TRANSFORMATIONS OF NETS*

BY

V. G. GROVE

Introduction

It is the purpose of this paper to extend some of the ideas relating to transformations of surfaces and conjugate nets to transformations of an arbitrary net. When two nets and the congruence of lines joining corresponding points are so related that the developables of the congruence cut the sustaining surfaces of the nets in the curves of the net, the nets and the congruences will be said to be in *relation C*. Either net will be said to be a C transform of the other. If the given nets are conjugate nets, then the transformation C becomes a transformation F.† If corresponding points of two nets in relation C separate harmonically the focal points on the line joining corresponding points, we shall call the given nets K_{-1} *transforms*. If the given nets are conjugate nets, then the transformation K_{-1} becomes the transformation K of Koenig.‡ We shall consider these and related transformations in some detail.

1. The differential equations

Let

$$y^{(k)} = y^{(k)}(u,v), \quad z^{(k)} = z^{(k)}(u,v) \qquad (k = 1,2,3,4)$$

be the parametric equations of surfaces S_y and S_z but assume that neither surface is a focal surface of the congruence G of lines g joining the points y and z. Suppose that the parametric nets N_y and N_z on S_y and S_z and the congruence G are in relation C. Evidently the tangents to the curves $u=$ const. on S_y and S_z are coplanar and similarly the tangents to $v=$ const. are coplanar. The functions $y^{(k)}$ and $z^{(k)}$ will satisfy a system of differential equations of the form§

* Presented to the Society, December 31, 1926; received by the editors in October, 1927.

† L. P. Eisenhart, *Transformations of Surfaces*, Princeton University Press, 1923, p. 34. Hereafter referred to as Eisenhart, *Surfaces*.

‡ Eisenhart, *Surfaces*, p. 57.

§ G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), p. 149. Hereafter referred to as Green, *Surfaces*.

$$
(1)\qquad
\begin{aligned}
y_{uu} &= a^{(11)}y_u + b^{(11)}y_v + c^{(11)}y + d^{(11)}z,\\
y_{uv} &= a^{(12)}y_u + b^{(12)}y_v + c^{(12)}y + d^{(12)}z,\\
y_{vv} &= a^{(22)}y_u + b^{(22)}y_v + c^{(22)}y + d^{(22)}z,\\
z_u &= my_u \qquad\qquad\quad + py \quad + qz,\\
z_v &= \qquad\qquad n'y_v + p'y \quad + q'z.
\end{aligned}
$$

The coefficients of system (1) satisfy the following integrability conditions*:

$$
(2a)\qquad
\begin{aligned}
&a_u^{(12)} - a_v^{(11)} + a^{(12)}b^{(12)} - a^{(22)}b^{(11)} + c^{(12)}\\
&\qquad\qquad = -md^{(12)},\\
&b_u^{(12)} - b_v^{(11)} + a^{(12)}b^{(11)} + (b^{(12)} - a^{(11)})b^{(12)} - b^{(11)}b^{(22)} - c^{(11)}\\
&\qquad\qquad = n'd^{(11)},\\
&c_u^{(12)} - c_v^{(11)} + a^{(12)}c^{(11)} + (b^{(12)} - a^{(11)})c^{(12)} - b^{(11)}c^{(22)}\\
&\qquad\qquad = p'd^{(11)} - pd^{(12)},\\
&d_u^{(12)} - d_v^{(11)} + a^{(12)}d^{(11)} + (b^{(12)} - a^{(11)})d^{(12)} - b^{(11)}d^{(22)}\\
&\qquad\qquad = q'd^{(11)} - qd^{(12)};
\end{aligned}
$$

$$
(2b)\qquad
\begin{aligned}
&a_u^{(22)} - a_v^{(12)} + a^{(22)}a^{(11)} + (b^{(22)} - a^{(12)})a^{(12)} - b^{(12)}a^{(22)} + c^{(22)}\\
&\qquad\qquad = -md^{(22)},\\
&b_u^{(22)} - b_v^{(12)} + a^{(22)}b^{(11)} - a^{(12)}b^{(12)} - c^{(12)}\\
&\qquad\qquad = n'd^{(12)},\\
&c_u^{(22)} - c_v^{(12)} + a^{(22)}c^{(11)} + (b^{(22)} - a^{(12)})c^{(12)} - b^{(12)}c^{(22)}\\
&\qquad\qquad = p'd^{(12)} - pd^{(22)},\\
&d_u^{(22)} - d_v^{(12)} + a^{(22)}d^{(11)} + (b^{(22)} - a^{(12)})d^{(12)} - b^{(12)}d^{(22)}\\
&\qquad\qquad = q'd^{(12)} - qd^{(22)};
\end{aligned}
$$

$$
(2c)\qquad
\begin{aligned}
-m_v + (n'-m)a^{(12)} + p' &= \qquad -q'm,\\
n'_u \qquad + (n'-m)b^{(12)} - p &= qn',\\
p'_u - p_v + (n'-m)c^{(12)} \qquad &= qp' - pq',\\
q'_u - q_v + (n'-m)d^{(12)} \qquad &= 0.
\end{aligned}
$$

