SECOND-ORDER DIFFERENTIAL SYSTEMS WITH INTEGRAL AND k-POINT BOUNDARY CONDITIONS*

BY

WILLIAM M. WHYBURN

Tamarkin† and C. E. Wilder‡ have considered the expansion problem for differential systems where the boundary conditions are of the integral and k-point types. Von Mises§ attacked the problem of proving existence and oscillation theorems for a special second-order equation with two integral conditions. However, von Mises did not carry his investigation to the point of establishing definite existence and oscillation theorems for his system. It is the purpose of the present paper to develop properties of second-order differential systems whose coefficients are Lebesgue summable functions and whose boundary conditions either contain definite integrals or apply to more than two points of the interval of definition.

1. Summable functions

Let $F(x)$ be any function that is defined on the finite interval $X: a \leqq x \leqq b$, and has the property that the Lebesgue integral $\int_a^b F(x)\,dx$ exists.

Let the interval X be divided into n subintervals $I_{in} = (x_{in},\ x_{i+1,n})$ of length Δ_{in} by means of the $n+1$ distinct points $a \equiv x_{0n},\ x_{1n},\ \cdots,\ x_{nn} \equiv b$. Let max $\Delta(n)$ represent the greatest Δ_{in} for each n. Then the sets of subdivision I_{in} represent a *fine* subdivision of X if $\lim_{n\to\infty} \max \Delta(n) = 0$. For a given set of subintervals I_{in} of X, $h_n(x)$ is a horizontal function of index n on X, if $h_n(x)$ has a constant value h_{in} for each value of x interior to I_{in}. The sum $\sum_{i=0,\cdots,n-1} h_{in}\Delta_{in}$ is called|| the *area* under $h_n(x)$ on X.

Theorem I. *There exists a sequence of horizontal functions $H_n(x)$ that is defined for every method of fine subdivision and has the following properties:*

(1) *For every method of fine subdivision of X, $\lim_{n\to\infty} H_n(x) = F(x)$ almost everywhere on X.*

---

* Presented to the Society, February 26, 1927, and April 15, 1927; received by the editors in May, 1927.

† *On some General Problems of the Theory*, etc., Petrograd, 1917.

‡ These Transactions, vol. 18 (1917), pp. 415–442.

§ *Festschrift Heinrich Weber*, Berlin, 1912, pp. 252–282.

|| Cf. Ettlinger, American Journal of Mathematics, vol. 48 (1926), pp. 216–222.

(2) *If ϵ is a given arbitrary positive constant, there exist positive numbers M_ϵ and N_ϵ such that, if F_n denotes the set of points for which $|H_n(x)|>M_\epsilon$, then for all $n>N_\epsilon$, $\sum_{(F_n)}|H_{in}|\Delta_{in}<\epsilon$.*

(3) *If A_n denotes the area under $H_n(x)$ for each n, then the limit as n increases indefinitely of A_n exists independent of the method of fine subdivision and this limit is the Lebesgue integral of $F(x)$ taken over X.*

A sequence of horizontal functions that is readily seen to have these properties is defined by

$$H_{in} = \left[\int_{x_{in}}^{x_{i+1,n}} F(x)dx\right]/\Delta_{in} \qquad (i = 1,2, \cdots, n).$$

THEOREM II. *If $h_n(x)$ is any sequence of horizontal functions that is defined for all methods of fine subdivision of X and has properties* (1) *and* (2) *of Theorem* I, *then $h_n(x)$ also has property* (3) *of that theorem.*

Theorem II follows from an application of the Duhamel-Moore theorem or the equivalent theorem given by Ettlinger.* The details of this proof are omitted here as the reader will have no difficulty in supplying them.

2. DIFFERENTIAL EQUATIONS WITH SUMMABLE COEFFICIENTS

Consider the pair of differential equations

$$\frac{dy}{dx} = K(x)z, \quad \frac{dz}{dx} = G(x)y, \tag{2.1}$$

where $K(x)>0$, $G(x)<0$, K and G are summable on X and have finite values at a and b.

