TRANSVERSALITY IN SPACE OF THREE DIMENSIONS*

BY

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So-called transversality relations arose first in the calculus of variations and later in the theory of infinitesimal contact transformations. For the plane such relations are of arbitrary character, but in all spaces of more than two dimensions only correspondences of certain specific types can be identified with transversality relations. Similarly, systems of extremals, which are of arbitrary character for the simplest problem of the calculus of variations, are of peculiar geometric character for all higher problems.†

In this paper we find a simple geometric criterion for testing when a given correspondence between surface elements and line elements in three-space is of the transversality type. ***Briefly stated, a certain induced homography must be involutorial. This is both necessary and sufficient.***

The result applies to simple integrals

$$\int G(x,y,z,y',z')dx = \text{minimum}, \tag{1}$$

to double integrals

$$\int\int F(x,y,z,p,q)dxdy = \text{minimum}, \tag{2}$$

and to infinitesimal contact transformations, defined by a characteristic function

$$W(x,y,z,p,q). \tag{3}$$

Incidentally we obtain a *principle of transference connecting simple and double integrals in the calculus of variations*: we may associate problems (1) and (2) when they produce identical transversalities. Given the integrand function G, the other integrand function F is determined up to a factor of

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† The easiest of the higher problems, in this connection, is $\int F(x, y, y', y'')dx$ for which I have given a necessary geometric criterion in the Bulletin of the American Mathematical Society, vol. 13 (1907), pp. 289–292.

form $\mu(x, y, z)$, and vice versa. As is well known, transversality is not affected by such a factor in any of the three problems (1), (2), (3).

Contact transformations. The infinitesimal contact transformation defined by (3) carries any surface element (x, y, z, p, q) into a neighboring surface element $(x+\delta x, \cdots, q+\delta q)$, where

$$(3') \qquad \begin{aligned} \delta x &= W_p \delta t, \\ \delta y &= W_q \delta t, \\ \delta z &= (pW_p + qW_q - W)\delta t. \end{aligned}$$

If we connect the point of the old element with the point of the new element, we obtain a definite direction element (or lineal element), namely (x, y, z, y', z'), where

$$(4) \qquad y' = \frac{W_q}{W_p}, \quad z' = \frac{pW_p + qW_q - W}{W_p}.$$

This lineal element is said to be transversal* to the surface element. Thus at any given point (x, y, z) we have a definite relationship or correspondence between the ∞^2 lines or directions (y', z') and the ∞^2 planes (p, q).

It is obvious that a correspondence of the peculiar analytic form (4) must have *some* geometric peculiarity, for (4) involves only one arbitrary function W of five arguments while a random correspondence would be of the general form

$$5) \qquad y' = \alpha(x, y, z, p, q), \quad z' = \beta(x, y, z, p, q),$$

involving two arbitrary functions. Before studying the appropriate geometric property, we make sure that essentially the same form (4) arises in problems (1) and (2).

Double integrals. If in the calculus of variations the boundary curve of the surface minimizing the double integral (2) is required to lie on a given surface $\phi(x, y, z) = 0$, then in the standard notation ϕ must fulfill the Kneser condition of being "transversal" to the extremal surface; and this relation

* The term transversal in this connection was apparently proposed first by the writer in his paper *The infinitesimal contact transformations of mechanics*, Bulletin of the American Mathematical Society, vol. 16 (1910), pp. 408–412. Vessiot, in his fundamental study of the connection between problems (1) and (3), used the term *conjugué* (see Bulletin de la Société Mathématique de France, vol. 34 (1906), pp. 230–269). Douglas presents this connection in a novel and interesting form, for both transversals and extremals, in these Transactions, vol. 29 (1927), pp. 401–420; he employs the term transversality for *any* correspondence between elements at a common point; those arising from contact transformations are described as "of special character." The relationship to double integrals was first observed by the writer.

is expressed by the fact that in every point of the boundary curve, ϕ goes through a certain direction (y', z') determined by (and duly termed transversal to) the element (p, q) of the extremal surface at that point. The formulas expressing y' and z' in terms of p and q are of exactly the same form as (1) with F in place of W.

