TRANSFORMATIONS OF NETS*

BY

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INTRODUCTION

It is the purpose of this paper to extend some of the ideas relating to transformations of surfaces and conjugate nets to transformations of an arbitrary net. When two nets and the congruence of lines joining corresponding points are so related that the developables of the congruence cut the sustaining surfaces of the nets in the curves of the net, the nets and the congruences will be said to be in relation C. Either net will be said to be a C transform of the other. If the given nets are conjugate nets, then the transformation C becomes a transformation F.† If corresponding points of two nets in relation C separate harmonically the focal points on the line joining corresponding points, we shall call the given nets \( K_{-1} \) transforms. If the given nets are conjugate nets, then the transformation \( K_{-1} \) becomes the transformation \( K \) of Koenig.‡ We shall consider these and related transformations in some detail.

1. THE DIFFERENTIAL EQUATIONS

Let

\[
y^{(k)} = y^{(k)}(u, v), \quad z^{(k)} = z^{(k)}(u, v) \quad (k = 1, 2, 3, 4)
\]

be the parametric equations of surfaces \( S_y \) and \( S_z \) but assume that neither surface is a focal surface of the congruence \( G \) of lines \( g \) joining the points \( y \) and \( z \). Suppose that the parametric nets \( N_y \) and \( N_z \) on \( S_y \) and \( S_z \) and the congruence \( G \) are in relation \( C \). Evidently the tangents to the curves \( u = \text{const.} \) on \( S_y \) and \( S_z \) are coplanar and similarly the tangents to \( v = \text{const.} \) are coplanar. The functions \( y^{(k)} \) and \( z^{(k)} \) will satisfy a system of differential equations of the form§

* Presented to the Society, December 31, 1926; received by the editors in October, 1927.
† L. P. Eisenhart, Transformations of Surfaces, Princeton University Press, 1923, p. 34. Hereafter referred to as Eisenhart, Surfaces.
‡ Eisenhart, Surfaces, p. 57.
§ G. M. Green, Memoir on the general theory of surfaces and rectilinear congruences, these Transactions, vol. 20 (1919), p. 149. Hereafter referred to as Green, Surfaces.
\[ y_{uu} = a^{(11)}y_u + b^{(11)}y_v + c^{(11)}y + d^{(11)}z, \]
\[ y_{uv} = a^{(12)}y_u + b^{(12)}y_v + c^{(12)}y + d^{(12)}z, \]
\[ y_{vv} = a^{(22)}y_u + b^{(22)}y_v + c^{(22)}y + d^{(22)}z, \]
\[ z_u = m'y_u + p'y + qz, \]
\[ z_v = n'y_v + p'y + q'z. \]

The coefficients of system (1) satisfy the following integrability conditions*:

\[ a^{(11)} - a^{(11)} + a^{(12)}b^{(12)} - a^{(22)}b^{(11)} + c^{(12)} = -md^{(12)}, \]
\[ b^{(11)} - b^{(11)} + a^{(12)}b^{(11)} + (b^{(12)} - a^{(11)})b^{(12)} - b^{(11)}b^{(22)} - c^{(11)} = n'd^{(11)}, \]
\[ c^{(12)} - c^{(11)} + a^{(12)}c^{(11)} + (b^{(12)} - a^{(11)})c^{(12)} - b^{(11)}c^{(22)} = p'd^{(11)} - pd^{(12)}, \]
\[ d^{(12)} - d^{(11)} + a^{(12)}d^{(11)} + (b^{(12)} - a^{(11)})d^{(12)} - b^{(11)}d^{(22)} = q'd^{(11)} - qd^{(12)}; \]
\[ a^{(22)} - a^{(12)} + a^{(22)}a^{(11)} + b^{(22)} - a^{(12)}b^{(12)} - b^{(12)}a^{(22)} + c^{(22)} = -md^{(22)}, \]
\[ b^{(22)} - b^{(12)} + a^{(22)}b^{(11)} - a^{(12)}b^{(12)} - c^{(12)} = n'd^{(12)}, \]
\[ c^{(22)} - c^{(12)} + a^{(22)}c^{(11)} + b^{(22)} - a^{(12)}c^{(12)} - b^{(11)}c^{(22)} = p'd^{(12)} - pd^{(22)}, \]
\[ d^{(22)} - d^{(12)} + a^{(22)}d^{(11)} + b^{(22)} - a^{(12)}d^{(12)} - b^{(11)}d^{(22)} = q'd^{(12)} - qd^{(22)}; \]

\[ -m + (n' - m)a^{(12)} + p' = -q'm, \]
\[ n' + (n' - m)b^{(12)} - p = qn', \]
\[ p' - p + (n' - m)c^{(12)} = qp' - pq', \]
\[ q' - q + (n' - m)d^{(12)} = 0. \]

If the fourth and fifth equation of system (1) are solved for \( y_u \) and \( y_v \), and if the resulting two equations are differentiated with respect to \( u \) and \( v \), we obtain the following equivalent system:

* Green, *Surfaces*, p. 150.
$z_{uu} = \alpha^{(11)} z_u + \beta^{(11)} z_v + \gamma^{(11)} z + \delta^{(11)} y,$
$z_{uv} = \alpha^{(12)} z_u + \beta^{(12)} z_v + \gamma^{(12)} z + \delta^{(12)} y,$
$z_{vv} = \alpha^{(22)} z_u + \beta^{(22)} z_v + \gamma^{(22)} z + \delta^{(22)} y,$
$y_u = \mu z_u + \pi z + \kappa y,$
$y_v = \nu z_v + \pi' z + \kappa' y,$

wherein

$$\alpha^{(11)} = a^{(11)} + \frac{\rho}{m} + \frac{m_u}{m} + q,$$
$$\beta^{(11)} = \frac{m}{n'} b^{(11)},$$

(3a)

$$\delta^{(11)} = -m \left[ a^{(11)} \frac{\rho}{m} + b^{(11)} \frac{\rho'}{n'} + \left( \frac{\rho}{m} \right)^2 - c^{(11)} - \frac{\partial}{\partial u} \left( \frac{\rho}{m} \right) \right],$$

$$\alpha^{(12)} = a^{(12)} + \frac{m_v}{m}, \quad \beta^{(12)} = b^{(12)} + \frac{n_u}{n'},$$

(3b)

$$\delta^{(12)} = -m \left[ a^{(12)} \frac{\rho}{m} + b^{(12)} \frac{\rho'}{n'} - \frac{\rho \rho'}{mn'} - c^{(12)} - \frac{\partial}{\partial u} \left( \frac{\rho}{m} \right) \right]$$
$$= -n' \left[ b^{(12)} \frac{\rho'}{n'} + a^{(12)} \frac{\rho}{m} - \frac{\rho \rho'}{mn'} - c^{(12)} - \frac{\partial}{\partial u} \left( \frac{\rho}{n'} \right) \right],$$

$$\alpha^{(22)} = \frac{n'}{m} a^{(22)}, \quad \beta^{(22)} = b^{(22)} + \frac{\rho'}{n'} + \frac{n'_u}{n'} + q',$$

(3c)

$$\delta^{(22)} = -n' \left[ a^{(22)} \frac{\rho}{m} + b^{(22)} \frac{\rho'}{n'} + \left( \frac{\rho'}{n'} \right)^2 - c^{(22)} - \frac{\partial}{\partial u} \left( \frac{\rho'}{n'} \right) \right],$$

(3d)

$$\mu = \frac{1}{m}, \quad \nu' = \frac{1}{n'}, \quad \kappa = -\frac{\rho}{m}, \quad \kappa' = -\frac{\rho'}{n'}, \quad \pi = -\frac{q}{m}, \quad \pi' = -\frac{q'}{n'}.$$

The differential equation of the asymptotic curves* on $S_y$ is

$$d^{(11)} du^2 + 2d^{(12)} du dv + d^{(22)} dv^2 = 0,$$

and the asymptotic curves on $S_x$ are given by the equation

$$d^{(11)} du^2 + 2d^{(12)} du dv + d^{(22)} dv^2 = 0.$$

The surface $S_y$ is therefore a developable if $d^{(11)} d^{(22)} - (d^{(12)})^2 = 0$. A similar statement holds for $S_x$. We shall assume hereafter that neither $S_y$ nor $S_x$ is developable. If $d^{(12)} = 0$ the parametric net on $S_y$ is conjugate; if $d^{(11)} = d^{(22)} = 0$ the parametric net is asymptotic. Similar statements hold for the parametric net on $S_x$.

* Green, Surfaces, p. 151.
2. The focal nets of $G$

Any point $P$ on $g$ is defined by an expression of the form

$$P = y + \theta z.$$  

As $y(z)$ describes a curve $v=v(u)$ on $S_y(S_z)$, $g$ generates a ruled surface. If this surface is a developable, the point $dP/du$ lies on $g$. Imposing this condition we find that the developables are given by the differential equation

$$\frac{(n' - m)}{m}dudv = 0$$  

and the focal points by

$$mn'^2 + (m + n')\theta + 1 = 0.$$  

The focal points of $g$ are therefore

$$\tau = my - z,$$

$$\tau' = n'y - z.$$  

We shall call the nets $N_r$ and $N_r'$ the focal nets of $G$. If the focal surfaces $S_r$ and $S_r'$ are not developable, the nets $N_r$ and $N_r'$ are conjugate. Equation (6) shows that the developables are indeterminate if $m - n' = 0$. In this case we shall call $N_r$ and $N_r'$ radial transforms. Unless explicitly stated to the contrary, we shall hereafter assume that $N_r$ and $N_r'$ are not radial transforms.