THEOREM III. *If G/K is a non-increasing function of x on X and if c and $d(c<d)$ are consecutive† zeros of $y(x)$ on X, then $|z(c)|\leq|z(d)|$.*

Multiply the second equation of (2.1) by z to get $zz'=Gyy'/K$ almost everywhere on X. Since $z(x)$ is absolutely continuous‡ on X, $\int_c^d zz'dt$ exists and hence $I=\int_c^d yy'G/Kdt$ exists and $I=\int_c^d zz'dt=(1/2)[z^2(d)-z^2(c)]$. Since G/K is monotonic and finite valued at a and b, it is bounded on X and from this fact, together with the summability of K and G, follows the summability of G/K. Let the points $x_{0n}=c$, x_{1n}, $\cdots$, $x_{nn}=d$ define a *fine*

* Loc. cit., p. 216.

† A proof that y and z have only a finite number of zeros on X is given by Ettlinger, Proceedings of the National Academy of Sciences, vol. 12 (1926), p. 540.

‡ See Carathéodory, *Vorlesungen über reele Funktionen*, Berlin, 1918, p. 678.

subdivision of $\bar{X}$: $c \leqq x \leqq d$ and let $H_{in} = [\int_{x_{in}}^{x_{i+1,n}} (G/K)dt]/\Delta_{in}$. The function $H_n(x)$ is bounded on $\bar{X}$ and $\lim_{n\to\infty} H_n(x) = G(x)/K(x)$ almost everywhere on this interval.

Let $H^*(x)$ be defined† by $H^*_{in} = [y_{in}\int_{x_{in}}^{x_{i+1,n}} y'(t)dt]/\Delta_{in}$. The function $H^*(x)$ has all three of the properties of Theorem I with respect to the summable function yy'. If we define a new sequence of horizontal functions $h_n(x)$ by letting $h_{in} = H_{in}H^*_{in}$, we see that $h_n(x)$ has properties (1) and (2) of Theorem I with respect to the summable function $yy'G/K$. It follows from Theorem II that

$$I = \lim_{n\to\infty} \sum_{i=0}^{n-1} h_{in}\Delta_{in}.$$

Now

$$\Delta[H_{in}y_{in}^2] = H_{in}[\Delta y_{in}]^2 + 2H_{in}y_{in}\,\Delta y_{in} + y_{i+1,n}^2\,\Delta H_{in}$$

and

$$\sum_{i=0}^{n-1}\Delta[H_{in}y_{in}^2] = \sum_{i=0}^{n-1}H_{in}[\Delta y_{in}]^2 + 2\sum_{i=0}^{n-1}H_{in}y_{in}\Delta y_{in} + \sum_{i=0}^{n-1}y_{i+1,n}^2\Delta H_{in}.$$

Since $y_{0n} = y(c) = y_{nn} = y(d) = 0$, the left hand side of the foregoing equation is zero and $2\sum_{i=0,\cdots,n-1} H_{in}y_{in}\Delta y_{in} = S_n + s_n$, where $S_n = -\sum_{i=0,\cdots,n-1} H_{in}\cdot[\Delta y_{in}]^2$ and $s_n = -\sum_{i=0,\cdots,n-1}\Delta H_{in}y_{i+1,n}^2$. If we let $\bar{h}_{in} = \Delta y_{in}H_{in}\Delta y_{in}/\Delta_{in}$, we note immediately that $\bar{h}_n(x)$ has property (2) and $\lim_{n\to\infty} \bar{h}_n(x) = 0$ almost everywhere on $\bar{X}$. It follows from Theorem II that $\lim_{n\to\infty} S_n = 0$. Hence

$$2I = 2\lim_{n\to\infty} \sum_{i=0}^{n-1}h_{in}\Delta_{in} = 2\lim_{n\to\infty} \sum_{i=0}^{n-1}H_{in}y_{in}\,\Delta y_{in} = 2\lim_{n\to\infty} s_n.$$

Since $\Delta H_{in} \leqq 0$, it follows that $s_n \geqq 0$, and $z^2(d) - z^2(c) = 2I \geqq 0$, or $|z(d)| \geqq |z(c)|$. This completes the proof of Theorem III.‡

The foregoing argument proves that $\lim_{n\to\infty} s_n = \int_c^d (yy'G/K)dt$. An examination of s_n shows that the above limit is the Stieltjes integral $\int_c^d y^2\, d(G/K)$. Hence Stieltjes $\int_c^d y^2 d(G/K) =$ Lebesgue $\int_c^d (yy'G/K)dt$.