Simple integrals. The third kind of transversality is a correspondence from line elements to surface elements. It arises from the condition which has to be fulfilled at the end point of the curve minimizing the integral (1) if this end point is required to lie on a given surface $\psi(x, y, z)=0$. The transversality of the element (p, q) of ψ to the direction (y', z') of the curve at that point is expressed by the relations

$$p = \frac{y'G_{y'} + z'G_{z'} - G}{G_{z'}}, \quad q = -\frac{G_{y'}}{G_{z'}}. \tag{6}$$

By means of the simple transformation

$$\begin{aligned} P &= -z', \quad Q = y', \\ Y' &= q, \qquad Z' = -p, \\ \overline{G}(P,Q) &= G(Q, -P), \end{aligned} \tag{7}$$

we see that (6) is reduced to a correspondence of the form (4) from the surface elements (P, Q) to the line elements (Y', Z'), with the new function $\overline{G}$ taking the place of W.

Thus the three kinds of transversality essentially reduce to one, expressed by the form (4) . *To every function $W(x, y, z, p, q)$ corresponds a definite transversality, but conversely a given transversality determines W only up to an arbitrary function of x, y, z as a factor.*

The geometric criterion. The peculiar character of (4) admits a simple geometric interpretation. Let us take a general analytic correspondence (5) associating surface and line elements. Denote it by T.

Consider an arbitrary point and through it a plane determined by (p, q) as fixed. By means of T a certain *induced* correspondence H arises in the pencil of directions determined by the point and the plane. It associates with the intersection of (p, q) and the neighboring element $(p+dp, q+dq)$ the intersection of (p, q) with the plane determined by the two directions (y', z') and $(y'+dy', z'+dz')$ corresponding to (p, q) and $(p+dp, q+dq)$ by T. This correspondence is easily seen to be a homography for every transformation T. When (5) is of the special type (4) we shall show that *this homography is involutorial*, and that the converse is also valid.

Transversality is therefore distinguished geometrically among all transformations (5) *by the property that the induced homography H arising in the above described manner in every pencil of directions is an involution.*

The calculations leading to the characteristic involution are as follows. We first discuss the equations

$$\alpha = \frac{W_q}{W_p}, \quad \beta = \frac{pW_{p+q}W_q - W}{W_p},$$

obtained by comparing (4) and (5). The easiest way to eliminate W is to form

$$\frac{W_p}{W} = \frac{1}{p + q\alpha - \beta}, \quad \frac{W_q}{W} = \frac{\alpha}{p + q\alpha - \beta},$$

and then write the condition for integrability

$$\frac{\partial}{\partial q}\left(\frac{1}{p + q\alpha - \beta}\right) - \frac{\partial}{\partial p}\left(\frac{\alpha}{p + q\alpha - \beta}\right) = 0.$$

We obtain

$$\alpha + q\alpha_q - \beta_q + \alpha_p(p + q\alpha - \beta) - \alpha(1 + q\alpha_p - \beta_p) = 0,$$

or, finally,

(8) $$(p - \beta)\alpha_p + q\alpha_q + \alpha\beta_p - \beta_q = 0,$$

which is therefore the criterion for transversality in analytic form.

The induced homography H. Consider an arbitrary correspondence

(9) $$y' = \alpha(p,q), \quad z' = \beta(p,q)$$

between the surface elements and direction elements at a given point. (Since the point is fixed it will not be necessary to write the x, y, z arguments in α and β.) To the element (p, q) whose plane is

$$p\delta x + q\delta y - \delta z = 0$$

corresponds the direction

$$1 : \alpha : \beta ;$$

to a neighboring element $(p+dp, q+dq)$, corresponds the direction

$$1 : \alpha + d\alpha : \beta + d\beta,$$

where $d\alpha = \alpha_p dp + \alpha_q dq$, $d\beta = \beta_p dp + \beta_q dq$. The plane of the two direction elements is

$$(\alpha d\beta - \beta d\alpha)\delta x - d\beta \delta y + d\alpha \delta z = 0.$$

It cuts the plane $p\delta x+q\delta y-\delta z=0$ in the direction

$$(m) \qquad qd\alpha - d\beta : (\beta - p)d\alpha - \alpha d\beta : q\beta d\alpha - (p + q\alpha)d\beta.$$

On the other hand $p\delta x+q\delta y-\delta z=0$ cuts the neighboring plane $(p+dp)\delta x+(q+dq)\delta y-\delta z=0$ in

$$(M) \qquad dq : - dp : pdq - qdp.$$

Thus we have in the plane (p, q) supposed fixed a definite induced correspondence H from the direction (m) to the direction (M). This H is obviously a homography since the differentials dp, dq enter linearly and p and q are fixed. The result holds for all analytic transformations (5) or (9).

