ON RELATIVE CONTENT AND GREEN’S LEMMA*

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It has been shown† that if the line integral \( \int_C x \, dy \) exists over a simple closed plane curve \( C \), then the content of \( K \), the interior of \( C \), exists equal to that integral. This result may be thought of as the special case of Green’s lemma

\[
\iint_K P_x(xy) \, dx \, dy = \int_C P(xy) \, dy
\]

in which \( P(xy) = x \); and it is to be noted that here \( C \) does not need to be rectifiable.

In the present paper a definition of relative content is given which makes it possible to prove that if \( P \) and \( P_x \) are subject to certain conditions, the content of \( K \), relative to a certain non-additive function of rectangles derived from \( P \), exists equal to the double integral on the left of (1) and also equal to the line integral on the right of (1) whenever that integral exists. This result includes as a special case the form of Green’s lemma for rectifiable \( C \) obtained by Gross,‡ except that in our result \( P_x \) is deliberately restricted to be properly Riemann integrable instead of summable. In the last section sufficient conditions for the existence of the line integral are given which yield Green’s lemma for an important case in which \( C \) does not need to be rectifiable.

1. Definitions and elementary theorems. Let \( \mathfrak{B} \) denote a class of partitions \( \Pi \) of the rectangle \( R_0 : a \leq x \leq b, c \leq y \leq d \), such that (1) each partition \( \Pi \) is formed by dividing \( R_0 \) into vertical and horizontal strips; and (2) the (greatest) lower bound of the norms of the partitions \( \Pi \) of \( \mathfrak{B} \) is zero; here by the norm of a partition \( \Pi \) of \( \mathfrak{B} \) is meant the (least) upper bound of the lengths of the diagonals of the rectangles of which \( \Pi \) consists.

Moreover let \( f(R) \) be a function (not necessarily single-valued) defined for every rectangle \( R : x' \leq x \leq x'', y' \leq y \leq y'' \) lying in \( R_0 \). Also, if \( K_1 \) and \( K_2 \) are any two sets in \( R_0 \), let \( \epsilon(K_1, K_2) = 1 \) if \( K_1 \) and \( K_2 \) have at least one point

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† H. L. Smith, these Transactions, vol. 27, p. 498.

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in common, and let $\epsilon(K_1, K_2) = 0$ if $K_1$ and $K_2$ have no point in common. Finally let $\Delta II$ denote any one of the rectangles of which $II$ consists and let $N II$ denote the norm of $II$.

Then if $K$ is a set in $R_0$ and

$$\lim_{N II \to 0} \sum_{\Delta II} f(\Delta II) \epsilon(K, \Delta II)$$

exists, it is called the outer content of $K$ relative to $f$. In the above the summation is naturally over all $\Delta II$. If

$$\lim_{N II \to 0} \sum_{\Delta II} f(\Delta II) [1 - \epsilon(R_0 - K, \Delta II)]$$

exists, it is called the inner content of $K$ relative to $f$. If both the outer and the inner contents of $K$ relative to $f$ exist and are equal, their common value is called the content of $K$ relative to $f$.

The outer content of $K$ relative to $f$ exists absolutely if it not only exists relative to $f$ but also relative to $|f|$. Similar definitions are given to the absolute existence of the inner content and of the content itself.

A set $K$ is squarable relative to $f$ if its content exists and equals zero; it is absolutely squarable relative to $f$ if its content exists absolutely relative to $f$ and is zero.

We mention the following obvious theorem:

**Theorem 1.** If a set $K$ is absolutely squarable every subset of $K$ is absolutely squarable.

We now prove

**Theorem 2.** If the boundary of a set $K$, entirely interior to $R_0$, is absolutely squarable relative to $f$, then the content of $K$ exists if either the inner content or the outer content exists relative to $f$.

For

$$\sum_{\Delta II} f(\Delta II) \epsilon(K, \Delta II) = \sum_{\Delta II} f(\Delta II) [1 - \epsilon(R_0 - K, \Delta II)] + T(\Pi),$$

where

$$T(\Pi) = \sum_{\Delta II} f(\Delta II) \epsilon(K, \Delta II) \epsilon(R_0 - K, \Delta II).$$

But

$$|T(\Pi)| \leq \sum_{\Delta II} |f(\Delta II)| \epsilon(K_b, \Delta II),$$

where $K_b$ denotes the boundary of $K$. Hence $\lim_{N II \to 0} T(\Pi) = 0$, from which the theorem follows.
Theorem 3. If $K_1$ and $K_2$ are both interior to $R_0$ and have no points in common and if each has inner content $(f)$ and one of their boundaries is squarable absolutely $(f)$, then the inner content $(f)$ of $K_1 + K_2$ exists equal to the sum of the inner contents $(f)$ of $K_1$ and $K_2$.