COROLLARY 1. *If G/K is a non-decreasing function of x on X, then* $|z(d)| \leqq |z(c)|$.

COROLLARY 2. *If G/K actually decreases over some sub-interval of $\bar{X}$, then*

$$|z(d)| > |z(c)|.$$

† $y_{in} = y(x_{in})$ and $\Delta y_{in} = y_{i+1,n} - y_{in}$. In general, $\Delta f_{in} = f_{i+1,n} - f_{in}$.

‡ If we add the requirement that $(G/K)'$ exist almost everywhere and be summable on X, this theorem follows from an application of partial integration.

COROLLARY 3. *If* $y(c)\neq 0$ *and* d *is the first zero of* $y(x)$ *such that* $d>c$ *then under the hypotheses of Theorem* III, $|z(c)|<|z(d)|$.

An examination of the proof for Theorem III shows that $I\geqq -(G/K)[y(c)]^2$ and this corollary follows.

The symmetry that exists between y and z yields the following:

THEOREM IV. *If* K/G *is a non-increasing function of* x *on* X *and if* e *and* f $(e<f)$ *are consecutive zeros of* $z(x)$ *on* X, *then* $|y(e)|\leqq|y(f)|$.

Theorem IV also yields corollaries analogous to these of Theorem III.

3. Differential systems with integral boundary conditions

Consider the differential system

$$\frac{dy}{dx}=K(x,\lambda)z,\qquad \frac{dz}{dx}=G(x,\lambda)y\,; \tag{3.1}$$

$$\alpha(\lambda)z(a,\lambda)-\beta(\lambda)y(a,\lambda)=0,\quad F(b,\lambda)=0\,; \tag{3.2}$$

where $F(x,\lambda)=\int_a^x A(t,\lambda)y(t,\lambda)dt$, K, G and A are continuous functions of λ on L: $L_1<\lambda<L_2$ for each fixed x on X, summable functions of x on X for each fixed λ on L, and bounded numerically for all values of x and λ on XL by a function $M(x)$ that is summable on X. The functions $\alpha(\lambda)$ and $\beta(\lambda)$ are continuous in λ on L. We suppose that $K(x,\lambda)$ is positive on XL and that the coefficients K, G, α, and β satisfy conditions that are sufficient to insure the validity of existence and oscillation theorems* for the system (3.1), (3.3), where

$$\alpha(\lambda)z(a,\lambda)-\beta(\lambda)y(a,\lambda)=0,\quad y(b,\lambda)=0. \tag{3.3}$$

Case I. $\alpha(\lambda)\neq 0$ on L.

Carathéodory's existence theorem† insures the existence of a pair of functions, $y(x,\lambda)$ and $z(x,\lambda)$, that are continuous in both variables on XL, absolutely continuous in x for each fixed λ on L, satisfy (3.1) almost everywhere on X, and satisfy the conditions $y(a,\lambda)=\alpha(\lambda)$, $z(a,\lambda)=\beta(\lambda)$ for every λ on L. Consider this pair of functions.

THEOREM V. *If for a fixed value of* λ *on* L, G/K *is negative and a non-increasing function of* x *on* X, *and* A/G *is positive and a non-decreasing function of* x *on* X, *then the zeros of* $F(x)$ *and the zeros of* $y(x)$ *separate each other on* X.

---

* See Bôcher, *Leçons sur les Méthodes de Sturm*, Paris, 1917, p. 66.

† Loc. cit., p. 678.

Since the zeros of $y(x)$ on X are simple and finite in number, we will let $x_1<x_2<\cdots<x_n$ be these zeros. Let $I_0=\int_a^{x_1}Ay\,dt$ and $I_i=\int_{x_i}^{x_{i+1}}Ay\,dt$. Clearly $I_jI_{j+1}<0$ for $j=0,1,\cdots,n-1$. Hence if we show that $|I_0|<|I_1|\leqq|I_2|\leqq|I_3|\leqq\cdots\leqq|I_i|\leqq\cdots\leqq|I_{n-1}|$, it will follow that $F(x_i)\cdot F(x_{i+1})<0$, and since $F(x)$ is a continuous, increasing (or decreasing) function of x on $x_i\leqq x\leqq x_{i+1}$, we can conclude that $F(x)$ has one and only one zero on the interval $x_i\leqq x\leqq x_{i+1}$. To show that $|I_i|\leqq|I_{i+1}|$, $i>0$, we consider two cases. In the first case let I_i be positive. We wish to show that $I_i\leqq -I_{i+1}$. Replace y in the integrands by its value from (3.1). Since $Ay=Az'/G$ almost everywhere on X, it follows that the integrals are not changed by this substitution. From the monotonic character of A/G,

