ANALYTIC FUNCTIONS OF HYPERCOMPLEX VARIABLES*

BY

P. W. KETCHUM

1. **Introduction.** Scheffers† has shown that many of the important theorems of monogenic functions of complex variables may be extended to functions of hypercomplex variables in algebras which are commutative and associative. Autonne‡ has studied monogenic functions of variables in non-commutative algebras. Berloty,§ basing his work on previous results of Weierstrass,‖ has also studied functions of hypercomplex variables, but his conclusions only hold for variables in Weierstrass algebras. Bechk-Widmanstetter¶ has attempted to make a practical application of Scheffer's results by trying to solve Laplace's equation in three variables.

The purpose of the present paper is to complete the work of Scheffers be extending to functions of hypercomplex variables all the important elementary theorems of complex variables, as far as is possible. Also Bechk-Widmanstetter's work is revised and extended by a study of the general problem of the solution of differential equations by analytic functions of hypercomplex variables.

2. **Preliminary definitions.** The general hypercomplex variable will be denoted by

$$(1)\qquad w = \sum_{i=1}^{n} w_i e_i ;\ e_i e_j = \sum_{s=1}^{n} \gamma_{ijs} e_s \qquad (i,j = 1, \cdots, n),$$

where the units e_i are linearly independent with respect to the complex domain, the constants γ_{ijs} are real numbers, and, except in the sections on differential equations, the w's are complex variables.

For simplicity of notation, whenever a summation is to be taken over all possible values the limits will be omitted. The index over which the sum is taken may also be omitted from under the summation sign if every index is

* Presented to the Society, September 7, 1928; received by the editors May 11, 1927.

† Leipziger Berichte, vol. 45 (1893), p. 828.

‡ Journal de Mathématiques, (6), vol. 3 (1907), p. 55. See also Encyclopédie des Sciences Mathématiques, vol. 1, part 1, p. 441; Hedrick and Ingold, these Transactions, vol. 27 (1925), p. 551.

§ Thesis, Paris, 1886.

‖ Göttinger Nachrichten, 1884, p. 395.

¶ Monatshefte für Mathematik und Physik, vol. 23 (1912), p. 257.

to be summed, or in other cases where there is no ambiguity. Also the bracket expression showing for what values of the subscripts a relation holds (as shown in equation (1)) will be omitted when every subscript not connected with a summation is to be taken for all possible values.

Multiplication is always distributive. It is commutative if and only if

$$\gamma_{iks} = \gamma_{kis}, \tag{2}$$

and associative if and only if

$$\sum_i \gamma_{iks}\gamma_{lji} = \sum_i \gamma_{lis}\gamma_{jki}. \tag{3}$$

Only those algebras are considered which possess a modulus or principal unit, ϵ. A modulus exists if and only if

$$\sum \gamma_{iks}\epsilon_k = \delta_{is}\,; \qquad \sum \gamma_{iks}\epsilon_i = \delta_{ks}, \tag{4}$$

where δ_{is} is Kronecker's symbol.

Division, defined as the inverse of multiplication, cannot fail for every number of the algebra. In particular it cannot fail for the modulus, the modulus being its own inverse. In a non-commutative algebra there are two kinds of division called right and left hand division. If there is some number which is not divisible by α, then α is called a nilfactor. A number α is a nilfactor if and only if

$$\Delta_\alpha = \left| \sum_i \alpha_i\gamma_{iks} \right| = 0, \text{ or } \Delta'_\alpha = \left| \sum_i \alpha_i\gamma_{kis} \right| = 0. \tag{5}$$

3. **Algebraic properties.** Geometric representation of hypercomplex numbers is the same as for complex variables, but the former are vectors in $2n$ instead of in 2 dimensions. Addition is done vectorially.

We have, as for complex numbers, the inequality for any two hypercomplex numbers

$$\big|\,|\alpha| - |\beta|\,\big| \leqq |\alpha + \beta| \leqq |\alpha| + |\beta|.$$

Also,

$$\sum |\alpha_i| \leqq n^{1/2}|\alpha|.$$

Hypercomplex numbers differ from complex numbers in that the factor law is not true in general; that is, if the product of two numbers, α and β, vanishes, it does not necessarily follow as in complex variables that either α or β vanish. In fact the theorem in complex numbers that the absolute value of the product (or quotient) of two numbers equals the product (or quotient) of their absolute values, does not hold in general for hypercomplex numbers.