For suppose $K_{1b}$, the boundary of $K_1$, is squarable absolutely $(f)$. Then, if we set $K = K_1 + K_2$,

$$ \sum_{\Delta \Pi} f(\Delta \Pi) [1 - \epsilon(R_0 - K, \Delta \Pi)] = \sum_{\Delta \Pi} f(\Delta \Pi) [1 - \epsilon(R_0 - K_1, \Delta \Pi)] + \sum_{\Delta \Pi} f(\Delta \Pi) [1 - \epsilon(R_0 - K_2, \Delta \Pi)] + T(\Pi), $$

where

$$ T(\Pi) = \sum_{\Delta \Pi} f(\Delta \Pi) e(K_1, \Delta \Pi) e(K_2, \Delta \Pi) [1 - \epsilon(R_0 - K, \Delta \Pi)]. $$

But

$$ |T(\Pi)| \leq \sum_{\Delta \Pi} |f(\Delta \Pi)| \epsilon(K_{1b}, \Delta \Pi). $$

Hence $\lim_{\Delta \Pi \to 0} T(\Pi) = 0$, from which the theorem follows.

Theorem 4. If $K_1$ and $K_2$ are interior to $R_0$ and have no points in common and if $K_1$ and $K_2$ each have outer content $(f)$ and one of their boundaries is squarable absolutely $(f)$, then the outer content $(f)$ of $K_1 + K_2$ exists equal to the sum of the outer contents $(f)$ of $K_1$ and $K_2$.

For suppose $K_{1b}$, the boundary of $K_1$, is absolutely squarable $(f)$. Then

$$ \sum_{\Delta \Pi} f(\Delta \Pi) e(K_1 + K_2, \Delta \Pi) = \sum_{\Delta \Pi} f(\Delta \Pi) e(K_1, \Delta \Pi) + \sum_{\Delta \Pi} f(\Delta \Pi) e(K_2, \Delta \Pi) - T(\Pi), $$

where

$$ T(\Pi) = \sum_{\Delta \Pi} f(\Delta \Pi) e(K_1, \Delta \Pi) e(K_2, \Delta \Pi). $$

But

$$ |T(\Pi)| \leq \sum_{\Delta \Pi} |f(\Delta \Pi)| \epsilon(K_{1b}, \Delta \Pi). $$

Hence $\lim_{\Delta \Pi \to 0} T(\Pi) = 0$; from which the theorem follows.

Theorems 2, 3 and 4 now give

Theorem 5. If $K_1$ and $K_2$ are interior to $R_0$ and have no points in common and if $K_1$ and $K_2$ each have content $(f)$ and one of their boundaries is absolutely squarable $(f)$, then the content of $K_1 + K_2$ exists $(f)$ and equals the sum of the contents $(f)$ of $K_1$ and $K_2$.

We shall need the following special case of Theorem 5.
**Theorem 6.** If $K$ is the interior of a simple closed curve $C$ which is interior to $R_0$ and $K$ has inner content $(f)$ and $C$ is absolutely squarable $(f)$, then $K$ and $C+K$ each have content $(f)$ and their contents are equal.

2. **On the existence of relative content.** Let $P(xy)$ and $Q(xy)$ be defined on $R_0$. Then let two (multiply-valued) functions $P^{(s)}(R)$ and $Q^{(v)}(R)$ be defined as follows:

$$P^{(s)}(R) = P(x''y) - P(x'y), \quad y' \leq y \leq y'';$$

$$Q^{(v)}(R) = Q(xy'') - Q(xy'), \quad x'' \leq x \leq x';$$

where $R$ is the rectangle $x' \leq x \leq x''$, $y' \leq y \leq y''$, which is assumed to be in $R_0$. In this section we shall be interested in content relative to $P^{(s)}Q^{(v)}$ in the special case where $Q(xy) = y$.

**Theorem 7.** If $P_x$, the first partial derivative of $P$ with respect to $x$, exists on $K$, the interior of a simple closed squarable curve $C$ in $R_0$, and if $P_x$ is bounded and integrable on $K$, then the inner content of $K$ relative to $P^{(s)}y^{(v)}$ exists absolutely and equals $\int_K P_x \, dx \, dy$.

To prove this, note that by the mean value theorem

$$\sum_{\Delta \Pi} P^{(s)}(\cdot \Delta \Pi) y^{(v)}(\cdot \Delta \Pi) [1 - \epsilon(R_0 - K, \Delta \Pi)]$$

$$= \sum_{\Delta \Pi} P_x(\cdot \Delta \Pi) \Delta \Pi [1 - \epsilon(R_0 - K, \Delta \Pi)],$$

where $\cdot \Delta \Pi$ is a point $(xy)$ in $\Delta \Pi$. But

$$\sum_{\Delta \Pi} P_x(\cdot \Delta \Pi) \Delta \Pi [1 - \epsilon(R_0 - K, \Delta \Pi)] = \sum_{\Delta \Pi} P_x(\cdot \Delta \Pi)(K \cdot \Delta \Pi) \epsilon(K, \Delta \Pi) - T(\Pi),$$

where $K \cdot \Delta \Pi$ denotes the set of points common to $K$ and $\Delta \Pi$ and where

$$T(\Pi) = \sum_{\Delta \Pi} P_x(\cdot \Delta \Pi)(K \cdot \Delta \Pi) \epsilon(R_0 - K, \Delta \Pi) \epsilon(K, \Delta \Pi);$$