$$\int_{x_i}^{x_{i+1}}(Az'/G)dt\leqq\frac{A(x_{i+1})}{G(x_{i+1})}\int_{x_i}^{x_{i+1}}z'dt\,;$$

$$-\int_{x_{i+1}}^{x_{i+2}}(Az'/G)dt\geqq-\frac{A(x_{i+1})}{G(x_{i+1})}\int_{x_{i+1}}^{x_{i+2}}z'dt,$$

and hence we will prove a stronger inequality if we show that

$$-\frac{A(x_{i+1})}{G(x_{i+1})}\int_{x_{i+1}}^{x_{i+2}}z'dt\geqq\frac{A(x_{i+1})}{G(x_{i+1})}\int_{x_i}^{x_{i+1}}z'dt,$$

or $-z(x_{i+2})+z(x_{i+1})\geqq z(x_{i+1})-z(x_i)$, which is the same as $-z(x_{i+2})\geqq -z(x_i)$. Since $z(x)$ vanishes once on $x_{i+1}<x<x_{i+2}$ (its zeros separate those of y) and z' is negative wherever it exists on this interval, we have $z(x_{i+2})<0$. A similar argument gives $z(x_i)<0$. An application of Theorem III yields $|z(x_{i+2})|\geqq|z(x_{i+1})|\geqq|z(x_i)|$ and hence $-z(x_{i+2})\geqq -z(x_i)$. This establishes our inequality for the case $I_i>0$. The case $I_i<0$ is treated in the same way to get $z(x_{i+2})\geqq z(x_i)$, where $z(x_{i+2})$ and $z(x_i)$ are both positive. The validity of this inequality is a consequence of Theorem III.

A similar argument to that given above reduces the proof that $|I_0|<|I_1|$ to an application of Corollary 3, Theorem III. This completes the proof of our separation theorem.

THEOREM VI. *If* $\lambda_0<\lambda_1<\lambda_2<\cdots$ *are the ordered characteristic numbers of the system* (3.1), (3.3) *and if the hypotheses of Theorem* V *are satisfied for every* λ *on* $\lambda_0\leqq\lambda<L_2$, *then there exists an infinite set of characteristic values,** $k_0, k_1, k_2,\cdots$, *for the system* (3.1), (3.2) *such that* $L_1<\lambda_0<k_0<\lambda_1<k_1<\cdots<L_2$.

* k_i may be an aggregate of a finite or infinite number of values.

$y(x, \lambda)$ and $z(x, \lambda)$ have been chosen so that the condition $\alpha(\lambda)z(a, \lambda) - \beta(\lambda)y(a, \lambda) = 0$ is satisfied for every λ on L. An implicit function theorem that was stated and proved "im kleinen" by Ettlinger* and extended by Sturdivant† to hold "im grossen," insures that the zeros of $F(x, \lambda)$ are continuous functions of λ on L. We extend the definitions of the coefficients of our system to apply to the interval $\overline{X}$: $a \leqq x \leqq b'$ $(b' > b)$ by making them have the same values outside of X that they have at $x = b$. For any fixed λ that is greater than (or equal to) λ_0, the zeros of F and y separate each other on X. Let us fix our attention on the ith zero that $F(x, \lambda)$ has on $a < x \leqq b'$ and let λ increase from λ_i to λ_{i+1}. If $x_i(\lambda)$ denotes this zero, then $x_i(\lambda_i) > b$ while $x_i(\lambda_{i+1}) < b$. Since $x_i(\lambda)$ is a continuous function, there must be at least one value of λ between λ_i and λ_{i+1} for which $x_i(\lambda) = b$. Let k_i be the set of all values of λ for which $x_i(\lambda) = b$. It follows from the continuity of $x_i(\lambda)$ and the separation of the zeros of F and y, that $\lambda_i < k_i < \lambda_{i+1}$.