If the fourth and fifth equation of system (1) are solved for y_u and y_v, and if the resulting two equations are differentiated with respect to u and v, we obtain the following equivalent system:

* Green, *Surfaces*, p. 150.

$$
(3)\qquad
\begin{aligned}
z_{uu} &= \alpha^{(11)}z_u + \beta^{(11)}z_v + \gamma^{(11)}z + \delta^{(11)}y,\\
z_{uv} &= \alpha^{(12)}z_u + \beta^{(12)}z_v + \gamma^{(12)}z + \delta^{(12)}y,\\
z_{vv} &= \alpha^{(22)}z_u + \beta^{(22)}z_v + \gamma^{(22)}z + \delta^{(22)}y,\\
y_u &= \mu z_u \qquad\qquad\quad + \pi z \quad + \kappa y,\\
y_v &= \qquad\qquad\quad \nu' z_v + \pi' z \quad + \kappa' y,
\end{aligned}
$$

wherein

$$
(3a)\qquad
\begin{aligned}
&\alpha^{(11)} = a^{(11)} + \frac{p}{m} + \frac{m_u}{m} + q, \qquad \beta^{(11)} = \frac{m}{n'}b^{(11)},\\
&\delta^{(11)} = -m\left[a^{(11)}\frac{p}{m} + b^{(11)}\frac{p'}{n'} + \left(\frac{p}{m}\right)^2 - c^{(11)} - \frac{\partial}{\partial u}\left(\frac{p}{m}\right)\right],
\end{aligned}
$$

$$
(3b)\qquad
\begin{aligned}
&\alpha^{(12)} = a^{(12)} + \frac{m_v}{m}, \quad \beta^{(12)} = b^{(12)} + \frac{n'_u}{n'},\\
&\delta^{(12)} = -m\left[a^{(12)}\frac{p}{m} + b^{(12)}\frac{p'}{n'} - \frac{pp'}{mn'} - c^{(12)} - \frac{\partial}{\partial v}\left(\frac{p}{m}\right)\right]\\
&\qquad = -n'\left[b^{(12)}\frac{p'}{n'} + a^{(12)}\frac{p}{m} - \frac{pp'}{mn'} - c^{(12)} - \frac{\partial}{\partial u}\left(\frac{p'}{n'}\right)\right],
\end{aligned}
$$

$$
(3c)\qquad
\begin{aligned}
&\alpha^{(22)} = \frac{n'}{m}a^{(22)}, \beta^{(22)} = b^{(22)} + \frac{p'}{n'} + \frac{n'_v}{n'} + q',\\
&\delta^{(22)} = -n'\left[a^{(22)}\frac{p}{m} + b^{(22)}\frac{p'}{n'} + \left(\frac{p'}{n'}\right)^2 - c^{(22)} - \frac{\partial}{\partial v}\left(\frac{p'}{n'}\right)\right],
\end{aligned}
$$

$$
(3d)\qquad \mu = \frac{1}{m}, \quad \nu' = \frac{1}{n'}, \quad \kappa = -\frac{p}{m}, \quad \kappa' = -\frac{p'}{n'}, \quad \pi = -\frac{q}{m}, \quad \pi' = -\frac{q'}{n'}.
$$

The differential equation of the asymptotic curves* on S_y is

$$
(4)\qquad d^{(11)}du^2 + 2d^{(12)}dudv + d^{(22)}dv^2 = 0,
$$

and the asymptotic curves on S_z are given by the equation

$$
(5)\qquad \delta^{(11)}du^2 + 2\delta^{(12)}dudv + \delta^{(22)}dv^2 = 0.
$$

The surface S_y is therefore a developable if $d^{(11)}d^{(22)} - (d^{(12)})^2 = 0$. A similar statement holds for S_z. We shall assume hereafter that neither S_y nor S_z is developable. If $d^{(12)} = 0$ the parametric net on S_y is conjugate; if $d^{(11)} = d^{(22)} = 0$ the parametric net is asymptotic. Similar statements hold for the parametric net on S_z.

* Green, *Surfaces*, p. 151.

2. The focal nets of G

Any point P on g is defined by an expression of the form

$$P = y + \theta z.$$

As $y(z)$ describes a curve $v=v(u)$ on $S_y(S_z)$, g generates a ruled surface. If this surface is a developable, the point dP/du lies on g. Imposing this condition we find that the developables are given by the differential equation

$$(n' - m)du dv = 0 \tag{6}$$

and the focal points by

$$mn'\theta^2 + (m + n')\theta + 1 = 0. \tag{7}$$

The focal points of g are therefore

$$\begin{aligned} \tau &= my - z, \\ \tau' &= n'y - z. \end{aligned} \tag{8}$$

We shall call the nets N_τ and $N_{\tau'}$ *the focal nets of* G. If the focal surfaces S_τ and $S_{\tau'}$ are not developable, the nets N_τ and $N_{\tau'}$ are conjugate. Equation (6) shows that the developables are indeterminate if $m-n'=0$. In this case we shall call N_y and N_z *radial transforms*. Unless explicitly stated to the contrary, we shall hereafter assume that N_y and N_z are not radial transforms.