here $\cdot \Delta \Pi$ has already been defined if $\epsilon(R_0 - K, \Delta \Pi) = 0$ and is defined as any point $(xy)$ in $K \cdot \Delta \Pi$ otherwise. But then

$$\lim_{\Delta \Pi \to 0} \sum_{\Delta \Pi} P_x(\cdot \Delta \Pi)(K \cdot \Delta \Pi) \epsilon(K \cdot \Delta \Pi) = \int_k P_x \, dx \, dy$$

and since

$$|T(\Pi)| \leq N \sum_{\Delta \Pi} (\Delta \Pi) \epsilon(C, \Delta \Pi),$$

where $N$ is the least upper bound of $|P_x|$ on $K$, it follows that

$$\lim_{N \Pi} T(\Pi) = 0.$$

This proves the theorem.
Theorem 8. If $P_x$ exists and is bounded and integrable on $R_0$ and if $C$ is a simple closed squarable curve interior to $R_0$, then $C$ is absolutely squarable $(P^{(x)}_y(y))$.

For

$$\sum_{\Delta \Pi} \left| P^{(x)}(\Delta \Pi) y^{(y)}(\Delta \Pi) \right| \varepsilon(C, \Delta \Pi) = \sum_{\Delta \Pi} \left| P_x^{(x)}(\Delta \Pi) \right| (\Delta \Pi) \varepsilon(C, \Delta \Pi) \leq N \cdot \sum_{\Delta \Pi} (\Delta \Pi) \varepsilon(C, \Delta \Pi).$$

Hence

$$\lim_{N \to \infty} \sum_{\Delta \Pi} \left| P^{(x)}(\Delta \Pi) y^{(y)}(\Delta \Pi) \right| \varepsilon(C, \Delta \Pi) = 0,$$

which is the theorem.

Theorem 9. If $K$ is a region bounded by a simple closed squarable curve $C$ and $P_x$ exists and is bounded and integrable on $R_0$, then $K$ and $K+C$ each has content equal to $\int K P_x dx dy$ relative to $P^{(x)}_y(y)$.

This follows from Theorems 6, 7 and 8.

It has now become necessary to make an additional assumption concerning $\mathfrak{B}$. We say a partition $\Pi$ of $R_0$ is of type (A) if $x^{(z)}(\Delta \Pi)$ is constant for all $\Delta \Pi$ of $\Pi$ and if $y^{(y)}(\Delta \Pi) \leq x^{(z)}(\Delta \Pi)$ for every $\Delta \Pi$ of $\Pi$. A partition $\Pi$ of $R_0$ is of type (B) if it can be obtained from a partition of type (A) by subdividing some or all of the cells of that partition into at most three parts each by means of vertical lines. We assume throughout the remainder of the paper that $\mathfrak{B}$ consists of all partitions of $R_0$ of type (A) or type (B).

We are now in a position to prove

Lemma 1. If $C$ is a simple closed rectifiable curve interior to $R_0$, then

$$\sum_{\Delta \Pi} y^{(y)}(\Delta \Pi) \varepsilon(C, \Delta \Pi) \leq 6 \cdot 2^{1/2} C,$$

for every $\Pi$ with norm sufficiently small, where $C$ denotes the length of $C$.

To prove this we note that if $r$ is less than one-half the diameter of $C$, $C_{(r)} \leq 2rC, \dagger$

where $C_{(r)}$ denotes the outer content of the set of all points distant by not more than $r$ from $C$. But if also $r = 2^{1/2} x^{(z)}(\Delta \Pi)$, and $\Pi$ is of type (A),

* Naturally we are here considering only partitions of $R$ which satisfy condition (1) of §1.

† This follows from a similar inequality for a simple arc stated by Gross, Monatshefte, vol. 29, p. 177. The proof given by Gross is incomplete; a correct proof is to be found in the author's Chicago dissertation.
\[ \sum_{\Delta \Pi} e(C, \Delta \Pi) \leq C_\epsilon \leq 2rC = 2 \cdot 2^{1/2}x^{(x)}(\Delta \Pi)C. \]

Hence since \( \Delta \Pi = x^{(x)}(\Delta \Pi)y^{(y)}(\Delta \Pi) \) and \( x^{(x)}(\Delta \Pi) \) is constant,
\[ \sum_{\Delta \Pi} y^{(y)}(\Delta \Pi)e(C, \Delta \Pi) \leq 2 \cdot 2^{1/2}C, \]
which proves the result for type (A); from this the result easily follows for type (B).

**Theorem 10.** If \( P \) is continuous in \( x \) uniformly as to \((xy)\) on \( R_0 \) and \( C \) is a simple closed rectifiable curve interior to \( R_0 \), then \( C \) is absolutely squarable \((P^{(x)}y^{(y)})\).

For
\[ \sum_{\Delta \Pi} | P^{(x)}(\Delta \Pi)y^{(y)}(\Delta \Pi) | e(C, \Delta \Pi) \leq a(\Pi)\sum_{\Delta \Pi} y^{(y)}(\Delta \Pi)e(C, \Delta \Pi) \leq a(\Pi)6 \cdot 2^{1/2}C, \]
where \( a(\Pi) \) is the largest value of \( |P^{(x)}(\Delta \Pi)| \) for all \( \Delta \Pi \) of \( \Pi \). But on account of the uniform continuity,
\[ \lim_{\Delta \Pi} a(\Pi) = 0, \]
from which the theorem follows.

