A THEOREM ON ORTHOGONAL SEQUENCES*

BY

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1. **Introduction.** In this paper we shall be dealing with infinite sequences of real numbers, and we shall use the functional form of notation. Thus, such a sequence may be considered as a real-valued function $\sigma(i)$ of a variable i, the range of i being the class of positive integers.

A sequence will be said to be *zero* if all its terms are zero, *positive* if all its terms are positive, *negative*, if all its terms are negative, *M-definite* if it has some terms of one sign and no terms of the opposite sign,† *completely signed* if it contains positive terms and negative terms.

If e is any positive number greater than unity, and $\sigma'(i)$ and $\sigma''(i)$ are two sequences such that the two series

$$(1)\qquad \sum_{i=1}^{\infty} |\sigma'(i)|^{e}, \qquad \sum_{i=1}^{\infty} |\sigma''(i)|^{e/(e-1)}$$

converge, then the series

$$(2)\qquad \sum_{i=1}^{\infty} \sigma'(i)\sigma''(i)$$

converges absolutely.‡ If the sum of the series (2) is zero, the sequences σ' and σ'' will be said to be *orthogonal*.

Assuming e to have any *fixed* value greater than unity, let us denote by $\mathfrak{S}'$ the class of all sequences σ' for which the first series of (1) converges, and by $\mathfrak{S}''$ the class of all sequences σ'' for which the second series of (1) converges.

Each of these classes is closed under the operation of linear combination. That is, if $\sigma_1(i), \sigma_2(i), \cdots, \sigma_m(i)$ are sequences of one of these classes, then the sequence $\sigma(i)$ defined by

$$\sigma(i) \equiv \sum_{j=1}^{m} c_j \sigma_j(i),$$

where the c_j are real constants, is a sequence of the same class.

* Presented to the Society, September 8, 1927; received by the editors August 18, 1927.

† An M-definite sequence is an instance of the more general M-definite *function* defined in an earlier paper, *On sets of functions of a general variable*, these Transactions, vol. 29, p. 463.

‡ Cf. F. Riesz, *Les Systèmes d'Equations Linéaires à une Infinité d'Inconnues*, p. 45.

The purpose of the present paper is to prove the

THEOREM.* *A necessary and sufficient condition that there exist in* $\mathfrak{S}''$ *a positive sequence* $\sigma''(i)$ *orthogonal to each of a given set of sequences* $\sigma_1'(i)$, $\sigma_2'(i), \cdots, \sigma_m'(i)$ *in* $\mathfrak{S}'$ *is that no linear combination of the given set shall be M-definite.*

2. An outline of the proof. The necessity of the condition is almost obvious. For if $\sigma''(i)$ is orthogonal to each of the given sequences, then it is orthogonal to every linear combination of them; that is

$$\sum_{i=1}^{\infty}\sigma'(i)\sigma''(i) = 0, \tag{3}$$

where $\sigma'(i)$ is any such combination. And the equality (3) is manifestly impossible if $\sigma'(i)$ is M-definite and $\sigma''(i)$ is positive.

To prove the sufficiency of the condition we proceed as follows. By a certain well defined process, the given set of m sequences is replaced by another set of m sequences, called a reduced set. The essential features of the reduction are the following:

(a) One sequence of the reduced set is zero.

(b) If the reduced set admits a positive sequence orthogonal to each of its members, then the same is true of the original set.

The reduction process may be repeated, the property (b) persisting, and the property (a) introducing a new zero sequence at each repetition; so that after m reductions the resulting set contains only sequences which are zero. Since any positive sequence is orthogonal to these zero sequences, it follows from (b) that the given set admits a positive orthogonal sequence. A complication presents itself however in the fact that the sequences of the "reduced sets" are *multiple* sequences, of a type which we now proceed to describe.

