3p+3$. Thus $s=pb/(3p+3)$. Again since s is an integer this is an impossible equation.

Thus a primitive H_{r+1} of degree $7p+7$ cannot lead to a doubly transitive group of degree $7p+8$.

16. We shall now take up the case $p=11$. Suppose that F has an alternating constituent. Then the subgroup E_1 (III, §30) exists. Now H_{r+1} is of degree not greater than $7p+7$ except when E_1 has a transitive constituent simply isomorphic to its alternating constituent of degree p (compare footnote of §4). In this case E_1 has exactly three constituents of degrees p, p, $p(p-1)/2$. Then H_{r+1} is of degree not greater than $7p+14$, and F has two transitive constituents. An alternating constituent of F cannot involve letters of more than one cycle of A_1(III, §28). Then F has a transitive constituent of degree $6p+k$, $k=12$ or $\leqq 6$, and one of degree p, $p+1$, or $p+2$. This consideration eliminates F of degree $7p+13$, $7p+12$, $7p+11$, $7p+10$, and $7p+9$. Then if F is of degree $7p+8$, the only possible partition of its

degree is $6p+6$, $p+2$. If F is of degree $7p+7$, the only possible partitions of its degree are $6p+6$, $p+1$, and $6p+5$, $p+2$. These three partitions are immediately impossible by Theorem 1. Thus if F has an alternating constituent, the degree of H_{r+1} does not exceed $7p+7$. Moreover, H_{r+1} of degree $7p+7$ cannot lead to a doubly transitive group of degree $7p+8$, for the reasoning used in §15 depended in no way upon the value of p.

The partitions of the degree of F are then the same as when p was assumed >11. Now Theorem 5 was the only theorem used in eliminating partitions which depended upon the value of p. However, all the partitions eliminated by this theorem contain a constituent of degree $p+2$ and a constituent of degree $mp+n(2\leqq m\leqq 5,\ 0\leqq n<13)$. Since $p+2$ is the prime number 13, the partitions of degree $mp+n$ contain a substitution of order 13 and of degree $13m-13$ at most. Such a substitution does not respect systems of imprimitivity if the constituent is imprimitive. Then the constituents of degree $mp+n$ are primitive. Moreover, they are alternating (see theorems quoted in §1). However, if F has an alternating constituent which involves more than one cycle of A_1, G contains a substitution of order 11 and of degree <77 (III, §28). Hence these partitions are also impossible when $p=11$.

Now H_{r+1} of degree $7p+11$ was shown to be impossible (§10) except when $p=11$. We shall now consider this case. The partitions of the degree of F are the following:

$5p+10$,	$2p$		
$5p+6$,	$2p+4$		
$4p+8$,	$3p+2$		
$4p+4$,	$3p+6$		
$5p+10$,	p,	p	
$5p+6$,	$p+2$,	$p+2$	
$4p+8$,	$2p+2$,	p	
$4p+8$,	$2p+1$,	$p+1$	
$4p+8$,	$2p$,	$p+2$	
$4p+4$,	$2p+4$,	$p+2$	
$3p+6$,	$3p+3$,	$p+1$	
$3p+6$,	$3p+2$,	$p+2$	
$3p+6$,	$2p+4$,	$2p$	
$3p+6$,	$2p+2$,	$2p+2$	
$3p+2$,	$2p+4$,	$2p+4$	
$4p+8$,	$p+2$,	p,	p
$4p+8$,	$p+1$,	$p+1$,	p
$4p+4$,	$p+2$,	$p+2$,	$p+2$
$3p+6$,	$2p+4$,	p,	p