THEOREM VII. *If p_i is any value of λ that belongs to k_i, then, on $a < x < b$, $F(x, p_i)$ has exactly i zeros, $y(x, p_i)$ has exactly $i+1$ zeros, and $z(x, p_i)$ has either i, $i+1$, or $i+2$ zeros.*

Since $y(a, \lambda) \neq 0$, it follows that no zeros of y or of F are lost from X. Since one and only one zero of y enters the X interval every time λ passes a characteristic value of the system (3.1), (3.3) a direct count reveals the validity of this theorem for $y(x, p_i)$. The remainder of the theorem follows from the separation of the zeros of $F(x, p_i)$ and $z(x, p_i)$, respectively, by the zeros of $y(x, p_i)$.

Case II. $\alpha(\lambda) \equiv 0$ and for every λ on L there exists a neighborhood of $x = a$ that is a subset of X, throughout which one of the quantities A/G, K/G actually increases as x increases.

The treatment of this case is the same as that given for Case I. The hypothesis that either A/G or K/G actually increase over some sub-interval of X that has a for one end point enables us to prove the same separation, existence, and oscillation theorems that were proved in Case I.

Case III. $\alpha(\lambda) \equiv 0$ and A/G and K/G are constant throughout X for every fixed λ on L.

In this case the zeros of F coincide with zeros of y. If $\lambda_0, \lambda_1, \lambda_2, \cdots$ are the characteristic numbers of the system (3.1), (3.3), then the characteristic numbers of the system (3.1), (3.2) are $\lambda_1, \lambda_3, \lambda_5, \cdots$. The function $y(x, \lambda_i)$ has exactly i zeros on $a < x < b$ while $F(x, \lambda_i)$ has $(i-1)/2$ zeros on this interval.

---

* *Existence theorems for implicit functions*, etc., Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 315–318.

† These Transactions, vol. 30 (1928), pp. 560–566.

Now consider the system (3.1) together with the conditions

$$\begin{aligned} \alpha(\lambda)z(a,\lambda)-\beta(\lambda)y(a,\lambda)&=0, \\ V(b,\lambda)&=0, \end{aligned} \tag{3.4}$$

where $V(b, \lambda)=\int_a^x B(t, \lambda)\, z(t, \lambda)dt$, $B(x, \lambda)$ is summable in x on X for each fixed λ on L, continuous in λ on L for each fixed x on X, and bounded numerically for all x and λ on XL by a function $M(x)$ that is summable on X. The functions $\alpha(\lambda)$ and $\beta(\lambda)$ are continuous in λ on L and the coefficients of the system (3.1), (3.4) are assumed to satisfy conditions that are sufficient to insure the validity of existence and oscillation theorems for the system (3.1), (3.5), where

$$\alpha(\lambda)z(a,\lambda) - \beta(\lambda)y(a,\lambda) = 0, \quad z(b,\lambda) = 0. \tag{3.5}$$

The symmetry that exists between y and z together with a repetition of the arguments used in treating system (3.1), (3.2) establishes the validity of separation, existence, and oscillation theorems for system (3.1), (3.4). These theorems are analogous to Theorems V, VI, and VII. The hypotheses on G/K and A/G are replaced by the hypotheses that B/K and $-G/K$ be positive and non-decreasing functions of x on X for each fixed λ on L.

Let

$$\begin{aligned} \phi(x,\lambda) &= \alpha(x,\lambda)z(x,\lambda) - \beta(x,\lambda)y(x,\lambda), \\ \psi(x,\lambda) &= \gamma(x,\lambda)z(x,\lambda) - \delta(x,\lambda)y(x,\lambda), \end{aligned} \tag{3.6}$$

where α, β, γ, and δ are absolutely continuous functions of x on X for each fixed λ on L and continuous in λ on L for each fixed x on X. We suppose that $\alpha\delta-\beta\gamma\neq 0$ on XL and that this quantity has been made identically equal to 1 by dividing through by a non-vanishing divisor. Let $\{\alpha\beta\}=\alpha\beta'-\beta\alpha'-\alpha^2G+\beta^2K$, $\{\gamma\delta\}=\gamma\delta'-\delta\gamma'-\gamma^2G+\delta^2K$, and $\{\alpha\beta\gamma\delta\}=\beta\gamma'-\alpha\delta'-\beta\delta K+\alpha\gamma G$. We suppose that $\{\alpha\beta\}$ and $\{\gamma\delta\}$ are bounded numerically for all x and λ on XL by a summable function of x. Equations (3.1) and (3.6) yield