3. Classes of multiple sequences. A k-tuple sequence may be thought of as a function of k variables, each of which ranges independently over the positive integers, or as a function of a single k-partite variable $q=(i_1, i_2, \cdots, i_k)$ which varies over a composite range $\mathfrak{Q}$ consisting of all k-tuples of positive integers.

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* An analogous theorem relative to continuous functions of a real variable has been proved in an earlier paper, *A theorem on orthogonal functions with an application to integral inequalities*, these Transactions, vol. 30, p. 425. The two theorems justify a certain generalizing postulate which has been used in a theory of linear inequalities in general analysis. Cf. Bulletin of the American Mathematical Society, vol. 33, p. 698.

Any class $\mathfrak{S}$ of simple sequences gives rise to a class of k-tuple sequences which we denote by $\mathfrak{S}_*^k$, and define as follows: The class $\mathfrak{S}_*^k$ consists of all k-tuple sequences $\sigma(i_1, i_2, \cdots, i_k)$ for which there exist sequences $\sigma_1(i_1), \sigma_2(i_2), \cdots, \sigma_k(i_k)$ of $\mathfrak{S}$ such that

$$| \sigma(i_1, i_2, \cdots, i_k) | \leqq \sigma_1(i_1)\sigma_2(i_2) \cdots \sigma_k(i_k).$$

Thus the classes $\mathfrak{S}'$ and $\mathfrak{S}''$ defined in §1 give rise to classes of multiple sequences $\mathfrak{S}_*'^k$ and $\mathfrak{S}_*''^k$. Relative to these two classes we note the following properties:

(i) Each of the classes $\mathfrak{S}_*'^k$ and $\mathfrak{S}_*''^k$ is closed under the process of linear combination.

(ii) If $\sigma'(i_1, i_2, \cdots, i_k)$ and $\sigma''(i_1, i_2, \cdots, i_k)$ are sequences of $\mathfrak{S}_*'^k$ and $\mathfrak{S}_*''^k$ respectively, then the multiple series

$$\sum_{i_1 i_2 \cdots i_k} \sigma'(i_1, i_2, \cdots, i_k)\sigma''(i_1, i_2, \cdots, i_k) \tag{4}$$

converges absolutely.

(iii) If $\sigma'(i_1, i_2, \cdots, i_k)$ belongs to $\mathfrak{S}_*'^k$ and $\sigma''(i_1, i_2, \cdots, i_l)$ belongs to $\mathfrak{S}_*''^l$ where $l = k + h$, then the series

$$\sum_{i_1 i_2 \cdots i_k} \sigma'(i_1, i_2, \cdots, i_k)\sigma''(i_1, i_2, \cdots, i_l)$$

converges absolutely for every $(i_{k+1}, i_{k+2}, \cdots, i_l)$, and the resulting sum is a sequence $\sigma''(i_{k+1}, i_{k+2}, \cdots, i_l)$ of the class $\mathfrak{S}_*''^h$.

Definition. If the sum of the series (4) is zero, then the k-tuple sequences σ' and σ'' will be said to be *orthogonal*.

4. **Reduction.** Consider any k-tuple* sequence $\rho(q) = \rho(i_1, i_2, \cdots, i_k)$ which is completely signed (that is, which contains at least one positive and one negative term), the symbol ρ being suggestive of the special rôle of *reducing sequence.*

Relative to the sequence $\rho(q)$, the range $\mathfrak{Q}$ of the variable q can be divided into three well defined sub-classes:

$$\mathfrak{Q}_P^{(\rho)} \equiv [\text{all } q \text{ such that } \rho(q) > 0],$$

$$\mathfrak{Q}_Z^{(\rho)} \equiv [\text{all } q \text{ such that } \rho(q) = 0],$$

$$\mathfrak{Q}_N^{(\rho)} \equiv [\text{all } q \text{ such that } \rho(q) < 0].$$

* A simple sequence if $k = 1$.