3. The transformations $L$ and $L'$

From the last two equations of system (1) we find that the points of intersection of corresponding tangents to the curves of nets in relation $C$ are defined by the expressions

$$r = y_u + \frac{p}{m}y = \mu(z_u - qz),$$

$$s = y_v + \frac{p'}{n'}y = v'(z_v - q'z).$$  

The line $h$ of intersection of the tangent planes to $S_y$ and $S_z$ at corresponding points $y$ and $z$ generates a congruence $H$. We may in the usual way show that the differential equation of the net corresponding to the developables of the congruence $H$ is

$$\left( m d^{(11)} \delta^{(12)} - n' d^{(12)} \delta^{(11)} \right) du^2 + \left[ m (d^{(11)} \delta^{(22)} + d^{(12)} \delta^{(12)}) - n' (d^{(22)} \delta^{(11)} + d^{(12)} \delta^{(12)}) \right] dudv - (n' d^{(22)} \delta^{(12)} - md^{(12)} \delta^{(22)}) dv^2 = 0.$$  

The focal points of $h$ are defined by the expressions
\[ P_{1,2} = x + \theta_{1,2} s, \]
where $\theta_{1,2}$ are the roots of
\[
(11) \quad m(d^{(23)}[22] - d^{(13)}[12])\varphi^2 + [n'(d^{(23)}[11] - d^{(12)}[12])
- m(d^{(11)}[22] - d^{(12)}[12])\varphi - n'(d^{(11)}[12] - d^{(12)}[11]) = 0.
\]

The harmonic invariant of (4) and (10) is
\[ (m - n')(d^{(11)}[22] - d^{(12)}[12]) = 0. \]

Hence the congruence $H$ will be harmonic to $S_\nu$ if and only if $G$ is conjugate to $S_\nu$. Similarly $H$ will be harmonic to $S_\nu$ if and only if $G$ is conjugate to $S_\nu$.

We shall call nets $N_\nu$ and $N_\nu$ L transforms or in relation L if they are in relation $C$ and if they correspond to the developables of $H$. Nets $N_\nu$ and $N_\nu$ in relation $C$ will be L transforms if
\[
\]
Hence if $N_\nu$ and $N_\nu$ are in relation $F$ they are L transforms. If $N_\nu$ and $N_\nu$ are asymptotic nets, they are L transforms. Suppose $N_\nu$ and $N_\nu$ are neither conjugate nor asymptotic. Then from (12) we obtain
\[ \frac{d^{(11)}d^{(22)}}{d^{(12)}^2} = \frac{d^{(11)}[22]}{d^{(12)}[12]} = \frac{d^{(11)}[22]}{d^{(12)}[12]} = 0. \]

Hence the tangents to the curves of non-asymptotic nets in relation L form the same cross ratios with the asymptotic tangents associated with the surfaces on which the nets lie. In particular if the tangents to the curves of a net on a surface form a constant cross ratio with the asymptotic tangents, any L transform of the net has the same property, the constant ratio being the same for all nets in relation $L_*$.\(^*\)

From (1) we see that the focal points of $h$ lie on the tangents to the curves of nets in relation $C$ if and only if
\[
\]
We shall call such nets $N_\nu$ and $N_\nu$ L transforms or nets in relation $L_*$. Hence if $N_\nu$ and $N_\nu$ are F transforms they are L transforms; if they are asymptotic nets in relation $C$ they are L transforms; if they are in relation $C$ and the asymptotic curves on the surfaces on which the nets lie correspond then the nets are L transforms.

\(^*\) This statement of the theorem includes the case in which the nets are conjugate.
From equations (13) we find that
\[ \frac{d^{(11)}d^{(22)}}{d^{(12)^2}} = \frac{\delta^{(11)}\delta^{(22)}}{\delta^{(12)^2}} . \]

Hence the tangents to the curves of non-asymptotic nets in relation \( L' \) form the same cross ratio with the tangents to the asymptotic curves on the surfaces on which the nets lie. In particular if the tangents to the curves of a net on a surface form a constant cross ratio with the asymptotic tangents, any \( L' \) transform has the same property, the constant ratio being the same for all nets in relation \( L' \).*

From (12) and (13) we see that nets in relation \( C \) are both \( L \) and \( L' \) transforms if they are in relation \( F \) or are both asymptotic nets. From (11) we find that the focal points of \( h \) corresponding to non-conjugate nets \( N_v \) and \( N_s \) are indeterminate if and only if \( N_v \) and \( N_s \) are radial transforms and the asymptotic curves on \( S_v \) and \( S_z \) correspond. In this case the net (10) is indeterminate. If \( N_v \) and \( N_s \) are conjugate nets the focal points of \( h \) are indeterminate if these nets are radial transforms and if the asymptotic nets on \( S_v \) and \( S_z \) correspond.