We can, however, establish an upper bound to the absolute value of the product as follows:

We have the relation

$$|(\alpha\beta)_s| \leqq \sum_{ij} |\gamma_{ijs}\alpha_i\beta_j| .$$

Hence

$$(6) \qquad |(\alpha\beta)_s| \leqq \sum_{ij} |\gamma_{ijs}| \cdot |\alpha| \cdot |\beta| .$$

Also

$$(7) \qquad |\alpha\beta| \leqq \sum_{ijs} |\gamma_{ijs}| \cdot |\alpha_i| \cdot |\beta_j| \leqq \sum_{ijs} |\gamma_{ijs}| \cdot |\alpha| \cdot |\beta| .$$

There is no simple generalization of the fact that in complex numbers the amplitude of the product (or quotient) of two numbers is the sum (or difference) of their respective amplitudes. Hence there is in general no simple rule for the calculation of powers of a hypercomplex number corresponding to DeMoivre's theorem. The ratio of the product of the absolute values to the absolute value of the product of two numbers, and the direction of the product depend only on the direction of the two numbers and not on their absolute values.

4. **Elementary functional properties.** The terms function (denoted by $f(w)$), conjugate functions, limit,* continuity, uniform continuity, are defined as in complex variables so that theorems dealing with such quantities hold for hypercomplex variables as for complex variables.†

The terms series, power series,‡ convergence, divergence, absolute convergence, uniform convergence, double series, are defined as in complex variables, and many of the elementary theorems based on these quantities hold for hypercomplex variables as for complex variables.§

A function is said to be *analytic at a point* α, if it can be expressed as the sum of a power series in $(w-\alpha)$ in the neighborhood of α. An *analytic func-*

* The two fundamental theorems on the sum and product of limits follow for hypercomplex variables as for complex variables. The fundamental theorem for a quotient also holds if the limit of the divisor is not a nilfactor.

† E.g. in Townsend, *Functions of a Complex Variable*, every theorem of Chap. II holds for hypercomplex variables as for complex variables.

‡ In case the algebra is not associative it may be necessary to distinguish different kinds of powers depending on the order of multiplication. In that case there would be different kinds of power series to correspond to the different kinds of powers, and the position of the constants.

§ E.g. in Townsend, loc. cit., all theorems in §§42–45 inclusive hold for hypercomplex variables. The proof of the first part of Theorem III, p. 208, is slightly changed due to the necessity of using equation (6) above. It is also to be noted that nilfactors in hypercomplex variables correspond to zero in complex variables so that β_1 cannot be a nilfactor in Theorem IV, p. 212.

tion is defined as the sum of a given power series together with all the values which can be obtained by analytic continuation of the conjugate functions of that series.

If the algebra is not commutative there will be two difference quotients defined as follows:

$$\begin{aligned} \Delta w \cdot F_r(w, \Delta w) &= f(w + \Delta w) - f(w) = \Delta f, \\ F_l(w, \Delta w) \cdot \Delta w &= f(w + \Delta w) - f(w) = \Delta f. \end{aligned} \tag{8}$$

From these we obtain two derivatives, $f'(w)$ and $'f(w)$ respectively, on taking limits as $\Delta w \to 0$.*

Differentials are defined as in complex variables so that

$$dwf' = df = 'fdw.$$

If dw is taken equal to ϵ it follows that $f' = 'f$ so that right and left hand derivatives must be equal and a distinction between them is no longer necessary.

The integral of a function, which is continuous along an arc C, along C, is defined as in complex variables by the relation

$$\begin{aligned} \int_C f(w)dw &= \lim_{\substack{m \to \infty \\ \Delta_k w \to 0}} \sum_{k=1}^{m} f(\zeta_k)\Delta_k w \\ &= \sum_s e_s \int_C \sum_{ij} f_i \gamma_{ijs} dw_j. \end{aligned} \tag{9}$$

The geometric interpretation of integration is the same as in complex variables, and most of the elementary theorems hold without change.† Also,

$$\left| \int_\alpha^\beta f(w)dw \right| \leqq \sum_{ijs} |\gamma_{ijs}| \int_\alpha^\beta |f(w)| \cdot |dw| \leqq \sum_{ijs} |\gamma_{ijs}| ML, \tag{10}$$

where M is the maximum value of $f(w)$, on C, and L is the length of C.