The three classes are mutually exclusive, and are complementary, that is, $\mathfrak{Q} \equiv \mathfrak{Q}_P^{(\rho)} + \mathfrak{Q}_Z^{(\rho)} + \mathfrak{Q}_N^{(\rho)}$. The elements of the respective classes will be denoted by the appropriate small letters:

$$\mathfrak{Q}_P^{(\rho)} \equiv [p], \quad \mathfrak{Q}_Z^{(\rho)} \equiv [z], \quad \mathfrak{Q}_N^{(\rho)} \equiv [n].$$

Corresponding to the division of the range $\mathfrak{Q}$, any k-tuple sequence $\sigma(q)$ can be divided into three well defined *sections*: $\sigma(p)$, $\sigma(z)$, $\sigma(n)$; and it will be convenient to use the notation

$$\sigma(q) \equiv [\sigma(p),\ \sigma(z),\ \sigma(n)]$$

to bring the sections into evidence. The reducing sequence, which for simplicity is omitted from the notation, will always be known from the context or by explicit statement.

In the sequel it will sometimes be desirable to replace one or more of the sections of a sequence by the corresponding sections of the identically zero sequence $\omega(q)$:

$$\omega(q) \equiv [\omega(p),\ \omega(z),\ \omega(n)] \equiv 0.$$

Of particular importance are the *reduced sequences* of the following two special types:

$$\sigma_{PZ}(q) \equiv [\sigma(p),\ \sigma(z),\ \omega(n)], \quad \sigma_N(q) \equiv [\omega(p),\ \omega(z),\ \sigma(n)]. \tag{5}$$

We note the obvious property that if $\sigma(q)$ belongs to the class $\mathfrak{S}_*{}'^k$, then each of the reduced sequences (5) belongs to this class.

5. **The reduced outer product.** The reducing k-tuple sequence $\rho(q)$ determines with any second k-tuple sequence $\sigma(q)$ a certain $2k$-tuple sequence called their *reduced outer product*, denoted by $((\rho\sigma))$, and defined as follows:

$$((\rho\sigma)) \equiv \rho_{PZ}(q_1)\sigma_P(q_2) - \sigma_{PZ}(q_1)\rho_N(q_2), \tag{6}$$

where the variables q_1 and q_2 vary independently over the range $\mathfrak{Q}$ and the reductions are made with respect to the first factor $\rho(q)$ as reducing sequence.

This type of combination of sequences is clearly not in general commutative. It does possess the following noteworthy properties:

(i) If σ is zero, or if $\sigma = \rho$, then $((\rho\sigma))$ is zero.

(ii) If ρ and σ belong to $\mathfrak{S}_*{}'^k$, then $((\rho\sigma))$ belongs to $\mathfrak{S}_*{}'^{2k}$.

6. **Reduction of a set of sequences.** Suppose we have a set of m k-tuple sequences

$$\sigma_1'(q), \quad \sigma_2'(q), \cdots, \sigma_m'(q), \tag{7}$$

belonging to $\mathfrak{S}_*{}'^k$. Then relative to any one of them which is completely signed, say $\sigma_r'(q)$, we may form a *reduced set*, viz. a set of m $2k$-tuple sequences

$$(8)\qquad ((\sigma_r'\sigma_1')),\ ((\sigma_r'\sigma_2')),\ \cdots,\ ((\sigma_r'\sigma_m')),$$

each of which is the reduced outer product of the corresponding sequence of the given set by $\sigma_r'(q)$.

We shall make use of the following three properties of this reduction process:

PROPERTY (a). *The rth sequence of the reduced set is zero.*

PROPERTY (b). *If there is in $\mathfrak{S}_*''^{2k}$ a positive $2k$-tuple sequence which is orthogonal to each sequence of the reduced set* (8), *then there is in $\mathfrak{S}_*''^{k}$ a positive k-tuple sequence which is orthogonal to each sequence of the set* (7).

PROPERTY (c). *If the set* (7) *admits no M-definite linear combination, then the same is true of the reduced set* (8).

Property (a) is an immediate consequence of §5 (i).