3. On the \( x \)-linear extension of \( P \). Its uniform continuity. So far we have been considering the function \( P(xy) \) as defined on the entire rectangle \( R_0 \). We now suppose that \( P(xy) \) is defined on a closed set \( S \) interior to \( R_0 \) and show how to extend its definition to the entire plane. To this end let \((x_0y_0)\) be a point not in \( S \). Let \( z_0 \) be the lower bound of all \( x' \) such that for \( x' \leq x \leq x_0 \) the point \((xy_0)\) is not in \( S \). Since \( S \) is closed it is clear that if \( z_0 \) is finite the point \((z_0y_0)\) is in \( S \). Similarly let \( \bar{z}_0 \) be the upper bound of all \( x'' \) such that for \( x_0 \leq x \leq x'' \) the point \((xy_0)\) is not in \( S \); if \( \bar{z}_0 \) is finite, the point \((z_0y_0)\) is in \( S \). We now define \( P(x_0y_0) \) as follows:
\[ P(x_0y_0) = P(z_0y_0) + (x_0 - z_0)P(z_0\bar{x}_0, y_0), \]
if \( z_0, \bar{z}_0 \) are both finite, where
\[ P(z_0\bar{x}_0, y_0) = [P(\bar{x}_0y_0) - P(z_0y_0)]/[\bar{x}_0 - z_0]. \]
We also make the following definitions: \( P(x_0y_0) = P(z_0y_0) \) if \( z_0 \) only is infinite; \( P(x_0y_0) = P(z_0y_0) \) if \( \bar{z}_0 \) only is infinite; \( P(x_0y_0) = 0 \) if both \( z_0 \) and \( \bar{z}_0 \) are infinite. We call the function whose definition has been thus extended the \( x \)-linear extension of \( P \).
Theorem 11. If $P$ is defined on and interior to a simple closed curve $C$, is continuous on $C$ and has a bounded first partial derivative $P_x$ on $K$, the interior of $C$, then the $x$-linear extension of $P$ is continuous as to $x$ uniformly as to $(x, y)$ on $R_0$.

To prove this we note that since $P$ is continuous on $C$ it is uniformly continuous there, that is, there is a system $(d'_i, \epsilon)$ such that

$$|P(x_1, y_1) - P(x_2, y_2)| \leq \epsilon/3$$

for $(x_1, y_1), (x_2, y_2)$ on $C$ and $[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2} \leq d'_i$.

We say two points $(x_0', y_0)$, $(x_0'', y_0)$ are of the first kind if they are both on $C$; of the second kind if $(x_0, y_0)$ is inside $C$ for every $x$ between $x_0'$ and $x_0''$; of the third kind if $(x_0, y_0)$ is outside $C$ for every $x$ between $x_0'$ and $x_0''$.

Consider first a pair $(x_0', y_0), (x_0'', y_0)$ of the first kind. It follows from the above that for such a pair

$$|P(x_0', y_0) - P(x_0'', y_0)| \leq \epsilon/3$$

if $|x_0' - x_0''| \leq d'_i$.

Next consider a pair of the second kind. Here by the mean value theorem

$$|P(x_0', y_0) - P(x_0'', y_0)| \leq |P_x(x', y_0)| \ |x_0' - x_0''| \leq N \ |x_0' - x_0''| \leq \epsilon/3,$$

if $|x_0' - x_0''| \leq d''$, where $d''$ is the smaller of $d'_i$ and $\epsilon/(3N)$, $N$ being the least upper bound on $P_x$ on $K$, and $x'$ is between $x_0'$ and $x_0''$.

Consider next a pair of the third kind. In this case there is a pair of the first kind and also of the third kind* $(x_0', y_0), (x_0''', y_0)$ such that $x_0' \leq x_0', x_0''' \leq x_0'''$. But then by definition of x-linear extension

$$|P(x_0', y_0) - P(x_0'', y_0)| = |P(x_0', y_0) - P(x_0''', y_0)| \ |x_0' - x_0''|/|x_0' - x_0'''| \leq \epsilon/3$$

for $|x_0' - x_0''| \leq d_*$, where $d_*$ is the smaller of $d''_i$ and $ed'_i/(6M)$, $M$ being the least upper bound of $P$ on $C$.

We now consider an arbitrary pair of points $(x_0', y_0), (x_0'', y_0)$ in $R_0$ such that $|x_0' - x_0''| \leq d_*$. The interval $(x_0', x_0'')$ can be broken up into at most three sub-intervals each of which with $y_0$ gives rise to a pair of points either of the first or second or third kind. Hence

$$|P(x_0', y_0) - P(x_0'', y_0)| \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

for $|x_0' - x_0''| \leq d_*$, which proves the desired theorem.