4. COAXIAL NETS IN RELATION \( C \)

Let the curves of the two nets \( N_v \) and \( N_s \) be union curves of the congruence \( G \). We shall call such nets coaxial nets since they have the same axis. From the first and the third equations of (1) and (3) it follows that the nets \( N_v \) and \( N_s \) in relation \( C \) are coaxial nets if and only if \( \beta^{(11)} = \beta^{(22)} = \alpha^{(11)} = \alpha^{(22)} = 0 \). From (3a) and (3c) it follows that if a net \( N_v \) is in relation \( C \) to its axis congruence \( G \), any \( C \) transform \( N_s \) of \( N_v \) by the same congruence \( G \) has \( G \) for its axis congruence. \( N_v \) and \( N_s \) are coaxial nets in relation \( C \).

We may readily verify that the curve \( v = \text{const.} \) on \( S_v \) is a plane curve if and only if the invariant
\[ G_v = b^{(11)}a^{(11)}d^{(11)} + b^{(11)}d^{(12)} + qd^{(11)} + d_v^{(11)} \]
\[ - d^{(11)}(a^{(11)}b^{(11)} + b^{(11)}b^{(12)} + b_d^{(11)}) \]
vanishes; the curve \( u = \text{const.} \) on \( S_v \) is plane if and only if
\[ G_v' = a^{(22)}b^{(22)}d^{(22)} + a^{(22)}d^{(12)} + q'd^{(22)} + d_v^{(22)} \]
\[ - d^{(22)}(a^{(22)}b^{(22)} + a^{(22)}a^{(12)} + a_d^{(12)}) \]
vanishes. Similar expressions \( G_v \) and \( G_v' \) may be written for the curves \( v = \text{const.} \) and \( u = \text{const.} \) on \( S_v \). From (14) and (15) it follows that if \( N_v \)

* This statement of the theorem includes the case in which the nets are conjugate.
and \( N_{s} \) are coaxial nets in relation \( C \) they are nets of plane curves.* Moreover it is readily proved that if \( N_{v} \) is a net of plane curves and \( N_{v} \) and \( N_{s} \) are coaxial they are in relation \( C \), and \( N_{s} \) is also a net of plane curves.

Suppose that \( N_{v} \) and \( N_{s} \) are in relation \( C \) to the axis congruence of \( N_{v} \) and that \( N_{s} \) has for one (or both) of its component families one (or both) of the families of asymptotic curves of \( S_{z} \). Then that family of curves consists of a set of rulings on the ruled surface \( S_{z} \).

The ray of the point \( y \) with respect to the net \( N_{v} \) joins the points
\[
\rho_{v} = y_{v} - \frac{1}{d^{(11)}}(b^{(12)}d^{(11)} - b^{(11)}d^{(12)})y,
\]
(16)
\[
\sigma_{v} = y_{v} - \frac{1}{d^{(22)}}(a^{(12)}d^{(22)} - a^{(22)}d^{(12)})y.
\]

The ray of the point \( z \) with respect to \( N_{s} \) joins the points
\[
\rho_{s} = z_{u} - \frac{1}{\delta^{(11)}}(\beta^{(12)}\delta^{(11)} - \beta^{(11)}\delta^{(12)})z,
\]
(17)
\[
\sigma_{s} = z_{u} - \frac{1}{\delta^{(22)}}(\alpha^{(12)}\delta^{(22)} - \alpha^{(22)}\delta^{(12)})z.
\]

Suppose that \( \rho_{v} \) and \( \rho_{s} \) are distinct. The line joining \( \rho_{v} \) and \( \rho_{s} \) intersects \( g \) in the point
\[
P_{r} = \left[ \rho + \frac{m}{d^{(11)}}(b^{(12)}d^{(11)} - b^{(11)}d^{(12)}) \right] y
\]
(18)
\[\quad + \left[ q - \frac{1}{\delta^{(11)}}(\beta^{(12)}\delta^{(11)} - \beta^{(11)}\delta^{(12)}) \right] z.
\]

Similarly \( \sigma_{v}, \sigma_{s} \) intersects \( g \) in the point \( P_{s} \) defined by
\[
P_{s} = \left[ \rho' + \frac{n'}{d^{(22)}}(a^{(12)}d^{(22)} - a^{(22)}d^{(12)}) \right] y
\]
(19)
\[\quad + \left[ q' - \frac{1}{\delta^{(22)}}(\alpha^{(12)}\delta^{(22)} - \alpha^{(22)}\delta^{(12)}) \right] z.
\]

The point \( P_{s} \) will coincide with the focal point \( r' \) of \( g \) if and only if
\[
n' \left[ q - \frac{1}{\delta^{(11)}}(\beta^{(12)}\delta^{(11)} - \beta^{(11)}\delta^{(12)}) \right] = - \rho - \frac{m}{d^{(11)}}(b^{(12)}d^{(11)} - b^{(11)}d^{(12)}).
\]