3. The transformations L and L'

From the last two equations of system (1) we find that the points of intersection of corresponding tangents to the curves of nets in relation C are defined by the expressions

$$\begin{aligned} r &= y_u + \frac{p}{m}y = \mu(z_u - qz), \\ s &= y_v + \frac{p'}{n'}y = \nu'(z_v - q'z). \end{aligned} \tag{9}$$

The line h of intersection of the tangent planes to S_y and S_z at corresponding points y and z generates a congruence H. We may in the usual way show that the differential equation of the net corresponding to the developables of the congruence H is

$$\begin{aligned} (md^{(11)}\delta^{(12)} - n'd^{(12)}\delta^{(11)})du^2 + \big[m(d^{(11)}\delta^{(22)} + d^{(12)}\delta^{(12)}) \\ - n'(d^{(22)}\delta^{(11)} + d^{(12)}\delta^{(12)})\big]du dv - (n'd^{(22)}\delta^{(12)} - md^{(12)}\delta^{(22)})dv^2 = 0. \end{aligned} \tag{10}$$

The focal points of h are defined by the expressions

$$P_{1,2} = r + \theta_{1,2}s,$$

where $\theta_{1,2}$ are the roots of

$$\begin{aligned}(11)\quad & m(d^{(22)}\delta^{(12)} - d^{(12)}\delta^{(22)})\theta^2 + [n'(d^{(22)}\delta^{(11)} - d^{(12)}\delta^{(12)}) \\ & \qquad - m(d^{(11)}\delta^{(22)} - d^{(12)}\delta^{(12)})]\theta - n'(d^{(11)}\delta^{(12)} - d^{(12)}\delta^{(11)}) = 0.\end{aligned}$$

The harmonic invariant of (4) and (10) is

$$(m - n')(d^{(11)}d^{(22)} - d^{(12)^2})\delta^{(12)} = 0.$$

Hence *the congruence H will be harmonic to S_y if and only if G is conjugate to S_z. Similarly H will be harmonic to S_z if and only if G is conjugate to S_y.*

We shall call nets N_y and N_z *L transforms* or in *relation L* if they are in relation C and if they correspond to the developables of H. Nets N_y and N_z in relation C will be L transforms if

$$(12)\qquad md^{(11)}\delta^{(12)} - n'd^{(12)}\delta^{(11)} = n'd^{(22)}\delta^{(12)} - md^{(12)}\delta^{(22)} = 0.$$

Hence *if N_y and N_z are in relation F they are L transforms. If N_y and N_z are asymptotic nets, they are L transforms.* Suppose N_y and N_z are neither conjugate nor asymptotic. Then from (12) we obtain

$$\frac{d^{(11)}d^{(22)}}{d^{(12)^2}} = \frac{\delta^{(11)}\delta^{(22)}}{\delta^{(12)^2}}.$$

Hence *the tangents to the curves of non-asymptotic nets in relation L form the same cross ratios with the asymptotic tangents associated with the surfaces on which the nets lie.* In particular *if the tangents to the curves of a net on a surface form a constant cross ratio with the asymptotic tangents, any L transform of the net has the same property, the constant ratio being the same for all nets in relation L**.

From (1) we see that the focal points of h lie on the tangents to the curves of nets in relation C if and only if

$$(13)\qquad d^{(22)}\delta^{(12)} - d^{(12)}\delta^{(22)} = d^{(11)}\delta^{(12)} - d^{(12)}\delta^{(11)} = 0.$$

We shall call such nets N_y and N_z *L' transforms* or nets in *relation L'*. Hence *if N_y and N_z are F transforms they are L' transforms*; *if they are asymptotic nets in relation C they are L' transforms*; *if they are in relation C and the asymptotic curves on the surfaces on which the nets lie correspond then the nets are L' transforms.*

---

* This statement of the theorem includes the case in which the nets are conjugate.