Hypercomplex numbers can be represented by points in a $2n$-dimensional space. A functional relation denotes a one-to-one correspondence between the points of one space (the w space) and the points of all or part of another space (the f space). A function will now be defined as monogenic in a $2n$-dimensional region τ if it possesses a derivative for every value of w in τ, and if the conjugate functions are analytic in τ. An inner point of such a re-

* Compare Scheffers, loc. cit. It is easily shown that the definitions are equivalent.

† E.g., in Townsend, loc. cit., Ex. 1, p. 61; equations 1, 3, 4, 6, pp. 62, 63; equation 5, p. 65, hold for hypercomplex variables.

gion τ is called a *regular point.* All points which are not regular are called *singular points.* A *monogenic function* is a function which is monogenic in some region, τ, and which is defined outside of τ by analytic continuation of the conjugate functions. As an immediate consequence of these definitions it follows that *α is a singular point of $f(w)$ if and only if $(\alpha_1, \cdots, \alpha_n)$ is a singular point of at least one of the conjugate functions.*

5. **Derivatives.** As in real variables the derivative of a constant is zero, of w is ϵ, and of a sum is the sum of the separate derivatives. Also w^2 has the derivative $2w_0$ at w_0 if and only if w_0 is commutative with all numbers of the algebra. Hence $f(w) = w^2$ possesses a derivative equal to $2w$ where and only where w is commutative with every number of the algebra.* But in order for such values of w to form a $2n$-dimensional region τ the algebra would have to be commutative. Hence *in a non-commutative algebra w^2 is not monogenic.*

In view of this fundamental difference between variables in commutative and non-commutative algebras we shall confine our attention to those that are commutative.†

The function w^3 has a derivative equal to $3w_0^2$ at w_0 if and only if $w_0(ww_0) = ww_0^2$ for every value of w. This condition would certainly be satisfied if the algebra were associative.

Scheffers (loc. cit.) has shown that *for a function to be monogenic in a region τ, it is necessary and sufficient that the conjugate functions f_i satisfy the following equations*; for all values of w in τ:

$$\frac{\partial f_s}{\partial w_k} = \sum f_i' \gamma_{iks}, \tag{11}$$

or the equations

$$\frac{\partial f_s}{\partial w_k} = \sum \epsilon_j \gamma_{iks} \frac{\partial f_i}{\partial w_j} \cdot \tag{12}$$

He has also shown that all the derivatives of a monogenic function exist and are themselves monogenic.

6. **Associativity.** If a is any number of an associative algebra,

$$(af')dw = a(f'dw),$$

or

$$\sum a_j f_i' dw_k \gamma_{jli} \gamma_{iks} e_s = \sum a_j f_i' dw_k \gamma_{lki} \gamma_{jis} e_s.$$

* Compare Scheffers, loc. cit.

† See, however, §24 on restricted variables for further results in non-commutative algebras.

If f is monogenic this relation must hold for all values of $a_j dw_k$ so that

$$\sum_{il} f_l' \gamma_{jli}\gamma_{iks} = \sum_{il} f_l' \gamma_{lki}\gamma_{jis}.$$

By (11) this becomes

$$\sum_i \gamma_{iks}\frac{\partial f_i}{\partial w_j} = \sum_i \gamma_{jis}\frac{\partial f_i}{\partial w_k}. \tag{13}$$

Furthermore, if (13) is multiplied by ϵ_j and summed over j, equations (12) are obtained.

Hence, *in an associative algebra, in order for a function to be monogenic in a region τ it is necessary and sufficient that the conjugate functions satisfy* (13) *for all values of w in τ.**

7. **Power series expansion of monogenic functions.** Consider any function $f(w)$, which is monogenic in a region τ, whose conjugate functions are homogeneous and of the rth degree (r a positive integer) in the variables w_i. By Euler's formula,

$$\sum_k w_k\frac{\partial f_i}{\partial w_k} = rf_i. \tag{14}$$

Multiplying (11) by w_k and e_s and summing with respect to k and s gives, by virtue of (14), $f'w = rf$. But f' is also monogenic and its conjugate functions are homogeneous of the $(r-1)$st degree. Hence $f''w=(r-1)f'$. Continuing this process we get

$$r!f = [f^{(r)}w^r] \text{ where } [f^{(r)}w^r] = (\cdots((f^{(r)}w)w)\cdots w).$$

But $f^{(r+1)}\, w=0$ for every value of w in τ, so that $f^{(r)}$ is a constant. Hence for values of w in τ,

$$f = [\beta w^r],$$

where β is a constant.