Using the expressions for \( \beta^{(11)} \) and \( \beta^{(12)} \) from (3a) and (3b) this condition may be reduced to

\[
\frac{\beta^{(11)}d^{(12)}}{d^{(11)}} = \frac{\delta^{(12)}}{\delta^{(11)}}.
\]

Similarly the line \( \sigma_u \sigma_v \) meets \( g \) in the focal point \( \tau \) if and only if

\[
\frac{a^{(22)}d^{(12)}}{d^{(22)}} = \frac{\delta^{(12)}}{\delta^{(22)}}.
\]

Let us call the points \( \rho_u, \rho_v, \tau' \) focal points of the first rank, and \( \sigma_u, \sigma_v, \tau \) focal points of the second rank. It follows therefore that the lines joining corresponding focal points of the tangents to the curves of nets in relation \( C \) with respect to a congruence intersect the lines of the congruence in the focal points of the same rank if

(a) the nets are coaxial, or if
(b) the nets are in relation \( F^* \) or if
(c) the asymptotic curves on \( S_u \) and \( S_v \) correspond.

5. THE DERIVED CONGRUENCES OF \( G \). THE RELATION \( R \)

We shall call the tangent to \( C_u(C_v) \) on the focal surface \( S_{u'}(S_{v'}) \) the minus first (first) derived congruence of \( G \). We find that the minus first derived line of \( g \) joins the point \( \tau' \) to the point

\[
T_{\tau'}^{-1} = \frac{\tau' - q\tau'}{n' - m'} = y_u - b^{(12)}y.
\]

Similarly the first derived line of \( g \) joins \( \tau \) to the point

\[
T_{\tau}^{+1} = \frac{\tau - q'\tau}{m - n'} = y_v - a^{(12)}y.
\]

Comparing (20) and (16), we find that the minus first derived line of \( g \) intersects the tangent to \( C_u \) on \( S_v \) in the focal point if and only if \( b^{(11)}d^{(12)} = 0 \).

Similarly the first derived line of \( g \) meets the tangent to \( C_v \) on \( S_v \) in the focal point if and only if \( a^{(22)}d^{(12)} = 0 \).

Hence the minus first (first) derived line of \( g \) intersects the tangent to \( C_u(C_v) \) on \( S_v \) in the focal point on this tangent if \( g \) lies in the osculating plane to \( C_u(C_v) \) at \( y \), or if \( N_u \) is a conjugate net. A similar statement may be made of \( N_v \). Hence the derived lines of \( g \) meet the tangents to the curves of nets in relation \( C \)

in the focal points of these tangents if (a) the nets are in relation F or if (b) they are coaxial nets.

We may readily verify that the asymptotic nets on $S_{v}$ and $S_{u}$ are defined by the differential equations

\[ (p' - n' + n'q')du^2 - (n' - m)b^{1(1)}dv^2 = 0, \]
\[ (p - mu +mq)dv^2 - (m - n')a^{1(1)}du^2 = 0, \]

respectively. Therefore, if the line $g$ in relation $C$ to $N_{v}$ and $N_{u}$ lies in the osculating plane to $C_{v}(C_{u})$ on $S_{v}$ or $S_{u}$, the focal surface $S_{v}(S_{u})$ is developable. If $N_{v}$ and $N_{u}$ are coaxial nets in relation $C$ the focal surfaces of $G$ are both developable, the generators forming the lines of $G$.

The points $T_{v}^{-1}$ and $T_{v}^{+1}$ defined by (20) and (21) have an interesting interpretation. Any point $r$ on the tangent to $C_{u}$ at $y$ is defined by an expression of the form

\[ r = y - \lambda y. \]

As $y$ moves along $C_{v}$ the tangent to $C_{u}$ on $S_{v}$ generates a ruled surface $R^{(u)}$ and $r$ traces out a curve on $R^{(u)}$. A point on the tangent to this curve is

\[ \eta = r_{v} - ar = (a^{1(1)} - a)y_{u} - (b^{1(1)} - \lambda)y_{u} + (c^{1(1)} - \lambda + \alpha)y + d^{1(1)}z. \]

Hence the tangent to the curve traced by $r$ as $y$ moves along $C_{v}$ lies in the plane determined by $g_{v}$, $r_{v}$ and $y_{v}$ if and only if

\[ \lambda = b^{1(1)}. \]

This value of $\lambda$ determines the point

\[ T_{v}^{-1} = y_{v} - b^{1(1)}y. \]

The point $T_{v}^{+1}$ has a similar interpretation.

The line $T_{v}^{-1}T_{v}^{+1}$ is called the line in relation $R$ to $g$ with respect to $N_{v}$. The points

\[ (23) \quad T_{v}^{-1} = z_{v} - \beta^{1(1)}z, \quad T_{v}^{+1} = z_{v} - \alpha^{1(1)}z \]

may be similarly defined. If $N_{v}$ is a conjugate net the points $T_{v}^{-1}$ and $T_{v}^{+1}$ are the Laplace transforms of $y$. We may state our results as follows:

The derived lines of $g$ meet the tangents to the curves of non-conjugate nets in relation $C$ to $G$ in points which determine the line $g'$ in relation $R$ to $g$. The derived lines of $g$ meet the tangents to the curves of conjugate nets in relation $F$ to $G$ in the Laplace transforms of the surface point $y$.