Theorems 6, 7, 10, 11 now give

* This is true unless $P(x_0', y_0) = P(x_0'', y_0)$, in which case the result is obvious.
Theorem 12. If \( P \) is defined on a simple closed rectifiable curve \( C \) and its interior \( K \), is continuous on \( C \) and possesses a bounded integrable first partial derivative \( P_x \) on \( K \), then \( K \) and \( K + C \) each have content equal to \( \iint P_x \, dx \, dy \) relative to \( P^{(x)}y^{(y)} \) and \( C \) is absolutely squarable relative to \( P^{(x)}y^{(y)} \), where \( P_1 \) is the \( x \)-linear extension of \( P \).

4. The generalized Green's lemma. It is the purpose of this section to prove

Theorem 13. If \( P \) is defined on \( R_0 \) and \( C \), a simple closed curve interior to \( R_0 \), is absolutely squarable \( (P^{(x)}y^{(y)}) \), if \( K \), the interior of \( C \), has inner content \( (P^{(x)}y^{(y)}) \), and if, moreover, the integral \( \int_C P \, dy \) exists, then

\[
\int_C P \, dy = \text{cont}_{P^{(x)}y^{(y)}}(K + C) .
\]

Let

\[
C : \quad x = \phi(t), \; y = \psi(t) \quad (0 \leq t \leq 1)
\]

be parametric equations of \( C \) such that as \( t \) varies from 0 to 1, \( C \) is described in the positive sense. For brevity write

\[
P_0(t) = P[\phi(t), \psi(t)].
\]

Now let \( \epsilon \) be a fixed positive number and \( \pi_0 \) a fixed partition of \( (01) \) into intervals \( \Delta \pi_0 \),

\[
\pi_0 : \; \; t_0(=0), \; t_1, \; \cdots, \; t_{n-1}, \; t_n(=1),
\]

and suppose \( \pi_0 \) is such that if \( \pi F \pi_0 \),† that is, if \( \pi \) is a partition obtained by subdividing some or all of the intervals of \( \pi_0 \), then

\[
\left| \int_C P \, dy - S^*_{P_0} P \Delta \psi \right| \leq \frac{\epsilon}{4} ,
\]

where

\[
S^*_{P_0} P \Delta \psi = \sum_{\Delta \pi} \frac{1}{2} \left\{ P_0(\Delta \pi) + P_0(\Delta \pi) \right\} \psi(\Delta \pi) .
\]

Next let us form a partition \( \Pi \) of \( \Psi \) by dividing \( R_0 \) into horizontal strips \( \rho_1, \cdots, \rho_k \) closed and non-overlapping (except for boundary points) and also into equal vertical strips \( \sigma_1, \cdots, \sigma_k \) of the same character in such a way that

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* It is sufficient to assume the integral exists in the weak sense; that is all that is actually used. (Cf. the author's paper cited above.)

† Loc. cit., p. 492.
(1) the width of each horizontal strip is at most equal to the common width of the vertical strips;
(2) each of the points \((\phi(t_i), \psi(t_i))\) which corresponds to a division point \(t_i\) of \(\pi_0\) is on the common boundary of two adjacent horizontal strips;
(3) the inequality

\[
\sum_{\Delta II} |P^{(v)}(\Delta II)| \gamma^{(v)}(\Delta II) \epsilon(C, \Delta II) \leq \frac{\epsilon}{2}
\]

holds for every partition \(\Pi\); e

(4) the inequality

\[
|\text{cont}_{P^{(u), \psi}} K - \sum_{\Delta II} P^{(v)}(\Delta II) \gamma^{(v)}(\Delta II) [1 - \epsilon(R_0 - K, \Delta II)]| \leq \frac{\epsilon}{4}
\]

holds for every partition \(\Pi\).

Let us now consider the intersection \(K_r\) of \(K\) with the interior of \(\rho_r\).

Since it is a region (that is, set of inner points), it can be resolved (uniquely) into a finite or denumerably infinite number of connected regions:

\[K_r = Q_{r1} + Q_{r2} + Q_{r3} + \cdots\]

We say a region \(Q_{ri}\) is of the first kind if its boundary contains an arc of \(C\) which has points of intersection with both the upper and the lower boundaries of \(\rho_r\); otherwise \(Q_{ri}\) is of the second kind. It is easily shown that for given \(r\) there are but a finite number of \(Q_{ri}\) of the first kind; suppose the notation so chosen that they are \(Q_{ni}, \cdots, Q_{ri}\). Let \(Q_r\) be the sum of the \(Q_{ri}\) of the second kind. Then

\[K_r = Q_r + \sum_{i=1}^{ir} Q_{ri},\]

where all the \(Q_{ri}\) are of the first kind.

It is now easily shown that since \(Q_{ri}(i = 1, \cdots, ir)\) is connected, its boundary consists of (1) an arc \(a_{ri}'\) of \(C\) lying entirely within \(\rho_r\) except for its end points, of which the first* lies on the lower boundary of \(\rho_r\) and the second on the upper boundary of \(\rho_r\); (2) an arc \(a_{ri}''\) of the same character except that its first and second end points are respectively on the upper and lower boundaries of \(\rho_r\); (3) a finite or denumerably infinite number of arcs of \(C\) each with its end points on the same boundary line of \(\rho_r\); (4) a finite or denumerably infinite number of segments of the upper and lower boundaries of \(\rho_r\).