$$\begin{aligned} \phi' &= -\{\alpha\beta\gamma\delta\}\phi - \{\alpha\beta\}\psi, \\ \psi' &= \{\gamma\delta\}\phi + \{\alpha\beta\gamma\delta\}\psi. \end{aligned} \tag{3.7}$$

Let $\bar\phi, \bar\psi$, H, and J be defined by

$$\bar\phi = \phi e^{\omega}, \quad \bar\psi = \psi e^{-\omega}, \quad H = -\{\alpha\beta\}e^{2\omega}, \quad J = \{\gamma\delta\}e^{-2\omega},$$

where

$$\omega = \int_a^x \{\alpha\beta\gamma\delta\}\,dt.$$

We note that H and J, as functions of x and λ, have the same properties of summability and continuity that K and G have. From (3.7) we get

$$\bar{\phi}' = H(x,\lambda)\bar{\psi}, \qquad (3.8)$$
$$\bar{\psi}' = J(x,\lambda)\bar{\phi}.$$

Consider the system (3.1), (3.9), where

$$\phi(a,\lambda) = 0, \qquad (3.9)$$
$$W(b,\lambda) = 0,$$

and $W(x, \lambda) = \int_a^x C(t, \lambda)\phi(t, \lambda)dt$, where $C(x, \lambda)$ is continuous in λ on L for each fixed x on X, summable in x on X for each fixed λ on L, and bounded numerically by a summable function of x. If we impose the same conditions on H, J, and $We^{-\omega}$ that are imposed on K, G, and A, respectively, in treating system (3.1), (3.2), we get existence and oscillation theorems for the system (3.8), (3.9). Since the zeros of ϕ coincide with those of $\bar{\phi}$, we get the same existence and oscillation theorems for system (3.1), (3.9).

As special cases of the foregoing theory we get existence and oscillation theorems for many non-self-adjoint* systems. If we let $A(x, \lambda) \equiv G(x, \lambda)$ and $\alpha \neq 0$ in system (3.1), (3.2), we get existence and oscillation theorems for the system (3.1), $\alpha(\lambda)z(a, \lambda) - \beta(\lambda)y(a, \lambda) = 0$, $z(a, \lambda) = z(b, \lambda)$. Similarly, if we let $B \equiv K$ and $\beta(\lambda) \neq 0$ in system (3.1), (3.4) we get another non-self-adjoint system.

As an illustration of the foregoing work, we give the following example:

$$y' = \lambda xz, \quad z' = -\lambda xy, \quad y(0,\lambda) = 0,$$

$$\int_0^\pi \lambda t^3 y \, dt = 0.$$

Let λ be restricted to the interval $0 < \lambda < \infty$. We note that the solution of the first three equations of this system is $y = \sin(\lambda x^2/2)$, $z = \cos(\lambda x^2/2)$. The characteristic numbers of the system are the roots of the equation $\tan(\lambda\pi^2/2) = \lambda\pi^2/2$.

* See Bôcher, loc. cit., p. 39.

4. Differential systems with boundary conditions at k points

Let $a_1, a_2, \cdots, a_k, d$ be $k+1$ distinct points of X such that $a \leqq a_1 < a_2 < \cdots < a_k \leqq b$, $a_1 < d \leqq b$. Consider the system (3.1), (4.1), where

$$\sum_{i=1}^{i=k} \alpha_i(\lambda) y(a_i, \lambda) = 0, \quad y(d, \lambda) = 0, \tag{4.1}$$

and $\alpha_1, \alpha_2, \cdots, \alpha_k$ are continuous functions of λ on L.