Now let $F(w)=\sum F_i e_i$ be any function which is monogenic in the neighborhood of a point α. Since the conjugate functions are analytic they can be expanded in power series in the variables $\{w_i-\alpha_i\}$. Thus

$$F_i = \sum_{m=0}^{\infty} f_{mi},$$

where f_{mi} are homogeneous functions of degree m in $\{w_i-\alpha_i\}$. These series

* Proved in a different manner by Scheffers (loc. cit.)

converge in the neighborhood of $(\alpha_1, \cdots, \alpha_n)$ and the derived series will also converge in the neighborhood of that point. Hence

$$\frac{\partial F_i}{\partial w_k} = \sum_{m=0}^{\infty} \frac{\partial f_{mi}}{\partial w_k}.$$

But from (12) we have in the neighborhood of $(\alpha_1, \cdots, \alpha_n)$,

$$\sum_{m=0}^{\infty} \frac{\partial f_{ms}}{\partial w_k} = \sum_{ij} \sum_{m=0}^{\infty} \gamma_{iks} \epsilon_j \frac{\partial f_{mi}}{\partial w_j}.$$

Equating terms of like degree,

$$\frac{\partial f_{ms}}{\partial w_k} = \sum \gamma_{iks} \epsilon_j \frac{\partial f_{mi}}{\partial w_j}.$$

Hence by (12), $f_m = \sum f_{mi} e_i$ are monogenic functions in the neighborhood of the point $(\alpha_1, \cdots, \alpha_n)$. But the conjugate functions f_{mi} are homogeneous and of degree m in $\{w_i - \alpha_i\}$, so by the theorem just proved $f_m = [\beta_m (w-\alpha)^m]$. Hence for values of w in the neighborhood of α,

$$F = \sum_{m=0}^{\infty} [\beta_m (w - \alpha)^m]. \tag{15}$$

This shows that any function which is monogenic in the neighborhood of a point α can be expanded in a power series in $(w-\alpha)$ which converges absolutely in some region about α. Therefore *if a function is monogenic in a given region it is also analytic in that region.**

8. **Transformation of the units.** A transformation of the units is defined by the relation

$$E_i = \sum \alpha_{ij} e_j, \qquad |\alpha_{ij}| \neq 0,$$

where α_{ij} are arbitrary complex constants. In applying such a transformation to the general variable w, we require that

$$W = \sum W_i E_i = \sum w_j e_j = w. \tag{16}$$

From this relation the variables W_i can be found as linear functions of the variables w_j. Since $W = w$, $f(W) = f(w)$ and $F(W)$ has a derivative equal to $f'(w)$ wherever $f(w)$ is monogenic. Also the conjugate functions of $f(W)$ are linear functions of the conjugate functions of $f(w)$ so that they are analytic whenever $f(w)$ is monogenic. Hence, *under linear transformation of the units, monogenic functions remain monogenic.*

* Proved by Scheffers (loc. cit., p. 846) for an associative algebra.

The following is an important transformation:

$$(17) \qquad E_1 = \epsilon = \sum \epsilon_i e_i\,; \qquad E_j \text{ arbitrary for } j \neq 1.$$

9. **Linear independence of derivatives of conjugate functions.** Consider the function $f(w) = [w^m]$, $m = 1, 2, \cdots$. Suppose w is transformed according to (17). Then

$$f_i = \text{const.}\cdot W_i W_1^{m-1} + \text{terms of lower degree in } W_1\,;$$

therefore

$$p_i{}^t \equiv \frac{\partial^t f_i}{\partial w_1{}^t} = \text{const.}\cdot W_i W_1^{m-t-1} + \text{terms of lower degree in } W_1 \qquad (t < m-1).$$