* Green, Surfaces, p. 86, and footnote p. 87.
The tangent to $C_v$ at $T_v^{-1}$ on the surface generated by this point cuts $g$ in the point
\[(24) \quad (a^{(12)}b^{(12)} + c^{(12)} - b_v^{(12)})y + d^{(12)}z. \]
Similarly the tangent to $C_u$ at $T_u^{+1}$ cuts $g$ in the point
\[(25) \quad (a^{(12)}b^{(12)} + c^{(12)} - a_u^{(12)})y + d^{(12)}z. \]
The points determined by (24) and (25) coincide if $d^{(12)} = 0$ or if $a_u^{(12)} - b_v^{(12)} = 0^*$. In case $d^{(12)} \neq 0$ and $a_u^{(12)} - b_v^{(12)} = 0$ the developables of the congruence of lines $g'$ in relation $R$ to $g$ correspond to a conjugate net.

Comparing the expressions (20) and (21) with (9), we find that the line $h$ of intersection of corresponding tangent planes to $S_v$ and $S_u$ is in relation $R$ to $g$ with respect to $N_v$ if and only if
\[(26) \quad b^{(12)} + \frac{p}{m} = 0, \quad a^{(12)} + \frac{p'}{n'} = 0. \]
Similarly $h$ will be in relation $R$ to $g$ with respect to $N_u$ if and only if
\[(27) \quad \beta^{(12)} - q = 0, \quad \alpha^{(12)} - q' = 0. \]
From (3b) and (2c) we find that
\[
\alpha^{(12)} - q' = \frac{n'}{m} \left( a^{(12)} + \frac{p'}{n'} \right), \\
\beta^{(12)} - q = \frac{m}{n'} \left( b^{(12)} + \frac{p}{m} \right).
\]
Hence if $N_v, N_u, G$ are in relation $C$ and if $H$ is in relation $R$ to $G$ with respect to $N_v$ it is in relation $R$ to $G$ with respect to $N_u$.

From (16) and (17) we note that the focal points $\rho_v$ and $\rho_u$ coincide if and only if
\[
m(b^{(12)}d^{(11)} - b_v^{(11)}d^{(12)}) + p d^{(11)} = 0, \\
\beta^{(13)} - \beta^{(11)}q^{(12)} - q^{(11)} = 0.
\]
These conditions may be reduced by means of (3a) and (3b) to
\[(28) \quad b^{(12)} + \frac{p}{m} = b^{(11)}\frac{d^{(12)}}{d^{(11)}}, \quad b^{(12)} + \frac{p}{m} = b^{(11)}\frac{\delta^{(12)}}{\delta^{(11)}}. \]

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Similarly the focal points \( \sigma_1 \) and \( \sigma_2 \) coincide if and only if

\[
a^{(12)} + \frac{\beta'}{m'} = a^{(12)} \frac{d^{(12)}}{d^{(22)}}, \quad a^{(12)} + \frac{\beta'}{m'} = a^{(22)} \frac{\delta^{(12)}}{\delta^{(22)}}.
\]

We shall call nets \( N_1 \) and \( N_2 \) coradial nets if they have the same rays. Hence coaxial nets in relation \( C \) are coradial if and only if \( h \) and \( g \) are in relation \( R \). If the nets are in relation \( F \) they are coradial if and only if the two derived lines of \( g \) intersect \( h \). In order for nets which are non-conjugate and not coaxial to be coradial it is necessary that the asymptotic curves on the sustaining surfaces correspond.

We may readily verify that the lines in relation \( R \) to \( g \) with respect to \( N_1 \) and \( N_2 \) intersect \( h \) in the points

\[
(a^{(12)} + \frac{\beta'}{m'}) r - \left( b^{(12)} + \frac{\beta}{m} \right) s, \quad n' \left( a^{(12)} + \frac{\beta'}{n'} \right) r - m \left( b^{(12)} + \frac{\beta}{m} \right) s
\]

respectively. These points coincide if \( N_1 \) and \( N_2 \) are radial transforms. Suppose that neither of the points (30) coincide with \( r \) or \( s \). The cross ratio of the points \( r, s \) and the points (30) is \( m/n' \). Hence if the lines in relation \( R \) to \( g \) meet \( h \) in distinct points, the cross ratio of these points and the points \( r \) and \( s \), and the cross ratio of the points \( y, z, r, r' \) are equal.

Similarly we may prove that if \( N_1 \) and \( N_2 \) are \( L' \) transforms the rays of \( N_1 \) and \( N_2 \) cut \( h \) in points forming with \( r \) and \( s \) the same cross ratio that \( y \) and \( z \) form with the focal points of \( g \).