To prove property (b), we have by hypothesis a positive $2k$-tuple sequence $\sigma''(q_1, q_2)$ such that for $j=1, 2, \cdots, m$,

$$\sum_{q_1q_2}[\sigma_{rPZ}'(q_1)\sigma_{jN}'(q_2)-\sigma_{jPZ}'(q_1)\sigma_{rN}'(q_2)]\sigma''(q_1,q_2)=0.$$

Since the series on the left converges absolutely, we may write this in the form

$$(9)\quad \sum_{q_2}\left[\sum_{q_1}\sigma_{rPZ}'(q_1)\sigma''(q_1,q_2)\right]\sigma_{jN}'(q_2)-\sum_{q_1}\left[\sum_{q_2}\sigma_{rN}'(q_2)\sigma''(q_1,q_2)\right]\sigma_{jPZ}'(q_1)=0.$$

From §3 (iii), it follows that each of the expressions in square brackets is a sequence of $\mathfrak{S}_*''^{k}$, and it is clear that the first of these sequences is positive and the second negative. If we denote them for the moment by $\pi''(q)$ and $\nu''(q)$ respectively, we may write (9) in the form

$$(10)\qquad \sum_q\sigma_{jN}'(q)\pi''(q)-\sum_q\sigma_{jPZ}'(q)\nu''(q)=0.$$

Now, defining the k-tuple sequence $\sigma''(q)$ by the equality

$$\sigma''(q)=\pi_N''(q)-\nu_{PZ}''(q)$$

the reductions being relative to the reducing sequence $\sigma_r'(q)$, we replace (10) by the equivalent equation

$$\sum_q \sigma_j'(q)\sigma''(q) = 0. \tag{11}$$

The sequence $\sigma''(q)$ is a *positive* sequence of $\mathfrak{S}_*''^k$, and since the relation (11) holds for $j=1, 2, \cdots, m$, we have established property (b).

To prove property (c) indirectly, let us suppose there exist constants $c_1, c_2, \cdots, c_m$, such that*

$$\sum_{j=1}^{m} c_j((\sigma_r' \sigma_j')) \geqq{}' 0. \tag{12}$$

Since $((\sigma_r' \sigma_r'))$ is zero, the coefficient c_r is arbitrary, and the term corresponding to $j=r$ may be omitted from the summation. Denoting this omission by an apostrophe over the summation sign, and substituting their definitional values for $((\sigma_r' \sigma_j'))$, we write (12) in the form

$$\sum_{j=1}^{m}{}' c_j\{\sigma_{rPZ}'(q_1)\sigma_{jN}'(q_2) - \sigma_{jPZ}'(q_1)\sigma_{rN}'(q_2)\} \geqq{}' 0.$$

And to bring the sections of the reduced sequences into evidence we write in the more extended form

$$\begin{aligned}\sum_{j=1}^{m}{}' c_j\{ [\sigma_r'(p_1), \sigma_r'(z_1), \omega(n_1)][\omega(p_2), \omega(z_2), \sigma_j'(n_2)] \\ - [\sigma_j'(p_1), \sigma_j'(z_1), \omega(n_1)][\omega(p_2), \omega(z_2), \sigma_r'(n_2)]\} \geqq{}' 0.\end{aligned} \tag{13}$$

Recalling that the ranges of the variables $p_1, z_1, n_1, p_2, z_2, n_2$ do not overlap, and that the sections $\sigma_r'(z)$, $\omega(p)$, $\omega(z)$, $\omega(n)$ are identically zero, we may replace (13) by two simpler simultaneous inequalities

$$\sum_{j=1}^{m}{}' c_j\{\sigma_r'(p_1)\sigma_j'(n_2) - \sigma_j'(p_1)\sigma_r'(n_2)\} \geqq 0, \tag{14}$$

$$- \sum_{j=1}^{m}{}' c_j \sigma_j'(z_1)\sigma_r'(n_2) \geqq 0, \tag{15}$$

the symbol $\geqq$ having the significance of $\geqq{}'$ in at least one of them.