It thus appears that *for $f(w) = [w^m]$ the derivatives $p_i{}^t$ are linearly independent with respect to both i and t* ($t = 0, 1, \cdots, m-1$). The same statements can at once be made for every polynomial, $\sum_{i=0}^{m}[\alpha_i w^i]$, provided α_m is a multiple of the modulus. Also, for a fixed value of t the only terms of total degree q in $p_i{}^t$, for a function $h(w) \equiv \sum_{i=0}^{\infty} [\beta_i w^i]$, come from that term of the series which is of degree $q+t$. But for different values of i, the $p_i{}^t$ are linearly independent for the function $[w^{q+t}]$. Hence *the derivatives $p_i{}^t$ are linearly independent with respect to i, for every function, $h(w)$, for which there is at least one coefficient, β_i, $i > t$, which is a multiple of the modulus*; and for values of w in the region of convergence of the series.

10. **Monogenicity and analyticity.** Consider a function which is monogenic in a $2n$-dimensional region τ about a point α. By the substitution $w' = w - \alpha$ a function $f(w')$ is obtained which is monogenic in a region τ' about the origin. Since the conjugate functions of $f(w')$ remain analytic on transforming the variable according to (17), it follows that, for w in τ,

$$(18) \qquad \frac{\partial^2 f_s}{\partial W_l \partial W_k} = \frac{\partial^2 f_s}{\partial W_k \partial W_l}.$$

Applying equations (12) and (17) to both members of (18) we get, for w in τ,

$$\sum_{ij} \Gamma_{iks}\Gamma_{jli}\frac{\partial^2 f_j}{\partial W_1^2} = \sum_{ij}\Gamma_{ils}\Gamma_{jki}\frac{\partial^2 f_j}{\partial W_1^2},$$

where the Γ's are the multiplication constants of the transformed units. But the derivatives p_j^2 are linearly independent for all functions, $h(w)$, which satisfy the conditions of the last section. For such functions the last equation reduces to (3) showing that the algebra must be associative. Hence *in a non-associative algebra no such function could be monogenic in τ.*

From this point on we shall confine our attention to associative algebras.

11. Fundamental theorems on nilfactors. At a regular point, α, of any analytic function $f(w)$, when dw is a nilfactor df must also be one; and if $f'(\alpha)$ is not a nilfactor, dw must be a nilfactor whenever df is such.

Also, *at a regular point α, the increment Δf becomes a nilfactor whenever the increment Δw is a nilfactor.* For there must exist a positive number δ such that, for $|\Delta w| \leqq \delta$, Δf becomes a nilfactor whenever Δw is such. If this were not so it would be possible to pick a set of points w dense at α for which the difference quotient would be infinite and hence the derivative would not exist, which is contrary to the hypothesis. But when Δf is a nilfactor the determinant $\Delta_{\Delta f}$ vanishes. Hence $\Delta_{\Delta f}=0$ whenever Δw is a nilfactor, for $|\Delta w| < \delta$. But if Δw is a nilfactor then $\rho\Delta w$ is also a nilfactor, if ρ is a complex number. But this means that Δf is a nilfactor at all points of a segment of a straight line of length 2δ, and $\Delta_{\Delta f}$ must vanish over the same segment. But $\Delta_{\Delta f}$ is an analytic function of the complex variables w_k, so that by analytic continuation, if $\Delta_{\Delta f}$ vanishes over a segment of a straight line, it vanishes over the entire line. Hence Δf is a nilfactor at all points in the plane $\rho\Delta w$.

*Every commutative and associative algebra is the direct sum of t integral subalgebras, S^i, whose units, τ_j^i, have the following properties:**

$$\begin{aligned} \tau_0^i \tau_j^i &= \tau_j^i, \\ \tau_j^i \tau_k^i &= \sum^i \gamma_{jkl} \tau_l^i, \quad l > j, \quad l > k. \end{aligned} \tag{19}$$

The variable w may be written in terms of these units as follows:

$$w = \sum_{i=1}^{t} \sum_{k=0}^{s_i} W_k^i \tau_k^i, \tag{20}$$

where s_i is the number of units in the algebra S^i. *The variables W_k^i are linear functions of the w_j's, and it is easily shown that they are linearly independent.*

Expressing $f(w)$ in terms of the units τ_j^i we get

$$f(w) = \sum_{i=1}^{t} \sum_{k=0}^{s_i} F_k^i(W_0^i, \cdots, W_{s_i}^i)\tau_k^i, \tag{21}$$

where the functions F_k^i are linear functions of the f_j's.