Theorem VIII. *If for a fixed λ on L, either $K(x) \geqq 0$ on X, or $K(x) \leqq 0$ on X and in either case the equality sign does not hold over more than a null set, then $y(x)$ can have neither a maximum nor a minimum at a point of $a < x < b$ where $z(x)$ is different from zero.*

Let c be a point of $a < x < b$ for which $z(c) \neq 0$. Since y and z are absolutely continuous functions of x on X, we can determine a positive number h such that the interval $c-h \leqq x \leqq c+h$ is a subset of X and for every x on this interval $z(x) > 0$ (or $z(x) < 0$). Now

$$y(x) = \int_a^x Kz\,dt + C = \int_{c-h}^x Kz\,dt + \bar{C}, \quad C \text{ and } \bar{C} \text{ constants},$$

is an increasing (or decreasing) function of x on $c-h \leqq x \leqq c+h$, since $Kz > 0$ (or $Kz < 0$) everywhere on this interval with the exception of a null set. It follows that $y(c)$ can be neither a maximum nor a minimum of $y(x)$.

A similar argument to the above proves that under the hypothesis that $G(x)$ be negative on X, with the possible exception of a null set of points, $z(x)$ can have extremums only at the ends of the interval X and at the points where $y(x) = 0$.

Let $y_1(x, \lambda)$, $z_1(x, \lambda)$ be the solution of system (3.1) such that $y_1(a, \lambda) = 0$, $z_1(a, \lambda) = 1$ for all values of λ on L, and let $y_2(x, \lambda)$, $z_2(x, \lambda)$ be the solution of that system that satisfies the conditions $y_2(a, \lambda) = 1$, $z_2(a, \lambda) = 0$ for all values of λ on L. Since these two solutions are linearly independent, the general solution of (3.1) can be written as a linear combination of them. Let c be any point of the interval $a < x \leqq b$; then

$$\begin{aligned} y(x,\lambda) &= y_2(x,\lambda) y_1(c,\lambda) - y_1(x,\lambda)[y_2(c,\lambda) + 1], \\ z(x,\lambda) &= z_2(x,\lambda) y_1(c,\lambda) - z_1(x,\lambda)[y_2(c,\lambda) + 1] \end{aligned} \tag{4.2}$$

is the solution of (3.1) that satisfies the condition $y(a, \lambda) + y(c, \lambda) = 0$ for every λ on L. Concerning y and z as given by (4.2) we have

THEOREM IX. *If for a fixed value of* λ *on* L, $K(x)$ *is positive*, $G(x)$ *is negative*, K/G *is a non-decreasing function of* x *on* X *and there exists a subinterval*, $a \leqq x \leqq h$, *of* X *over which* K/G *actually increases, then the zeros of* $y(x)$ *and* $y_2(x)$ *separate each other on* X.

The corollary to Theorem IV that is the analogue of Corollary 1 to Theorem III, together with the hypothesis that K/G actually increase throughout a neighborhood of a, yields

$$| y_2(a) | > | y_2(x_1) | \geqq | y_2(x_2) | \geqq \cdots \geqq | y_2(x_j) | ,$$

where $a, x_1, x_2, \cdots, x_j$ are the ordered zeros of $z_2(x)$ on X. Theorem VIII shows that $y_2(x)$ attains its only extremums at the zeros of $z_2(x)$, hence $|y_2(x)|<1$ and $y_2(c)+1>0$ for every x on $a<x\leqq b$. Since y_1 and y_2 cannot both vanish at the same point, it follows from equations (4.2) and the inequality $y_2(x)+1>0$ that $y(x)$ and $y_2(x)$ cannot both vanish at the same point. Let r and s be any two zeros of $y_2(x)$ on X and assume that $y(x)$ does not vanish for any x on $\overline{X}$: $r\leqq x\leqq s$. The function y_2/y is continuous on $\overline{X}$ and has a summable derivative that is given by

$$\frac{d}{dx}(y_2/y) = KW(y_1,y_2)[y_2(c) + 1]/y^2$$

almost everywhere on X, where $W(y_1, y_2)=y_2z_1-y_1z_2$ is the wronskian of y_1 and y_2 and is positive on X. If we write y_2/y as the indefinite integral of this derivative, we note that y_2/y is an increasing function of x on $\overline{X}$ and hence cannot vanish at both $x=r$ and $x=s$. This contradicts our hypothesis that r and s were zeros of y_2 and proves that between every two zeros of $y_2(x)$ there is at least one zero of $y(x)$. A similar argument, when applied to y/y_2, shows that between every pair of zeros of $y(x)$ there is a zero of $y_2(x)$ and thus completes the proof of our theorem.