$3p+6$,	$2p+2$,	$p+2$,	p		
$3p+6$,	$2p+2$,	$p+1$,	$p+1$		
$3p+6$,	$2p+1$,	$p+2$,	$p+1$		
$3p+6$,	$2p$,	$p+2$,	$p+2$		
$3p+3$,	$2p+4$,	$p+2$,	$p+1$		
$3p+2$,	$2p+4$,	$p+2$,	$p+2$		
$2p+4$,	$2p+4$,	$2p+2$,	p		
$2p+4$,	$2p+4$,	$2p+1$,	$p+1$		
$2p+4$,	$2p+4$,	$2p$,	$p+2$		
$2p+4$,	$2p+2$,	$2p+2$,	$p+2$		
$3p+6$,	$p+2$,	$p+2$,	p,	p	
$3p+6$,	$p+2$,	$p+1$,	$p+1$,	p	
$3p+6$,	$p+1$,	$p+1$,	$p+1$,	$p+1$	
$3p+3$,	$p+2$,	$p+2$,	$p+2$,	$p+1$	
$3p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+2$	
$2p+4$,	$2p+4$,	$p+2$,	p,	p	
$2p+4$,	$2p+4$,	$p+1$,	$p+1$,	p	
$2p+4$,	$2p+2$,	$p+2$,	$p+2$,	p	
$2p+4$,	$2p+2$,	$p+2$,	$p+1$,	$p+1$	
$2p+4$,	$2p+1$,	$p+2$,	$p+2$,	$p+1$	
$2p+4$,	$2p$,	$p+2$,	$p+2$,	$p+2$	
$2p+2$,	$2p+2$,	$p+2$,	$p+2$,	$p+2$	
$2p+4$,	$p+2$,	$p+2$,	$p+2$,	p,	p
$2p+4$,	$p+2$,	$p+2$,	$p+1$,	$p+1$,	p
$2p+4$,	$p+2$,	$p+1$,	$p+1$,	$p+1$,	$p+1$
$2p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	p
$2p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$,	$p+1$
$2p+1$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+1$
$2p$,	$p+2$,	$p+2$,	$p+2$,	$p+2$,	$p+2$.

Since p^2 does not divide the order of F, J_1 is transitive of degree 11. Now the largest group of order $p^q(p-1)(q!)$ on the same letters in which $\{A_1\}$ is invariant has just one subgroup of order p^q (III, §22). Then J_1 has an invariant subgroup of degree and order p and consequently is of class $p-1=10$. Consider the partitions of the degree of F. The J group of a constituent of degree $5p+10$ is of class 5 at most (IV, §3). Thus the only possible partitions are the following: $3p+2$, $p+2$, $p+2$, $p+2$, $p+2$; $2p+2$, $2p+2$, $p+2$, $p+2$, $p+2$; $2p+2$, $p+2$, $p+2$, $p+2$, $p+2$, p; $2p$, $p+2$, $p+2$, $p+2$, $p+2$, $p+2$. However, since $p+2$ is the prime number 13, all of these partitions bring a substitution of order 11 and of degree < 77 into G.

This completes the proof of the case $q=7$. It has been shown that H_{r+1}

of degree $>7p+7$ $(p>7)$, does not exist. Moreover, H_{r+1} of degree $7p+7$ can lead to a doubly transitive group of degree $7p+8$ only if it is imprimitive. Thus the degree of G cannot exceed $7p+8$.

17. It will now be shown that the degree of a primitive group of class >3, which contains a substitution of prime order $p(p>7)$ and of degree $6p$ cannot exceed $6p+6$. The present limit of $6p+10$ given by Manning depends upon the possibility of the existence of a primitive H_{r+1} of degree $6p+9$ (IV, p. 73). Furthermore, H_{r+1} of this degree can exist only if the partitions of the degree F are $4p+4$, $2p+4$ or $2p+2$, $2p+2$, $p+2$, $p+2$. We find that the real difficulty lies in trying to eliminate the former partition.

Before these partitions are discussed, a correction in the list of the partitions of the degree of F should be made. The omitted partitions are the following:

F of degree $6p+11$:

$$2p+4,\quad p+2,\quad p+2,\quad p+2,\quad p+1;$$

F of degree $6p+9$:

$$\begin{array}{lllll} 2p+4, & p+2, & p+2, & p+1, & p \\ 2p+4, & p+2, & p+1, & p+1, & p+1 \\ 2p+2, & p+2, & p+2, & p+2, & p+1 \\ 2p+1, & p+2, & p+2, & p+2, & p+2; \end{array}$$

F of degree $6p+8$:

$$\begin{array}{lllll} 2p+4, & p+2, & p+2, & p, & p \\ 2p+4, & p+2, & p+1, & p+1, & p \\ 2p+4, & p+1, & p+1, & p+1, & p+1 \\ 2p+2, & p+2, & p+2, & p+2, & p \\ 2p+2, & p+2, & p+2, & p+1, & p+1 \\ 2p, & p+2, & p+2, & p+2, & p+2; \end{array}$$

F of degree $6p+7$:

$$\begin{array}{lllll} 2p+4, & p+2, & p+1, & p, & p \\ 2p+4, & p+1, & p+1, & p+1, & p \\ 2p+2, & p+2, & p+2, & p+1, & p \\ 2p+2, & p+2, & p+1, & p+1, & p+1 \\ 2p, & p+2, & p+2, & p+2, & p+1. \end{array}$$

However, all of these partitions except the two following: $2p+4$, $p+1$, $p+1$, $p+1$, p, and $2p+4$, $p+1$, $p+1$, $p+1$, $p+1$, are immediately impossible, because either F contains a negative substitution (see §7) or Theorem 5 is contradicted. The two partitions which remain are incompatible with any of the J groups for H_{r+1} of these degrees.