It is easily shown that $f(w)$ has a derivative if and only if the t functions $F^i(W^i)$ have derivatives, where

$$W^i = \sum_{k=0}^{s_i} W_k^i \tau_k^i, \quad F^i = \sum_{k=0}^{s_i} F_k^i \tau_k^i. \tag{22}$$

---

* Dickson, *Linear Algebras*, p. 57.

Hence *every monogenic function of w is the sum of t independent functions* F^i, *each of which is a monogenic function of a different hypervariable* W^i. Thus, since the component variables W^i act entirely independently of each other, their functions always remaining in the subalgebra S^i, the general problem of studying monogenic functions of hypervariables is reduced to a study of functions of hypervariables which contain only a single idempotent unit.

We shall now confine our attention (as far as §24) to monogenic functions of a hypervariable W^i. Since only one variable will be considered the superscripts will be omitted.

All values of W except multiples of the modulus, τ_0, *are nilpotent and therefore are nilfactors.* If s is the number of units of the variable, *the* $(2s-2)$-*dimensional flat,* $W_0=\alpha_0$, *constitutes exactly those points where* $W-\alpha$ *is a nilfactor. If a monogenic function vanishes at a regular point* α, *then the function value is a nilfactor at every point of the flat* $W_0=\alpha_0$.

Evidently $\tau_0/(W-\alpha)$ *is infinite for and only for values of W on the flat* $W_0=\alpha_0$. The same is true for $\tau_0/(W-\alpha)^k$.

12. **Linear fractional transformation.** The transformation $W'=W+\alpha$ is a translation. The transformation $W'=\beta W$ is a homogeneous strain of the entire space. If β is a nilfactor the transformation is singular, the transformed space being of lower dimensionality than the original space. If $\beta=\beta_0\tau_0$ the transformation is a mere expansion of the space, but if β has components in nilpotent directions, the space is sheared.

The points $W=W_0\tau_0$ are the only ones that remain finite under the transformation $W'=\tau_0/W$. These points behave exactly as under the corresponding transformation in complex variables.

The general linear fractional transformation is a combination of the above special cases.

13. **Extension of Cauchy's integral theorem.** *The integral of a function,* $F(W)$, *around a closed curve, C, vanishes, provided there exists a diaphragm* S (i.e., a region of a surface) whose complete boundary is C, such that* $F(W)$ *is monogenic at every point of S.*†

Generalizing Stokes' theorem to s dimensions‡ gives

---

* Only ordinary curves, surfaces, and spaces are considered in this paper.

† Just as in complex variables the curve C need not be a single curve, since cross cuts can be introduced so as to make the curve single and the diaphragm simply connected.

The proof which follows is a direct generalization of the proof in complex variables which is based on Green's theorem. See Forsyth, *Theory of Functions*, 3d edition, p. 28. Cauchy's theorem may also be proved by the method of Goursat (Whittaker and Watson, *Modern Analysis*, 3d edition, p. 85).

‡ See J. B. Shaw, these Transactions, vol. 14 (1922), p. 224.

$$\int_C \sum_{j=1}^{s} P_j\{W_i\}dW_j = \iint_S \sum_{pq=1}^{s}{}^{*}\left(\pm\frac{\partial P_p}{\partial W_q}\mp\frac{\partial P_q}{\partial W_p}\right)dW_p dW_q, \tag{23}$$

where for the present purpose the signs are immaterial except that they must be opposite in the two terms in the bracket. The starred sum is taken over only those values of p and q for which $p<q$. The P_j and their derivatives are any continuous functions. From (9) and (23),

$$\int_C F(W)dW = \sum_k \tau_k \iint_S \sum_{pq}{}^{*}\left(\pm\frac{\partial\sum_i \gamma_{ipk}F_i}{\partial W_q}\mp\frac{\partial\sum_i \gamma_{iqk}F_i}{\partial W_p}\right)dW_p dW_q. \tag{24}$$

But from (13) the right hand member of (24) vanishes, so the integral on the left also vanishes, and the theorem is proved.