* The first end point is the one which corresponds to the smaller value of \(t\).
We next form a certain partition $\pi$ of $(01)$. To this end let $I'_n$, $I''_n$ be the $t$-intervals corresponding to arcs $a'_n$, $a''_n$, respectively. If a division point $t_i$ of $\pi_0$ is not an end point of some $I'_n$ or $I''_n$, it is an end point of some $I_i$ which is the $t$-interval corresponding to an arc of $C$ which lies entirely in some strip $\rho_r$ and has its end points on the same horizontal boundary line of that strip. Now take $\pi_1$ as the partition whose points of division are the end points of the intervals $I'_n$, $I''_n$ and existent intervals $I_r$.

It can now be proved that if $\Delta \pi_1$ is an interval of $\pi_1$, then $\psi(\Delta \pi_1) = \psi(\Delta \pi_1) - \psi(\Delta \pi_1) = 0$ unless $\Delta \pi_1$ is an $I'_n$ or an $I''_n$. For then $\Delta \pi_1$ is either (1) an $I_r$, or (2) between an $I_r$ and an $I'_n$ or $I''_n$, or (3) between two intervals of types $I'_n$, $I''_n$. In case (1), $\psi(\Delta \pi_1) = 0$ obviously. The same also holds in case (2); for otherwise $\Delta \pi_1$ would correspond to an arc of $C$ with end points on different horizontal boundary lines of $\rho_r$. But then this arc would contain an arc lying entirely in some $\rho_r$ and with its end points on different horizontal boundary lines of that $\rho_r$ and would therefore correspond to an $I'_n$ or to an $I''_n$, that is, $\Delta \pi_1$ would contain some $I'_n$ or some $I''_n$, contrary to hypothesis. The case (3) is similar, and the conclusion is established.

From what has just been proved, it follows that

$$S^0_n P \Delta \psi = \sum_\delta \frac{1}{2} \left\{ P_0(I'_\delta) + P_0(I''_\delta) \right\} \psi(I'_\delta)$$

But

$$\psi(I'_\delta) = \psi(I''_\delta) = \gamma_r, \text{ say},$$

$$\psi(I'_\delta) = \psi(I''_\delta) = \gamma_r, \text{ say},$$

$$\psi(I'_\delta) = - \psi(I''_\delta) = \Delta \gamma_r, \text{ say}.$$

Moreover let us write

$$\phi(I'_\delta) = \bar{x}_r, \quad \phi(I''_\delta) = \bar{x}'_r,$$

$$\phi(I''_\delta) = \bar{x}_r', \quad \phi(I''_\delta) = \bar{x}_r''.$$

Then (1) may be written

$$S^0_n P \Delta \psi = \sum_\gamma \Delta \gamma_r \sum_{i=1}^r \frac{1}{2} \left\{ P(\bar{x}'_r \gamma_r) - P(\bar{x}_r' \gamma_r) \right\}$$

$$+ \sum_\gamma \Delta \gamma_r \sum_{i=1}^r \frac{1}{2} \left\{ P(\bar{x}'_r' \gamma_r) - P(\bar{x}_r'' \gamma_r) \right\}.$$
Now let $R'_i$, $R''_i$ be the rectangles

\[ K'_i : \quad \bar{z}_i' \leq x \leq \bar{z}_i'', \quad \bar{y}_i' \leq y \leq \bar{y}_i'' ; \]
\[ K''_i : \quad \bar{z}_i'' \leq x \leq \bar{z}_i', \quad \bar{y}_i'' \leq y \leq \bar{y}_i' . \]

Also let $\xi'_i$, $\xi''_i$ be respectively the upper and lower bounds of values of $x$ for points $(x,y)$ in the (existent) rectangle $\sigma_x R'_i$. Then

\[ (3) \quad [P(\bar{z}_i'\bar{y}_i) - P(\bar{z}_i''\bar{y}_i)] \Delta y_r = \sum_{i} [P(\xi'_i\bar{y}_i) - P(\xi''_i\bar{y}_i)] \Delta y_r. \]

Therefore

\[ \left| \text{cont}_P(\sigma_x) K - \sum_{r} \sum_{i} [P(\bar{z}_i'\bar{y}_i) - P(\bar{z}_i''\bar{y}_i)] \Delta y_r \right| \]

\[ \leq \left| \text{cont}_P(\sigma_x) K - \sum_{r} \sum_{i} [P(\xi''_i\bar{y}_i) - P(\xi'_i\bar{y}_i)] \cdot [1 - \epsilon(R_0 - Q_i, \sigma_x, R'_i) \Delta y_r] + \sum_{r} \sum_{i} \sum_{i} \left| P(\xi''_i\bar{y}_i) \right| \cdot \epsilon(R_0 - Q_i, \sigma_x, R'_i) \cdot \Delta y_r \leq \frac{\epsilon}{4} + \frac{\epsilon}{2} = \frac{3}{4} \epsilon. \]

Similarly

\[ \left| \text{cont}_P(\sigma_x) K - \sum_{r} \sum_{i} [P(\bar{z}_i'\bar{y}_i) - P(\bar{z}_i''\bar{y}_i)] \Delta y_r \right| \leq \frac{3}{4} \epsilon, \]

so that by (4)

\[ (4) \quad | \text{cont}_P(\sigma_x) K - S_{x_i} P_0 \Delta \psi | \leq \frac{3}{4} \epsilon. \]

But

\[ (5) \quad \left| \int_{C} P \cdot dy - S_{x_i} P_0 \Delta \psi \right| \leq \frac{1}{4} \epsilon. \]

From (4) and (5) the theorem follows, since $\epsilon$ is arbitrary.