To prove that for the special correspondence (4) this homography H is involutorial, it is sufficient to express one of the ratios of m in terms of the corresponding ratio of M. Putting

$$m_1 = \frac{(p-\beta)d\alpha + \alpha d\beta}{qd\alpha - d\beta} = \frac{\{(p-\beta)\alpha_p + \alpha\beta_p\}dp + \{(p-\beta)\alpha_q + \alpha\beta_q\}dq}{(q\alpha_p - \beta_p)dp + (q\alpha_q - \beta_q)dq}$$

and $M_1=dp/dq$ we have

$$m_1 = \frac{\{(p-\beta)\alpha_p + \alpha\beta_p\}M_1 + \{(p-\beta)\alpha_q + \alpha\beta_q\}}{(q\alpha_p - \beta_p)M_1 + (q\alpha_q - \beta_q)}.$$

The condition that this shall be an involution is

$$(p-\beta)\alpha_p+\alpha\beta_p+q\alpha_q-\beta_q=0.$$

This is identical with the condition for transversality obtained in (8). Therefore our theorem is proved.

For the general analytic transformation

$$p = \alpha(x,y,z,y',z'), \qquad q = \beta(x,y,z,y',z')$$

from a line element to a surface element, we find of course the corresponding dual result: it determines a homography in the pencil of planes through the direction (y', z') supposed fixed, and this homography is an involution when and only when the correspondence is a transversality.

Linear transversalities and geodesic systems. If a correspondence from plane elements to line elements is assumed to be projective, when will it satisfy the condition for transversality? Using the analytic test (8) we readily find that the correspondence is duality (with respect to any quadric cone at the given point). Of course this is not true for two dimensions, but it is true for all higher spaces.

The only case in which space transversality is a projective relation is where it reduces to duality.

The integrand functions G and F in (1) and (2), and the characteristic function (W) in (3) then reduce to the special form of the square root of any quadratic polynomial (so that the extremals are the geodesics of a Riemann manifold). This includes the more special case of natural families of trajectories,

$$\int \nu(x,y,z)(1+y'^2+z'^2)^{1/2}dx = \text{minimum},$$

where transversality is ordinary orthogonality.

I showed in 1910 that the ∞^4 extremals are then distinguished by the fact that the ∞^2 which are orthogonal to any base surface always admit ∞^1 orthogonal surfaces.* This has been proved more briefly by Schouten,† who uses the term conform-geodetic instead of my term natural. Douglas (analytically) and Blaschke (synthetically)‡ have recently shown that such a Kneser integrability test is valid in the characterization of any extremal family of curves (problem (1) above), orthogonality being of course replaced by transversality.

It follows from this theorem and the theorem above on polarities, that *any geodesic system* (*that is, any family of curves in flat space obtainable by arbitrary point representation of the geodesics of a Riemann space*) *may be characterized by the existence at each point of a projective* (*necessarily polar*) *transversality with respect to which the system has the integrability property.*

Synthetic treatment. I observe in conclusion that the geometric criterion of general transversality (involutorial character of H) may be deduced synthetically from another interpretation of the analytic test (8). If a line element at a given point corresponds to a surface element, and if we construct parallel surface elements at all the ∞^1 points of the straight line determined by the line element, we obtain in all ∞^3 surface elements, that is, a Pfaffian equation; the integrability of this equation (for every given point) is the necessary and sufficient criterion of transversality. (See also the equivalent statement quoted by Douglas, page 403.) The ∞^1 integral surfaces are of course parallel, and we may use the theory of conjugate directions to obtain the property above, which has the advantage of dealing only with the elements at a given point.

* This is the converse of the Thomson-Tait theorem, discussed in *the Theorem of Thomson and Tait and natural families of trajectories*, these Transactions, vol. 11 (1910), pp.121–140.

† Nieuw Archief voor Wiskunde, 1927.

‡ Douglas in the paper cited above, p. 404; Blaschke, *Ueber die Umkehrung der Kneser'schen Satz*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 1927.

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