6. The transformations \( K_R \)

We shall call nets \( N_1 \) and \( N_2 \) in relation \( C \) \( K_R \) transforms if

\[
\frac{\partial^2}{\partial u \partial v} \log R = 0,
\]

wherein the cross ratio \( (y, r', z, r) = m/n' = R \). If \( N_1 \) and \( N_2 \) are in relation \( F \) and are \( K_{-1} \) transforms, they are \( K \) transforms in the sense of Koenig since corresponding surface points separate the focal points of \( g \) harmonically.

From (3b) we find that

\[
\alpha_u^{(12)} - \beta_u^{(12)} = a_u^{(12)} - b_u^{(12)} + \frac{\partial^2}{\partial u \partial v} \log R.
\]

Hence if the line in relation \( R \) to \( g \) with respect to a net is harmonic to \( S_u \) the same property is enjoyed by any \( K_R \) transform of the given net.

Suppose that \( N_y \) and \( N_z \) are \( K \)-transforms. We find from the integrability conditions (2c) and \( m + n' = 0 \), that

\[
\frac{\partial}{\partial v} \log m + 2a^{(12)} + \frac{p'}{n'} = q',
\]

(32)

\[
\frac{\partial}{\partial v} \log n' + 2b^{(12)} + \frac{p}{m} = q.
\]

Differentiating (32) with respect to \( u \) and \( v \) respectively and subtracting, we find that

\[
2(a^{(12)} - b^{(12)}) + \frac{\partial}{\partial u} \left( \frac{p'}{n'} \right) - \frac{\partial}{\partial v} \left( \frac{p}{m} \right) + q_0 - q_0' = 0.
\]

But from (2c)

\[
q_0' - q_0 = (n' - m)d^{(12)};
\]

and from (3b)

\[
\frac{\partial}{\partial u} \left( \frac{p'}{n'} \right) - \frac{\partial}{\partial v} \left( \frac{p}{m} \right) = (\nu' - \mu)d^{(12)}.
\]

We have, therefore, the theorem of Koenig: \( K \) transforms have equal point invariants.

Let now \( N_y \) and \( N_z \) be coradial nets in relation \( F \). From (28) and (29) and (2c) it follows that

\[
\frac{\partial^2}{\partial u \partial v} \log R = 0.
\]

Hence if \( N_y \) and \( N_z \) are coradial nets in relation \( F \), they have equal point invariants.†

The ray curves of \( N_y \) form a conjugate net‡ if

\[
\frac{\partial}{\partial u} \left( a^{(12)} - \frac{a^{(22)}d^{(12)}}{d^{(22)}} \right) - \frac{\partial}{\partial v} \left( b^{(12)} - \frac{b^{(11)}d^{(12)}}{d^{(11)}} \right) = 0.
\]

The ray curves of \( N_z \) form a conjugate net if

\[
\frac{\partial}{\partial u} \left( \alpha^{(12)} - \frac{\alpha^{(22)}\delta^{(12)}}{\delta^{(22)}} \right) - \frac{\partial}{\partial v} \left( \beta^{(12)} - \frac{\beta^{(11)}\delta^{(12)}}{\delta^{(11)}} \right) = 0.
\]

Let \( N_y \) and \( N_z \) be \( L \) transforms. Equation (34) may be written

---

† Eisenhart, _Surfaces_, p. 68, exercise 16.
‡ See V. G. Grove, loc. cit.
Hence if two nets $N_y$ and $N_z$ are $L$ transforms in the relation of a transformation $K_B$, the ray curves of $N_y$ will form a conjugate system if and only if the ray curves of $N_z$ form a conjugate system.

7. Pencils of conics in the tangent planes

Consider the points

$$r' = y_u - \lambda y, \quad s' = y_v - \mu y,$$

lying on the tangent lines at $y$ to the curves of the net $N_y$. If use be made of (9), (35) may be written

$$r' = r - \left(\frac{\lambda + \frac{p}{m}}{m}\right)y, \quad s' = s - \left(\frac{\mu + \frac{p}{m}}{m}\right)y.$$

The pencil of conics tangent to the parametric tangents at $r'$ and $s'$ is therefore

$$2k_1x_2x_2 = \left[ x_1 + \left(\frac{\lambda + \frac{p}{m}}{m}\right)x_2 + \left(\frac{\mu + \frac{p}{m}}{n'}\right)x_3 \right]^2, \quad x_4 = 0,$$

the tetrahedron of reference being $y, r, s, z$.