The foregoing argument, together with an examination of (4.2), shows that when $y_1(c)>0$, $y(x)$ vanishes once between $x=a$ and the first zero of $y_2(x)$ on X and when $y_1(c)<0$, $y(x)$ does not vanish between $x=a$ and the first zero of $y_2(x)$ on X.

Let the coefficients of system (3.1) satisfy conditions that are sufficient to insure the existence of characteristic numbers, $\lambda_0, \lambda_1, \lambda_2, \cdots$ for the system (3.1), $y(a, \lambda)=1$, $y(d, \lambda)=0$, and let the hypotheses of Theorem IX be satisfied for every fixed λ on $\lambda_0\leqq\lambda<L_2$.

THEOREM X. *There exists an infinite set of characteristic values*, $k_1, k_2, \cdots$, *for the system* (3.1), (4.3) *such that* $\lambda_0<k_1<\lambda_1<k_2<\cdots$, *where*

$$y(a,\lambda) + y(c,\lambda) = 0, \quad y(d,\lambda) = 0. \tag{4.3}$$

The solution $y(x, \lambda)$, $z(x, \lambda)$ given by (4.2) satisfies the first condition of (4.3) for all values of λ on L. Ettlinger's† implicit function theorem can be applied to show that the zeros of $y(x, \lambda)$ are continuous functions of λ on L. Let the interval X^*: $a \leqq x \leqq d$ be extended in the manner indicated under Theorem VI and let x_i be the zero of y that lies between the ith and $(i+1)$st zeros of y_2 on this extended interval. Now $x_i(\lambda_{i-1}) > d$ and $x_i(\lambda_i) < d$, since the zeros of y_2 and y move onto the interval X^* at $x = d$. For at least one value of λ between λ_{i-1} and λ_i, $x_i(\lambda) = d$. Let k_i be the set of all values of λ for which $x_i(\lambda) = d$. Since the motion of the zeros of y_2 is monotonic, it follows that $\lambda_{i-1} < k_i < \lambda_i$.

THEOREM XI. *If p_i is any value of λ in k_i, $y_2(x, p_i)$ has i zeros on $a < x < d$ while $y(x, p_i)$ has i zeros on this interval if $y_1(c, p_i) > 0$ and $i-1$ zeros on it if $y_1(c, p_i) < 0$.*

Since y_2 cannot lose any zeros at $x = a$ or at any other point of X^*, we get the first part of this theorem by a count of the zeros of y_2 as they enter X^* at $x = d$. The remainder of the theorem follows from Theorem IX.

Now consider the system (3.1), (4.1). Let all of the α's be positive and let $\alpha_1 \geqq \sum_{i=2,\dots,k} \alpha_i$. The solution

$$y(x,\lambda) = y_2(x,\lambda) \sum_{i=1}^{i=k} \alpha_i(\lambda) y_1(a_i,\lambda) - y_1(x,\lambda) \sum_{i=1}^{i=k} \alpha_i(\lambda) y_2(a_i,\lambda),$$

$$z(x,\lambda) = z_2(x,\lambda) \sum_{i=1}^{i=k} \alpha_i(\lambda) y_1(a_i,\lambda) - z_1(x,\lambda) \sum_{i=1}^{i=k} \alpha_i(\lambda) y_2(a_i,\lambda)$$

satisfies the first condition of (4.1) for all values of λ on L. The condition $\alpha_1 \geqq \sum_{i=2,\dots,k} \alpha_i$ insures that the coefficient of $y_1(x, \lambda)$ in $y(x, \lambda)$ is positive and this enables us to prove the same separation theorem that was proved in the preceding case.

If we replace a by a_1 in the treatment of system (3.1), (4.3) and repeat the treatment given for this system, we get the same separation, existence, and oscillation theorems for system (3.1), (4.1) that were obtained in Theorems IX, X, and XI.

It is evident that the work of this section can be applied to treat systems where z or ϕ, a linear combination of y and z, replaces y in the boundarv conditions (4.1).

---

† Ettlinger, loc. cit.

UNIVERSITY OF TEXAS,
AUSTIN, TEXAS