From equations (13) we find that

$$\frac{d^{(11)}d^{(22)}}{d^{(12)^2}} = \frac{\delta^{(11)}\delta^{(22)}}{\delta^{(12)^2}}.$$

Hence *the tangents to the curves of non-asymptotic nets in relation L' form the same cross ratio with the tangents to the asymptotic curves on the surfaces on which the nets lie.* In particular *if the tangents to the curves of a net on a surface form a constant cross ratio with the asymptotic tangents, any L' transform has the same property, the constant ratio being the same for all nets in relation L'.**

From (12) and (13) we see that *nets in relation C are both L and L' transforms if they are in relation F or are both asymptotic nets.* From (11) we find that *the focal points of h corresponding to non-conjugate nets N_y and N_z are indeterminate if and only if N_y and N_z are radial transforms and the asymptotic curves on S_y and S_z correspond.* In this case *the net* (10) *is indeterminate. If N_y and N_z are conjugate nets the focal points of h are indeterminate if these nets are radial transforms and if the asymptotic nets on S_y and S_z correspond.*

4. Coaxial nets in relation C

Let the curves of the two nets N_y and N_z be union curves of the congruence G. We shall call such nets *coaxial* nets since they have the same axis. From the first and the third equations of (1) and (3) it follows that the nets N_y and N_z in relation C are coaxial nets if and only if $b^{(11)} = a^{(22)} = \beta^{(11)} = \alpha^{(22)} = 0$. From (3a) and (3c) it follows that *if a net N_y is in relation C to its axis congruence G, any C transform N_z of N_y by the same congruence G has G for its axis congruence. N_y and N_z are coaxial nets in relation C.*

We may readily verify that the curve $v = \text{const.}$ on S_y is a plane curve if and only if the invariant

$$(14)\qquad G_y = b^{(11)}(a^{(11)}d^{(11)} + b^{(11)}d^{(12)} + qd^{(11)} + d_u^{(11)}) - d^{(11)}(a^{(11)}b^{(11)} + b^{(11)}b^{(12)} + b_u^{(11)})$$

vanishes; the curve $u = \text{const.}$ on S_y is plane if and only if

$$(15)\qquad G_y' = a^{(22)}(b^{(22)}d^{(22)} + a^{(22)}d^{(12)} + q'd^{(22)} + d_v^{(22)}) - d^{(22)}(a^{(22)}b^{(22)} + a^{(22)}a^{(12)} + a_v^{(12)})$$

vanishes. Similar expressions G_z and G_z' may be written for the curves $v = \text{const.}$ and $u = \text{const.}$ on S_z. From (14) and (15) it follows that *if N_y*

* This statement of the theorem includes the case in which the nets are conjugate.

*and N_z are coaxial nets in relation C they are nets of plane curves.** Moreover it is readily proved that *if N_y is a net of plane curves and N_y and N_z are coaxial they are in relation C, and N_z is also a net of plane curves.*

Suppose that N_y and N_z are in relation C to the axis congruence of N_y and that N_z has for one (or both) of its component families one (or both) of the families of asymptotic curves of S_z. Then that family of curves consists of a set of rulings on the ruled surface S_z.

The ray of the point y with respect to the net N_y joins the points

$$\text{(16)}\qquad \begin{aligned} \rho_y &= y_u - \frac{1}{d^{(11)}}(b^{(12)}d^{(11)} - b^{(11)}d^{(12)})y, \\ \sigma_y &= y_v - \frac{1}{d^{(22)}}(a^{(12)}d^{(22)} - a^{(22)}d^{(12)})y. \end{aligned}$$

The ray of the point z with respect to N_z joins the points

$$\text{(17)}\qquad \begin{aligned} \rho_z &= z_u - \frac{1}{\delta^{(11)}}(\beta^{(12)}\delta^{(11)} - \beta^{(11)}\delta^{(12)})z, \\ \sigma_z &= z_v - \frac{1}{\delta^{(22)}}(\alpha^{(12)}\delta^{(22)} - \alpha^{(22)}\delta^{(12)})z. \end{aligned}$$

Suppose that ρ_y and ρ_z are distinct. The line joining ρ_y and ρ_z intersects g in the point

$$\text{(18)}\qquad \begin{aligned} P_\rho = &\left[p + \frac{m}{d^{(11)}}(b^{(12)}d^{(11)} - b^{(11)}d^{(12)})\right] y \\ &+ \left[q - \frac{1}{\delta^{(11)}}(\beta^{(12)}\delta^{(11)} - \beta^{(11)}\delta^{(12)})\right] z. \end{aligned}$$

Similarly $\sigma_y\sigma_z$ intersects g in the point P_σ defined by

$$\text{(19)}\qquad \begin{aligned} P_\sigma = &\left[p' + \frac{n'}{d^{(22)}}(a^{(12)}d^{(22)} - a^{(22)}d^{(12)})\right] y \\ &+ \left[q' - \frac{1}{\delta^{(22)}}(\alpha^{(12)}\delta^{(22)} - \alpha^{(22)}\delta^{(12)})\right] z. \end{aligned}$$

The point P_ρ will coincide with the focal point τ' of g if and only if

$$n'\left[q - \frac{1}{\delta^{(11)}}(\beta^{(12)}\delta^{(11)} - \beta^{(11)}\delta^{(12)})\right] = -p - \frac{m}{d^{(11)}}(b^{(12)}d^{(11)} - b^{(11)}d^{(12)}).$$

* G. M. Green, *Nets of space curves*, these Transactions, vol. 21 (1920), p. 231. Hereafter referred to as Green, *Nets*. See also G. M. Green, *Projective differential geometry of one-parameter families of curves*, etc., American Journal of Mathematics, vol. 38 (1916), p. 304.