Similarly the pencil of conics tangent at

$$r'' = z_u - \lambda'z = m_r - (\lambda' - q)z, \quad s'' = z_v - \mu'z = n's - (\mu' - q')z,$$

referred to the same tetrahedron is

$$2k_2x_2x_2 = \left[ \frac{\lambda - q}{m}x_2 + \frac{\mu - q'}{n'}x_3 + x_4 \right]^2, \quad x_1 = 0.$$

The conics (36) and (37) determine two involutions on the line $h$. These involutions will be identical if and only if

$$(\phi + \lambda m)(\mu' - q')^2 = (\phi' + \mu n')(\lambda' - q)^2.$$

Suppose that the points $r', s', r'', s''$ are the focal points on the lines on which they lie. From (16) and (17) it follows that

$$\lambda = \delta^{(12)} - \frac{\delta^{(11)}d^{(12)}}{d^{(11)}}, \quad \lambda' = \beta^{(12)} - \frac{\beta^{(11)}\delta^{(12)}}{\delta^{(11)}},$$

$$\mu = a^{(12)} - \frac{a^{(22)}d^{(12)}}{d^{(22)}}, \quad \mu' = a^{(12)} - \frac{a^{(22)}\delta^{(12)}}{\delta^{(22)}}.$$
Suppose now that $N_y$ and $N_z$ are $L'$ transforms. Equations (38) under conditions (39) may be written

$$
(p + mb^{(12)}) - \frac{mb^{(11)}d^{(12)}}{d^{(11)}})^2 \left(\frac{p' + n'a^{(12)}}{m} - \frac{n'a^{(22)}d^{(12)}}{md^{(22)}}\right)^2
$$

$$
= \left(p' + n'a^{(12)} - \frac{n'a^{(22)}d^{(12)}}{d^{(22)}}\right)^2 \left(\frac{p + mb^{(12)}}{n'} - \frac{mb^{(11)}d^{(12)}}{d^{(11)}}\right)^2.
$$

Hence

$$
m^2 - n'^2 = 0.
$$

We may state our results as follows: Let the parametric nets $N_y$ and $N_z$ be in relation $L'$. The pencils of conics tangent to the parametric tangents at their focal points determine the same involution on $h$ if and only if $L'$ is $K_{-1}$. If the parametric nets are in relation $F$, we have the theorem of Eisenhart*:

The two pencils of conics tangent to the parametric tangents to the curves of nets in relation $F$ determine the same involution on the line of intersection of corresponding tangent planes if and only if the nets are in relation $K$.

Let $N_y$ and $N_z$ be $K_{-1}$ transforms in relation $L'$. The two conics (36) and (37) under conditions (39) have a point in common if and only if

$$
k_1 = m^2k_2 + 2\left(\frac{p}{m}\right)\left(\mu + \frac{p'}{n'}\right).
$$

The pencil of quadrics determined by these conics under condition (40) is

$$
x_1^2 + \left(\lambda + \frac{p}{m}\right)x_2^2 + \left(\mu + \frac{p'}{n'}\right)x_3^2 + m^2x_4^2 + 2\left(\frac{p}{m}\right)x_1x_2
$$

$$+ 2\left(\mu + \frac{p'}{n'}\right)x_1x_3 + 2k_3x_1x_4 - 2\left[\left(\frac{p}{m}\right) + \lambda\right]\left(\frac{p'}{m}\right) + \mu
$$

$$+ m^2k_2\right) x_2x_3 - 2m\left(\frac{p}{m}\right)x_2x_4 + 2m\left(\mu + \frac{p'}{n'}\right)x_3x_4 = 0,
$$

the tetrahedron of reference being $y, r, s, z$. The two quadrics of the pencil (41) tangent to the line $g$ are determined by the parameter values $k_3 = \pm m$. These values of $k_3$ are the parameter values of the cones of the pencil provided all the quadrics of the pencil are not cones. The coordinates of the vertices of the cones are for

---

We readily verify that the first vertex of (42) lies on $\sigma_{x'T'}$, and the second vertex on $\rho_{x'T'}$. The pencil (41) cuts the line $g$ in an involution whose double points are the focal points of $g$. We may state our results as follows:

If $N_y$ and $N_z$ are $K_{-1}$ transforms in relation $L'$, any two of the conics (36) and (37) with values of $\lambda$ and $\mu$ given by (39) meeting on $h$ determine a pencil of quadrics which cut $g$ in an involution whose double points are the focal points of $g$. The two cones of the pencil are the quadrics tangent to $g$ at the focal points of $g$. The vertices of the cones are on the lines joining corresponding focal points on the parametric tangents.

If in the above theorem the restriction that $N_y$ and $N_z$ be $L'$ transforms be removed, and the condition of being coaxial be imposed on the nets the remainder of the theorem is true. Also let $N_y$ and $N_z$ be coaxial nets in relation $C$. It follows that the pencil of conics tangent to the parametric tangents at their focal points determine the same involution if and only if $C$ is $K_{-1}$.

Let now $N_y$ and $N_z$ be non-conjugate nets in relation $C$ not necessarily $L'$. The wording of the above theorem for parametric values $\lambda = b^{(12)}, \mu = a^{(12)}$ then holds.

The vertices of the cones in question lie in this case on the derived lines of $g$.

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