5. On the existence of the line integral of Green’s lemma. Let the curve $C$ and its parametric representation be as above with the additional requirement that the representation be one-to-one for $0 \leq t < 1$. Let $P(xy)$ be defined on $R_0$. We seek sufficient conditions that $\int_{C} P_0(t) d\psi(t)$ exist, where $P_0$ is as above. The first such condition is given by

**Theorem 14.** If $P$ is continuous on $C$, the integral $\int_{C} P_0(t) d\psi(t)$ exists if $\psi(t)$ is of limited variation, in particular if $C$ is rectifiable.
For then $P_0(t)$ is continuous and the theorem follows from a well-known theorem on the Stieltjes integral.

We next obtain a condition less restrictive on $C$; to this end we first prove two lemmas.

**Lemma 2.** In order that

$$
\sum_{\Delta\tau} (O_{\Delta\tau} P_0) | \psi(\Delta\tau) | = 0
$$

it is sufficient that

(i) $P$ satisfy the Lipschitz condition

$$
| P(x_1y_1) - P(x_2y_2) | \leq A | x_1 - x_2 | + B | y_1 - y_2 |
$$

for every pair of points $(x_1y_1), (x_2y_2)$ on $C$;

(ii) $\sum_{\Delta\tau} (O_{\Delta\tau}\psi) | \psi(\Delta\tau) | = 0$;

and

(iii)* $\sum_{\Delta\tau} (O_{\Delta\tau}\phi) | \psi(\Delta\tau) | = 0$.

To prove this note that for every partition $\pi$ of (01) there is a system $(t^1_\pi, t^2_\pi | \Delta\pi, \pi)$ where $t^1_\pi, t^2_\pi$ are in $\Delta\pi$, such that

$$
0 \leq O_{\Delta\tau} P_0 \leq 2 [ P_0(t^1_\pi) - P_0(t^2_\pi) ]
$$

$$
\leq 2A | \psi(t^1_\pi) - \psi(t^2_\pi) | + 2B | \psi(t^1_\pi) - \psi(t^2_\pi) |
$$

$$
\leq 2 [ A O_{\Delta\tau}\phi + B O_{\Delta\tau}\psi ].
$$

Hence

$$
0 \leq \sum_{\Delta\tau} (O_{\Delta\tau} P_0) | \psi(\Delta\tau) |
$$

$$
\leq \left[ A \sum_{\Delta\tau} (O_{\Delta\tau}\phi) | \psi(\Delta\tau) | + B \sum_{\Delta\tau} (O_{\Delta\tau}\psi) | \psi(\Delta\tau) | \right].
$$

Therefore

$$
\sum_{\Delta\tau} (O_{\Delta\tau} P_0) | \psi(\Delta\tau) | = 0,
$$

as was to be proved.

**Lemma 3.** The condition

$$
\sum_{\Delta\tau} (O_{\Delta\tau} P_0) | \psi(\Delta\tau) | = 0
$$

---

* The condition (iii) is equivalent to the same condition with $A$ omitted if $A \neq 0$; but it is desired to include the case when $P$ is independent of $y$, in which case $A$ may be taken to be 0; the condition is then satisfied for all $\psi$. A similar remark applies to (ii).
is sufficient for the existence of \( \int_0^1 P_0(t)\psi(t) \) provided that \( P \) is continuous on \( C \) and that \( P(x' y) \leq P(x'' y) \) for every pair of points \( (x' y), (x'' y) \) on \( C \) such that \( x' < x'' \).

For then the functions \( P_0(t), \psi(t) \) satisfy one of the sufficient conditions of Corollary 1, p. 505 of the author's paper cited above.

We can now state the desired condition as

**Theorem 15.** The integral \( \int_0^1 P_0(t)\psi(t) \) exists if

(i) \( P_s(xy) \) exists and is less in absolute value than a fixed number \( N \) on \( R_0 \), and for each \( y \), \( P_s(xy) \) is a continuous function of \( x \);

(ii) \( P, P_s \) satisfy the Lipschitz conditions

\[
\left| P(x_1 y) - P(x_2 y) \right| \leq C \left| y_1 - y_2 \right|
\]

\[
\left| P_s(x_1 y) - P_s(x_2 y) \right| \leq D \left| y_1 - y_2 \right|
\]

for every pair of points \( (x_1 y), (x_2 y) \) on \( R_0 \);

(iii) \[
\sum_{\Delta x} \left( O_{\Delta x} \phi \right) \left| \phi(\Delta x) \right| = 0
\]