We then turn to the consideration of the partitions of the degree of F that cause difficulty when H_{r+1} is of degree $6p+9$. There is an incorrect

statement (IV, p. 72) regarding the partition $3p+2$, $p+2$, $p+2$, $p+2$. However, it may immediately be dismissed from the discussion for it contradicts Theorem 5. Similarly, the partition $2p+2$, $2p+2$, $p+2$, $p+2$ is incompatible with Theorem 5. Then we need to consider only the partition $4p+4$, $2p+4$.

We recall that the only J_1' compatible with this partition is the octic group written out in §13 of this paper. Now apply Theorem 6 to this partition. Let $L(x)$ be the subgroup that fixes the letter x of H_{r+1}. Choose the constituent of degree $2p+4$ as the constituent of degree m on the letters $a_1, a_2, \cdots, a_{2p+4}$. We know (IV, §3) that $k_1=1$, and $k_2=2p+2$. Now $r=4p+4$. Then $s=(2p+4)/(4p+4)$ or $(2p+4)(2p+2)/(4p+4)$. Since s is an integer, the former of these two equations is impossible, and from the latter $s=p+2$. Thus x belongs to a transitive constituent of degree $p+2$ in $L(a_1)(a_2)$. Now $L(a_1)(a_2)$ contains substitutions of order p, for it is the subgroup that fixes one letter of the constituent of degree $4p+4$. Therefore the order of the constituent of degree $p+2$ is divisible by p, for if it were not, $L(a_1)(a_2)$ would contain a substitution of order p and of degree $<6p$. Thus the constituent of degree $p+2$ contributes a transposition to J_1'.

Let us see what J_1' demands of the subgroup $L(a_1)(a_2)$. First note that in this subgroup, the constituent of degree $4p+4$ in $L(a_1)$ can contribute at most a transposition to J_1'. J_1' then demands that the constituent of degree $2p+4$ contribute a substitution of order 2 and of degree 4 to it from this subgroup. Since the order of the constituent of degree $p+2$ in $L(a_1)(a_2)$ is divisible by p, the order of every transitive constituent of $L(a_1)(a_2)$ is divisible by p, for if it were not, the invariant subgroup generated by all the substitutions of order p in $L(a_1)(a_2)$ would fix the letters of the constituents whose order is not divisible by p, and this subgroup would bring a substitution of order 2 and of degree <6 into J_1'. Then the possible partitions of the degree of $L(a_1)(a_2)$ are the following: $p+2$, $p+2$, $3p+2$, $p+1$; $p+2$, $p+2$, $3p+2$, p; $p+2$, $p+2$, $3p+1$, $p+2$; $p+2$, $p+2$, $3p$, $p+2$; $p+2$, $p+2$, $2p+2$, $2p+1$; $p+2$, $p+2$, $2p+2$, $2p$; $p+2$, $p+2$, $2p+2$, $p+1$, p; $p+2$, $p+2$, $2p+2$, p, p; $p+2$, $p+2$, $2p+1$, $p+2$, p; $p+2$, $p+2$, $2p$, $p+2$, $p+1$; $p+2$, $p+2$, $2p$, $p+2$, p; $p+2$, $p+2$, $p+2$, $p+1$, p, p; $p+2$, $p+2$, $p+2$, p, p, p. Now $L(a_1)(a_2)$ has an invariant subgroup of the same degree generated by all of its substitutions of order p. The transitive constituents of this invariant subgroup are positive groups. Consequently, the partitions which contain constituents of degree $p+2$ and p (or $p+1$) at the same time are impossible (see §7). Thus there are only the following partitions: $p+2$, $p+2$, $3p+1$, $p+2$; $p+2$, $p+2$, $3p$, $p+2$; $p+2$, $p+2$, $2p+2$, $2p+1$; $p+2$, $p+2$, $2p+2$, $2p$, to be considered.

Consider the last two partitions first. Now apply Theorem 6 to the group $L(a_1)$, and let the constituent of degree $4p+4$ be the constituent of degree m. Then $k_i=1$, $2p+2$, $2p+1$, or $2p$, and $r=2p+4$. Consequently $s=(4p+4)/(2p+4)$, $(4p+4)(2p+2)/(2p+4)$, $(4p+4)(2p+1)/(2p+4)$, or $(4p+4)\cdot(2p)/(2p+4)$. Since s is an integer all of these equations are impossible.