Using the expressions for $\beta^{(11)}$ and $\beta^{(12)}$ from (3a) and (3b) this condition may be reduced to

$$b^{(11)}\frac{d^{(12)}}{d^{(11)}} = b^{(11)}\frac{\delta^{(12)}}{\delta^{(11)}}.$$

Similarly the line $\sigma_y\sigma_z$ meets g in the focal point τ if and only if

$$a^{(22)}\frac{d^{(12)}}{d^{(22)}} = a^{(22)}\frac{\delta^{(12)}}{\delta^{(22)}}.$$

Let us call the points ρ_y, ρ_z, τ' *focal points of the first rank*, and σ_y, σ_z, τ *focal points of the second rank. It follows therefore that the lines joining corresponding focal points of the tangents to the curves of nets in relation C with respect to a congruence intersect the lines of the congruence in the focal points of the same rank if*

(a) *the nets are coaxial, or if*

(b) *the nets are in relation F,* or if*

(c) *the asymptotic curves on S_y and S_z correspond.*

5. The derived congruences of G. The relation R

We shall call the tangent to $C_u(C_v)$ on the focal surface $S_{\tau'}(S_\tau)$ the minus first (first) derived congruence of G. We find that the minus first derived line of g joins the point τ' to the point

$$T_y^{-1} = \frac{\tau_u' - q\tau'}{n' - m} = y_u - b^{(12)}y. \tag{20}$$

Similarly the first derived line of g joins τ to the point

$$T_y^{+1} = \frac{\tau_v - q'\tau}{m - n'} = y_v - a^{(12)}y. \tag{21}$$

Comparing (20) and (16), we find that the minus first derived line of g intersects the tangent to C_u on S_y in the focal point if and only if $b^{(11)}d^{(12)}=0$.

Similarly the first derived line of g meets the tangent to C_v on S_y in the focal point if and only if $a^{(22)}d^{(12)}=0$.

Hence *the minus first (first) derived line of g intersects the tangent to $C_u(C_v)$ on S_y in the focal point on this tangent if g lies in the osculating plane to $C_u(C_v)$ at y, or if N_y is a conjugate net.* A similar statement may be made of N_z. Hence *the derived lines of g meet the tangents to the curves of nets in relation C*

* Eisenhart, *Surfaces*, p. 94, exercise 20.

in the focal points of these tangents if (a) *the nets are in relation F or if* (b) *they are coaxial nets.*

We may readily verify that the asymptotic nets on $S_{r'}$ and S_r are defined by the differential equations

$$(p' - n_v' + n'q')du^2 - (n' - m)b^{(11)}dv^2 = 0,$$
$$(p - m_u + mq)dv^2 - (m - n')a^{(22)}du^2 = 0, \tag{22}$$

respectively. Therefore, if the line g in relation C to N_y and N_z lies in the osculating plane to $C_u(C_v)$ on S_y or S_z, the focal surface $S_{r'}(S_r)$ is developable. *If N_y and N_z are coaxial nets in relation C the focal surfaces of G are both developables, the generators forming the lines of G.*

The points T_y^{-1} and T_y^{+1} defined by (20) and (21) have an interesting interpretation. Any point r on the tangent to C_u at y is defined by an expression of the form

$$r = y_u - \lambda y.$$

As y moves along C_v the tangent to C_u on S_y generates a ruled surface $R^{(u)}$ and r traces out a curve on $R^{(u)}$. A point on the tangent to this curve is

$$\eta = r_v - ar = (a^{(12)} - a)y_u - (b^{(12)} - \lambda)y_v + (c^{(12)} - \lambda_v + \alpha\lambda)y + d^{(12)}z.$$

Hence the tangent to the curve traced by r as y moves along C_v lies in the plane determined by g, r and y if and only if

$$\lambda = b^{(12)}.$$

This value of λ determines the point

$$T_y^{-1} = y_u - b^{(12)}y.$$

The point T_y^{+1} has a similar interpretation.

The line $T_y^{-1}T_y^{+1}$ is called *the line in relation R to g* with respect to N_y*. The points

$$T_z^{-1} = z_u - \beta^{(12)}z, \quad T_z^{+1} = z_v - \alpha^{(12)}z \tag{23}$$

may be similarly defined. If N_y is a conjugate net the points T_y^{-1} and T_y^{+1} are the Laplace transforms of y. We may state our results as follows:

The derived lines of g meet the tangents to the curves of non-conjugate nets in relation C to G in points which determine the line g′ in relation R to g. The derived lines of g meet the tangents to the curves of conjugate nets in relation F to G in the Laplace transforms of the surface point y.