Now since by definition the product $-\sigma_r'(p_1)\,\sigma_r'(n_2)$ is positive for every (p_1, n_2), we may obtain from (14) an equivalent inequality

$$\sum_{j=1}^{m}{}' c_j \frac{\sigma_j'(p_1)}{\sigma_r'(p_1)} \geqq \sum_{j=1}^{m}{}' c_j \frac{\sigma_j'(n_2)}{\sigma_r'(n_2)}. \tag{16}$$

* The symbol $\geqq{}'$ is to be read "is somewhere greater than and nowhere less than." Thus (12) is equivalent to the statement that the left side is M-definite.

The values on the left side of (16) have (for varying p_1) a greatest lower bound, and those on the right (for varying n_2) a least upper bound. These bounds may or may not coincide, but in any case we may choose the arbitrary c_r so that

$$\sum_{j=1}^{m}{}' c_j \frac{\sigma_j'(p_1)}{\sigma_r'(p_1)} \geqq -c_r \geqq \sum_{j=1}^{m}{}' c_j \frac{\sigma_j'(n_2)}{\sigma_r'(n_2)} \,.$$

From this double relation we obtain, since $\sigma_r'(p)>0$ and $\sigma_r'(n)<0$,

$$\sum_{j=1}^{m} c_j\sigma_j'(p_1) \geqq 0, \qquad \sum_{j=1}^{m} c_j\sigma_j'(n_2) \geqq 0. \tag{17}$$

Furthermore, from (15) we obtain

$$\sum_{j=1}^{m} c_j\sigma_j'(z_1) \geqq 0. \tag{18}$$

Since the ranges of p_1, z_1, n_2, when combined adjunctively, form the range $\mathfrak{Q}$, we may combine (17) and (18) in the single inequality

$$\sum_{j=1}^{m} c_j\sigma_j'(q) \geqq' 0.$$

This result contradicts the hypothesis of property (c), and the contradiction proves the desired conclusion.

7. **Completion of proof of the theorem.** We return now to the principal theorem of §1. By hypothesis we are given a set of sequences

$$\sigma_1'(i),\ \sigma_2'(i),\ \cdots,\ \sigma_m'(i),$$

of $\mathfrak{S}'$, of which no linear combination is M-definite. We are to prove that there is a positive sequence of $\mathfrak{S}''$ which is orthogonal to each sequence of the set.

First of all we may assume that not all the sequences of the set are zero, since in that case the theorem is obviously true. Also, the sequences which are not zero are completely signed, since they cannot be M-definite.

Suppose the first sequence σ_1' is not zero. We form the reduced set with respect to σ_1' as reducing function, and denote it by

$$0,\ \sigma_2'^{(1)},\ \sigma_3'^{(1)},\ \cdots,\ \sigma_m'^{(1)}.$$

It consists of m double sequences of $\mathfrak{S}_*'^2$, of which the first is zero. Other sequences of the set may be zero. If they are all zero, they all admit a positive orthogonal sequence, from which follows our theorem, by property (b). In any case, the non-zero sequences are completely signed, by property (c). We choose the first such sequence—suppose for definiteness it is $\sigma_2'^{(1)}$—as a reducing sequence, and form a second reduced set

$$0,\ 0,\ \sigma_3'^{(2)},\ \cdots,\ \sigma_m'^{(2)}.$$

The process is now obvious. We repeat the reduction process, until after $l(\leqq m)$ reductions we obtain a set of h-tuple sequences $(h=2^l)$, all of which are zero.

Any positive sequence of $\mathfrak{S}_*''^h$ is orthogonal to all sequences of this last reduced set, and the existence of such a positive sequence implies, by repeated application of property (b), the existence of a positive sequence in $\mathfrak{S}''$ orthogonal to each sequence of the given set. This completes the proof of the theorem.

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