14. **Fundamental integral theorem.** Consider the function

$$\phi(W) = \int_\alpha^W F(W)dW,$$

where $F(W)$ is any function which is monogenic in some simply connected region which includes W and α. It can be shown as in complex variables that $\phi(W)$ is single valued and monogenic and has the derivative $F(W)$.† *The fundamental theorem of the integral calculus therefore holds for hypercomplex variables*; thus,

$$\int_\alpha^\beta F(W)dW = \psi(\beta) - \psi(\alpha), \tag{25}$$

where $\psi(W)$ is any primitive function of $F(W)$.

It is easily shown that the function W^m (m a positive or negative integer) is monogenic for all values of W, with the exception of nilfactors when m is negative, and has the derivative mW^{m-1}.‡ Hence by the fundamental integral theorem

$$\int_\alpha^W W^m dW = \frac{W^{m+1}-\alpha^{m+1}}{m+1}, \quad m \neq -1. \tag{26}$$

If $W=\alpha$ this integral vanishes, so that for any closed curve the above integral vanishes as long as $m\neq -1$.

† Goursat, *Mathematical Analysis*, vol. 2, part 1, p. 72. Special analysis is required in case h is a nilfactor.

‡ The sum and the product of two monogenic functions are monogenic, and likewise the quotient as long as the divisor is not a nilfactor. The formulas for the derivative of a product and of a quotient are the same as in complex variables.

15. **Converse of Cauchy's theorem: Morera's theorem.** *Consider a function $F(W)$ which is continuous in a region S of any surface in the W space. If the conjugate functions are analytic and if the integral of $F(W)$ vanishes for every closed curve lying entirely within S, then $F(W)$ is monogenic.* The proof is as in complex variables.*

16. **Singularities.** An integral around a closed curve, C, depends only on the values of the function on C and is independent of the way in which the diaphragm (as used in the proof of Cauchy's theorem) is situated. But if the contour integral does not vanish, there must be some singular points on the diaphragm, S, and hence on *every* diaphragm, S. Hence there must exist a space of $2s-2$ dimensions, which is linked† by C, and every point of which is a singular point.

If every point of the flat, $W_0=\alpha_0$, is a singular point of a function $F(W)$, and if there are no other singular points in the neighborhood of that flat, then $F(W)$ is said to have an *ordinary singularity* at α.

The *residue* of a singularity is the value of

$$\frac{\tau_0}{2\pi i}\int_C F(W)dW,$$

where C is any curve linking that singularity and no other. If the curve C lies in the plane $W=W_0$ and incloses the origin of that plane, it will link the singularity of τ_0/W, and

$$\int_C \frac{\tau_0 dW}{W} = 2\pi i\tau_0, \tag{27}$$

if C is traversed in a counterclockwise direction. Hence *the residue of the singularity of τ_0/W is τ_0.*

17. **Liouville's theorem.** *If an analytic function is bounded throughout the finite region the function is a constant.* For, if the function has no singularities the conjugate functions also have none. But each conjugate function is an analytic function of s, or less, complex variables, and such functions must be constants. Since the conjugate functions are constants the function $F(W)$ is a constant.

---

* Goursat, loc. cit., p. 78.

† The term *linked* may be regarded as defined by the above conditions, i.e., if every diaphragm, of which a closed curve C forms the complete boundary, intersects a $(2s-2)$-dimensional space, then C is said to link the space.

18. **Cauchy's integral formula.** *Let C be a closed curve not linking or passing through any singularities of an analytic function $F(W)$. If α is any point such that the flat $W_0-\alpha_0$ is linked by C, then*

$$F(\alpha) = \frac{\tau_0}{2\pi i}\int_C \frac{F(W)dW}{W-\alpha}. \tag{28}$$

The proof is almost exactly the same as in complex variables if the auxiliary circle γ is taken to be in the plane $W=\alpha+W_0$.

It can be shown by the classical method of real variables, if $\phi(W, \alpha)$ is an analytic function of W and α, for α in any region V, and for W in a region including any curve C, that for α in V,

$$\frac{d}{d\alpha}\int_C \phi(W,\alpha)dW = \int_C \frac{\partial\phi(W,\alpha)}{\partial\alpha}dW. \tag{29}$$

Hence

$$F^{(m)}(\alpha) = \frac{m!\tau_0}{2\pi i}\int_C \frac{F(W)}{(W-\alpha)^{m+1}}dW. \tag{30}$$

19. **Infinite series.** The sum of a series of continuous functions which converges uniformly in a region S is continuous in S. A series of continuous functions which converges uniformly along an arc of a curve can be integrated term by term along that arc.*

A series may also be differentiated term by term if the resulting series is a uniformly convergent series of continuous functions.