---

* Green, *Surfaces*, p. 86, and footnote p. 87.

The tangent to C_v at T_y^{-1} on the surface generated by this point cuts g in the point

$$(24)\qquad (a^{(12)}b^{(12)} + c^{(12)} - b_v^{(12)})y + d^{(12)}z.$$

Similarly the tangent to C_u at T_y^{+1} cuts g in the point

$$(25)\qquad (a^{(12)}b^{(12)} + c^{(12)} - a_u^{(12)})y + d^{(12)}z.$$

The points determined by (24) and (25) coincide if $d^{(12)}=0$ or if $a_u^{(12)}-b_v^{(12)}=0$*. In case $d^{(12)}\neq 0$ and $a_u^{(12)}-b_v^{(12)}=0$ the developables of the congruence of lines g' in relation R to g correspond to a conjugate net.

Comparing the expressions (20) and (21) with (9), we find that the line h of intersection of corresponding tangent planes to S_y and S_z is in relation R to g with respect to N_y if and only if

$$(26)\qquad b^{(12)} + \frac{p}{m} = 0,\quad a^{(12)} + \frac{p'}{n'} = 0.$$

Similarly h will be in relation R to g with respect to N_z if and only if

$$(27)\qquad \beta^{(12)} - q = 0,\quad \alpha^{(12)} - q' = 0.$$

From (3b) and (2c) we find that

$$\alpha^{(12)} - q' = \frac{n'}{m}\left(a^{(12)} + \frac{p'}{n'}\right),$$

$$\beta^{(12)} - q = \frac{m}{n'}\left(b^{(12)} + \frac{p}{m}\right).$$

Hence *if N_y, N_z, and G are in relation C and if H is in relation R to G with respect to N_y it is in relation R to G with respect to N_z.*

From (16) and (17) we note that the focal points ρ_y and ρ_z coincide if and only if

$$m(b^{(12)}d^{(11)} - b^{(11)}d^{(12)}) + pd^{(11)} = 0,$$

$$\beta^{(12)}\delta^{(11)} - \beta^{(11)}\delta^{(12)} - q\delta^{(11)} = 0.$$

These conditions may be reduced by means of (3a) and (3b) to

$$(28)\qquad b^{(12)} + \frac{p}{m} = b^{(11)}\frac{d^{(12)}}{d^{(11)}},\quad b^{(12)} + \frac{p}{m} = b^{(11)}\frac{\delta^{(12)}}{\delta^{(11)}}.$$

---

* V. G. Grove, *A theory of a general net on a surface*, these Transactions, vol. 28 (1926), p. 495, footnote.

Similarly the focal points σ_y and σ_z coincide if and only if

$$a^{(12)} + \frac{p'}{n'} = a^{(22)}\frac{d^{(12)}}{d^{(22)}}, \quad a^{(12)} + \frac{p'}{n'} = a^{(22)}\frac{\delta^{(12)}}{\delta^{(22)}}. \tag{29}$$

We shall call nets N_y and N_z *coradial nets* if they have the same rays. Hence *coaxial nets in relation C are coradial if and only if h and g are in relation R. If the nets are in relation F they are coradial if and only if the two derived lines of g intersect h.* In order for nets which are non-conjugate and not coaxial to be coradial it is necessary that the asymptotic curves on the sustaining surfaces correspond.

We may readily verify that the lines in relation R to g with respect to N_y and N_z intersect h in the points

$$\left(a^{(12)} + \frac{p'}{n'}\right)r - \left(b^{(12)} + \frac{p}{m}\right)s, \quad n'\left(a^{(12)} + \frac{p'}{n'}\right)r - m\left(b^{(12)} + \frac{p}{m}\right)s \tag{30}$$

respectively. These points coincide if N_y and N_z are radial transforms. Suppose that neither of the points (30) coincide with r or s. The cross ratio of the points r, s and the points (30) is m/n'. Hence *if the lines in relation R to g meet h in distinct points, the cross ratio of these points and the points r and s, and the cross ratio of the points y, z, τ, τ' are equal.*

Similarly we may prove that if N_y and N_z are L' transforms the rays of N_y and N_z cut h in points forming with r and s the same cross ratio that y and z form with the focal points of g.

6. The transformations K_R

We shall call nets N_y and N_z in relation C K_R *transforms* if

$$\frac{\partial^2}{\partial u \partial v} \log R = 0, \tag{31}$$

wherein the cross ratio $(y, \tau', z, \tau) = m/n' = R$. If N_y and N_z are in relation F and are K_{-1} transforms, they are K transforms in the sense of Koenig* since corresponding surface points separate the focal points of g harmonically.

From (3b) we find that

$$\alpha_u^{(12)} - \beta_v^{(12)} = a_u^{(12)} - b_v^{(12)} + \frac{\partial^2}{\partial u \partial v} \log R.$$

Hence *if the line in relation R to g with respect to a net is harmonic to S_y the same property is enjoyed by any K_R transform of the given net.*

* Comptes Rendus, vol. 113 (1891), p. 1022. Eisenhart, *Surfaces*, p. 58.