(iv) \[
\sum_{\Delta x} \left( O_{\Delta x} \phi \right) \left| \phi(\Delta x) \right| = 0.
\]

To prove this let us form the functions

\[
P'(xy) = P(xy) + \frac{1}{2} \int_a^y \left\{ \left| P_s(uy) \right| + P_s(uy) \right\} du,
\]

\[
P''(xy) = \frac{1}{2} \int_a^y \left\{ \left| P_s(uy) \right| - P_s(uy) \right\} du.
\]

Clearly

(6) \[
P(xy) = P'(xy) - P''(xy)
\]

and

(7) \[
P'(x' y) \leq P'(x'' y), \quad P''(x' y) \leq P''(x'' y) \quad (x' < x'').
\]

Now consider the function \( P'(xy) \). We have

(8) \[
\left| P'(x_1 y) - P'(x_2 y) \right| = \frac{1}{2} \left| \int_{x_1}^{x_2} \left\{ \left| P_s(uy) \right| + P_s(uy) \right\} du \right| \leq N \left| x_1 - x_2 \right|.
\]
Also

\[ |P'(xy_1) - P'(xy_2)| \leq |P(ay_1) - P(ay_2)| + \frac{1}{2} \left| \int_a^z |P_s(uy)| \right| 
- |P_s(uy_2)| + P_s(uy_1) - P_s(uy_2)| \, du |\]

(9) \[ \leq |P(ay_1) - P(ay_2)| + \int_a^z |P_s(uy_1) - P_s(uy_2)| \, du \]

\[ \leq C |y_1 - y_2| + \int_a^z D |y_1 - y_2| \, du \]

\[ \leq C |y_1 - y_2| + (x - a)D |y_1 - y_2| \]

\[ \leq E |y_1 - y_2| , \]

where \( E = C + (b - a)D. \)

From (8) and (9) we get

(10) \[ |P'(x_1y_1) - P'(x_2y_2)| \leq |P'(x_1y_1) - P'(x_1y_2)| 
+ |P'(x_1y_2) - P'(x_2y_2)| \leq N |x_1 - x_2| + E |y_1 - y_2| . \]

If we write

\[ P^*_t(t) = P^*_t(\phi(t), \psi(t)), \]

we see, from (10) and the hypothesis, that the conditions of Lemma 2 are satisfied* and hence

(11) \[ \sum_{\Delta x} (O_{\Delta x} P^*_t)\psi(\Delta x) = 0 . \]

But (11) and (7) show (Lemma 3) that \( \int_P P^*_t(t) d\psi(t) \) exists. Similarly if \( P^*_t(t) = P''(\phi(t), \psi(t)) \), it can be shown that \( \int_P P^*_t(t) d\psi(t) \) exists. Hence, by (6), \( \int_P P_0(t) d\psi(t) \) exists and equals \( \int_P P^*_t(t) d\psi(t) - \int_P P''(t) d\psi(t) \) and the theorem is proved.

6. Two special cases. We can now state two important special cases of Green’s lemma. The first is given by

**Theorem 16.** If \( P(xy) \) is defined and continuous on a simple closed rectifiable curve \( C \) and is defined and possesses a bounded integrable partial derivative \( P_x(xy) \) on \( K \), the interior of \( C \), then \( \int_C P(xy) dy \) exists and

\[ \int_C P(xy) dy = \int_K P_s(xy) dx dy. \]

* The continuity of \( P \) in \( x \) and \( y \) together follows from (i) and (ii) which imply respectively that \( P \) is continuous in \( x \) for every \( y \) and in \( y \) uniformly as to \( x \).

† This is the result obtained by Gross, except that he assumes \( P(xy) \) to be summable instead of Riemann integrable.
This theorem follows from Theorems 12, 13, 14.

The second is given by

**Theorem 17.** If $P(xy)$ is defined on $R_0$ and possesses a partial derivative $P_x(xy)$ on the interior of $R_0$ and if $C$ is a simple closed squarable curve interior to $R_0$, and $K$ is its interior, then

$$
\int_C P(xy)dy = \int\int_K P_x(xy)dxdy,
$$

provided $\int_C P(xy)dy$ exists and $P_x(xy)$ is bounded and integrable on $R_0$. In particular, provided

(i) $P_x(xy)$ is, for each $y$, a continuous function of $x$;
(ii) $P$ and $P_x$ satisfy the Lipschitz conditions

$$
| P(xy_1) - P(xy_2) | \leq C | y_1 - y_2 | ,
$$

$$
| P_x(xy_1) - P_x(xy_2) | \leq D | y_1 - y_2 | ,
$$

for every pair of points $(xy_1), (xy_2)$ in $R_0$;

(iii) $\sum_{\Delta x} (O_{\Delta x}\psi) | \psi(\Delta x) | = 0$;

(iv) $\sum_{\Delta x} (O_{\Delta x}\psi) | \psi(\Delta x) | = 0$,

where $N$ is the upper bound of $|P_x|$ on $R_0$.

This theorem follows from Theorems 8, 9, 13, 15, since the hypothesis implies the continuity in $x$ and $y$ together of $P$ and $P_x$ (see first footnote, p. 418).

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