We also find the first two partitions to be impossible, for the subgroup that fixes one letter of the constituent of degree $4p+4$ cannot have constituents of the degrees given. We shall consider, then, a group of degree $4p+4$ whose subgroup that fixes one letter has constituents of degrees $3p+1$ and $p+2$ or of degrees $3p$ and $p+2$. Let $L(y)$ be the subgroup that fixes the letter y of the group of degree $4p+4$. Let $c_1, c_2, \cdots, c_{p+2}$ be the letters of the constituent of degree $p+2$ in $L(y)$. Then $L(y)(c_1)$ has a transitive constituent of degree $p+1$ on the letters $c_2, c_3, \cdots, c_{p+2}$. If the order of $L(y)$ is t, the order of $L(y)(c_1)$ is $t/(p+2)$ and the order of $L(y)(c_1)(c_2)$ is $t/[(p+2)(p+1)]$. In $L(c_1)$, y belongs to a transitive constituent of degree $p+2$. The $p+1$ letters $c_2, c_3, \cdots, c_{p+2}$ cannot form with y a transitive constituent of degree $p+2$ in $L(c_1)$, for, then, the group $\{L(y), L(c_1)\}$ has a transitive constituent of degree $p+3$, which brings a substitution of degree and order 3 into J_1'. Thus since the $p+1$ letters $c_2, c_3, \cdots, c_{p+2}$ cannot belong to the transitive constituent of degree $p+2$ in $L(c_1)$, they must belong to the transitive constituent of degree $3p+1$ or $3p$. Then the order of $L(c_1)(c_2)$ is $t/(3p+1)$ or $t/(3p)$ according as $L(c_1)$ has a constituent of degree $3p+1$ or $3p$. If y belongs to a transitive constituent of degree s in $L(c_1)(c_2)$, the order of $L(c_1)(c_2)(y)$ is $t/[(3p+1)(s)]$ or $t/(3ps)$. Then $s=(p+2)\cdot(p+1)/(3p+1)$ or $(p+2)(p+1)/(3p)$. However, s is an integer.

Thus it has been shown that a primitive group of class >3 which contains a substitution of prime order p $(p>7)$ and of degree $6p$ does not exist. The case $p=7$ will now be considered. Theorem 5 was the only theorem used in eliminating partitions which depended upon the value of p. The partitions thrown out by means of this theorem were $2p+1$, $p+2$, $p+2$, $p+2$, $p+2$; $3p+2$, $p+2$, $p+2$, $p+2$; $2p+2$, $2p+2$, $p+2$, $p+2$; $2p$, $p+2$, $p+2$, $p+2$, $p+2$. Manning has already shown that the third partition is impossible when $p=7$ (IV, p. 78). In the first partition the constituent of degree $2p+1$ $(=15)$ is primitive. Moreover it is doubly transitive, for a simply transitive primitive group of degree 15 whose subgroup that fixes one letter has two transitive constituents of degree 7 does not exist.*

This partition is then impossible by Theorem 1. In the second partition the constituent of degree $3p+2=23$, a prime number. Then G has a substitu-

* G. A. Miller, Proceedings of the London Mathematical Society, vol. 28 (1897), p. 540.

tion of degree and order 23 and consequently its degree does not exceed 25. In the last partition, the constituent of degree $2p(=14)$ is imprimitive, for a simply transitive group of degree 14 does not exist,* and a multiply transitive group of degree 14 is impossible by Theorem 1. Now the constituent of degree 9 is at least triply transitive and consequently is either the Mathieu group of order 504 or the group of order 1512 and of class 6. The constituent of degree 14 has systems of imprimitivity of two letters only. Its group in the systems is a primitive group of degree 7. Then the group of degree 9 cannot be simply isomorphic to this group in the systems, for these groups of degree 9 occur for the first time on 9 letters. The Mathieu group is then impossible, for it is a simple group. The only invariant subgroup of the group of order 1512 is the Mathieu group of order 504. Then the group in the systems of the constituent of degree 14 must have a quotient group of order 3. However, since the constituent of degree 14 contains more than one subgroup of order p, its group in the systems cannot have such a quotient group.

Thus the theorem for the case $q=6$ now reads

The degree of a primitive group of class >3 *which contains a substitution of prime order* $p(p>7)$ *and of degree* $6p$ *cannot exceed* $6p+6$. *If* $p=7$, *the true limit of the degree of* G *is* $6p+7$.

* G. A. Miller, Quarterly Journal of Mathematics, vol. 29 (1897), p. 242.

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