Suppose that N_y and N_z are K_{-1} transforms. We find from the integrability conditions (2c) and $m+n'=0$, that

$$
(32)\qquad
\begin{aligned}
\frac{\partial}{\partial v}\log m + 2a^{(12)} + \frac{p'}{n'} &= q',\\
\frac{\partial}{\partial v}\log n' + 2b^{(12)} + \frac{p}{m} &= q.
\end{aligned}
$$

Differentiating (32) with respect to u and v respectively and subtracting, we find that

$$
2(a_u^{(12)} - b_v^{(12)}) + \frac{\partial}{\partial u}\left(\frac{p'}{n'}\right) - \frac{\partial}{\partial v}\left(\frac{p}{m}\right) + q_v - q_u' = 0.
$$

But from (2c)

$$
q_u' - q_v = (n' - m)d^{(12)};
$$

and from (3b)

$$
\frac{\partial}{\partial u}\left(\frac{p'}{n'}\right) - \frac{\partial}{\partial v}\left(\frac{p}{m}\right) = (\nu' - \mu)\delta^{(12)}.
$$

We have, therefore, the theorem of Koenig*: *K transforms have equal point invariants.*

Let now N_y and N_z be coradial nets in relation F. From (28) and (29) and (2c) it follows that

$$
\frac{\partial^2}{\partial u \partial v}\log R = 0.
$$

Hence *if N_y and N_z are coradial nets in relation F, they have equal point invariants.*†

The ray curves of N_y form a conjugate net‡ if

$$
(33)\qquad \frac{\partial}{\partial u}\left(a^{(12)} - \frac{a^{(22)}d^{(12)}}{d^{(22)}}\right) - \frac{\partial}{\partial v}\left(b^{(12)} - \frac{b^{(11)}d^{(12)}}{d^{(11)}}\right) = 0.
$$

The ray curves of N_z form a conjugate net if

$$
(34)\qquad \frac{\partial}{\partial u}\left(\alpha^{(12)} - \frac{\alpha^{(22)}\delta^{(12)}}{\delta^{(22)}}\right) - \frac{\partial}{\partial v}\left(\beta^{(12)} - \frac{\beta^{(11)}\delta^{(12)}}{\delta^{(11)}}\right) = 0.
$$

Let N_y and N_z be L transforms. Equation (34) may be written

---

* Comptes Rendus, vol. 113 (1891), p. 1022.

† Eisenhart, *Surfaces*, p. 68, exercise 16.

‡ See V. G. Grove, loc. cit.

$$\frac{\partial}{\partial u}\left(a^{(12)} - a^{(22)}\frac{d^{(12)}}{d^{(22)}}\right) - \frac{\partial}{\partial v}\left(b^{(12)} - b^{(11)}\frac{d^{(12)}}{d^{(11)}}\right) = -\frac{\partial^2}{\partial u \partial v}\log R.$$

Hence *if two nets N_y and N_z are L transforms in the relation of a transformation K_R, the ray curves of N_z will form a conjugate system if and only if the ray curves of N_y form a conjugate system.*

7. Pencils of conics in the tangent planes

Consider the points

$$r' = y_u - \lambda y, \quad s' = y_v - \mu y, \tag{35}$$

lying on the tangent lines at y to the curves of the net N_y. If use be made of (9), (35) may be written

$$r' = r - \left(\lambda + \frac{p}{m}\right)y, \quad s' = s - \left(\mu + \frac{p}{m}\right)y.$$

The pencil of conics tangent to the parametric tangents at r' and s' is therefore

$$2k_1x_2x_3 = \left[x_1 + \left(\lambda + \frac{p}{m}\right)x_2 + \left(\mu + \frac{p'}{n'}\right)x_3\right]^2, \quad x_4 = 0, \tag{36}$$

the tetrahedron of reference being y, r, s, z.

Similarly the pencil of conics tangent at

$$r'' = z_u - \lambda' z = mr - (\lambda' - q)z,$$
$$s'' = z_v - \mu' z = n's - (\mu' - q')z,$$

referred to the same tetrahedron is

$$2k_2x_2x_3 = \left[\frac{\lambda' - q}{m}x_2 + \frac{\mu' - q'}{n'}x_3 + x_4\right]^2, \quad x_1 = 0. \tag{37}$$

The conics (36) and (37) determine two involutions on the line h. These involutions will be identical if and only if

$$(p + \lambda m)^2(\mu' - q')^2 = (p' + \mu n')^2(\lambda' - q)^2. \tag{38}$$

Suppose that the points r', s', r'', s'' are the focal points on the lines on which they lie. From (16) and (17) it follows that

$$\begin{aligned} \lambda &= b^{(12)} - \frac{b^{(11)}d^{(12)}}{d^{(11)}}, \quad \lambda' = \beta^{(12)} - \frac{\beta^{(11)}\delta^{(12)}}{\delta^{(11)}}, \\ \mu &= a^{(12)} - \frac{a^{(22)}d^{(12)}}{d^{(22)}}, \quad \mu' = \alpha^{(12)} - \frac{\alpha^{(22)}\delta^{(12)}}{\delta^{(22)}}. \end{aligned} \tag{39}$$

Suppose now that N_y and N_z are L' transforms. Equations (38) under conditions (39) may be written

$$\left(p + mb^{(12)} - \frac{mb^{(11)}d^{(12)}}{d^{(11)}}\right)^2 \left(\frac{p' + n'a^{(12)}}{m} - \frac{n'a^{(22)}d^{(12)}}{md^{(22)}}\right)^2$$
$$= \left(p' + n'a^{(12)} - \frac{n'a^{(22)}d^{(12)}}{d^{(22)}}\right)^2 \left(\frac{p + mb^{(12)}}{n'} - \frac{mb^{(11)}d^{(12)}}{d^{(11)}}\right)^2.$$

Hence

$$m^2 - n'^2 = 0.$$

We may state our results as follows: *Let the parametric nets N_y and N_z be in relation L'. The pencils of conics tangent to the parametric tangents at their focal points determine the same involution on h if and only if L' is K_{-1}.* If the parametric nets are in relation F, we have the theorem of Eisenhart*:

The two pencils of conics tangent to the parametric tangents to the curves of nets in relation F determine the same involution on the line of intersection of corresponding tangent planes if and only if the nets are in relation K.

Let N_y and N_z be K_{-1} transforms in relation L'. The two conics (36) and (37) under conditions (39) have a point in common if and only if

$$k_1 = m^2k_2 + 2\left(\lambda + \frac{p}{m}\right)\left(\mu + \frac{p'}{n'}\right). \tag{40}$$

The pencil of quadrics determined by these conics under condition (40) is

$$\begin{aligned} &x_1^2 + \left(\lambda + \frac{p}{m}\right)^2 x_2^2 + \left(\mu + \frac{p'}{n'}\right)^2 x_3^2 + m^2x_4^2 + 2\left(\lambda + \frac{p}{m}\right)x_1x_2 \\ &+ 2\left(\mu + \frac{p'}{n'}\right)x_1x_3 + 2k_3x_1x_4 - 2\left[\left(\frac{p}{m} + \lambda\right)\left(\frac{p'}{m'} + \mu\right)\right. \\ &\left. + m^2k_2\right]x_2x_3 - 2m\left(\lambda + \frac{p}{m}\right)x_2x_4 + 2m\left(\mu + \frac{p'}{n'}\right)x_3x_4 = 0, \end{aligned} \tag{41}$$

the tetrahedron of reference being y, r, s, z. The two quadrics of the pencil (41) tangent to the line g are determined by the parameter values $k_3 = \pm m$. These values of k_3 are the parameter values of the cones of the pencil provided all the quadrics of the pencil are not cones. The coördinates of the vertices of the cones are for

* L. P. Eisenhart, *Conjugate systems with equal point invariants*, Annals of Mathematics, (2), vol. 18 (1916), p. 14.

$$
(42) \qquad \begin{aligned} k_3 &= m, & \left[m^2 k_2,\ 0,\ 2m\left(\lambda + \frac{p}{m}\right),\ k_1\right], \\ k_3 &= -m, & \left[m^2 k_2,\ 2m\left(\mu + \frac{p'}{n'}\right),\ 0,\ k_1\right]. \end{aligned}
$$

We readily verify that the first vertex of (42) lies on $\sigma_y\tau$, and the second vertex on $\rho_y\tau'$. The pencil (41) cuts the line g in an involution whose double points are the focal points of g. We may state our results as follows:

If N_y and N_z are K_{-1} transforms in relation L', any two of the conics (36) *and* (37) *with values of λ and μ given by* (39) *meeting on h determine a pencil of quadrics which cut g in an involution whose double points are the focal points of g. The two cones of the pencil are the quadrics tangent to g at the focal points of g. The vertices of the cones are on the lines joining corresponding focal points on the parametric tangents.*

If in the above theorem the restriction that N_y and N_z be L' transforms be removed, and the condition of being coaxial be imposed on the nets the remainder of the theorem is true. Also *let N_y and N_z be coaxial nets in relation C. It follows that the pencil of conics tangent to the parametric tangents at their focal points determine the same involution if and only if C is K_{-1}.*

Let now N_y and N_z be non-conjugate nets in relation C not necessarily L'. The wording of the above theorem for parametric values $\lambda = b^{(12)}$, $\mu = a^{(12)}$ then holds.

The vertices of the cones in question lie in this case on the derived lines of g.

MICHIGAN STATE COLLEGE,
EAST LANSING, MICH.