Consider a uniformly convergent series whose terms are monogenic functions. Since the series can be integrated term by term, the sum $F(W)$ is continuous and the integral of $F(W)$ vanishes for every closed curve C in the region where the terms are monogenic. But the uniform convergence of the given series requires that the series of conjugate functions converge uniformly in the corresponding region and in that case the conjugate functions of $F(W)$ are analytic. Hence from Morera's theorem *every uniformly convergent series of monogenic functions defines a monogenic function.* In particular, every power series defines a monogenic function. Hence *the necessary and sufficient condition for a function to be monogenic is that it be analytic.* A distinction between analytic and monogenic functions is no longer necessary.

---

* See Townsend, loc. cit., p. 223. Equation (7) must be employed in the proof for hypercomplex variables.

The region of convergence of a power series will now be investigated. If $W=W_0\tau_0+W_N$, where W_N is the nilpotent part of W, any power series takes the form

$$(31)\quad \sum_{m=0}^{\infty}\alpha_m W^m = \sum_{m=s}^{\infty}\alpha_m \sum_{k=0}^{s}\binom{m}{k} W_0^{m-k}W_N^k + \text{finite number of terms},$$

where s is the number of nilpotent units and $\binom{m}{k}$ is the binomial coefficient.

Suppose the series (31) converges for $W=a\tau_0$ where a is an ordinary complex constant. Then $\sum\alpha_m W_0^m$ converges absolutely and uniformly for $|W_0|<|a|$. If the conjugate series of this series are written out and differentiated successively, it follows easily that the series $\sum_{m=0}^{\infty}\alpha_m\binom{m}{k}W_0^{m-k}$ converge absolutely and uniformly for $|W_0|<|a|$. Multiplying these series by W_N^k and summing over k from 0 to s will give a series which still converges absolutely and uniformly for $|W_0|<|a|$. But the result thus obtained is the right hand member of (31), except for a finite number of terms. Hence if the series in the left member of (31) converges for $W=a\tau_0$, it converges absolutely and uniformly for all values of W for which $|W_0|<|a|$.

If the coefficients α_m are written $\alpha_m=\sum\alpha_{im}\tau_i$, then either $\alpha_m=0$ or there is a least value of i, say k, for which $\alpha_{km}\neq 0$. In this case, it is evident from the properties of the units τ_i that

$$\alpha_m W_N \equiv \sum(\alpha_m W_N)_i\tau_i = 0 \text{ for } i \geqq k.$$

Hence the numbers α_m and $\alpha_m W_N$ are either linearly independent or zero. The corresponding terms,

$$\alpha_m W_0^m\tau_0 \text{ and } \alpha_m\binom{m}{k}W_0^{m-k}W_N^k,$$

must also be linearly independent; and as a consequence, in order for (31) to diverge it is sufficient that the series $\sum\alpha_m W_0^m$ diverge. But if the latter series diverges for $W_0=a$ it also diverges wherever $|W_0|>|a|$. Hence the series (31) converges if and only if $\sum\alpha_m W_0^m\tau_0$ converges; and *there will exist a complex number a such that* (31) *diverges if* $|W_0|>|a|$ *and converges absolutely and uniformly if* $|W_0|<|a|$. A corresponding statement can be made for a power series in $W-\alpha$.

20. **Taylor's expansion.** *If $F(W)$ is analytic in a region where W is such that $|W_0-\alpha_0|<|a|$, where α is any point and a is any complex constant, then for values of W in this region,*

$$F(W) = \sum_{m=0}^{\infty} \frac{(W-\alpha)^m}{2\pi i} \int_C \frac{F(t)dt}{(t-\alpha)^{m+1}}$$
$$(32) \qquad = \sum_{m=0}^{\infty} \frac{F^{(m)}(\alpha)}{m!}(W-\alpha)^m,$$

where C is a closed curve within this region which links the flat $W_0=\alpha_0$, and t represents values of W on C. The proof is analogous to the standard proofs for complex variables.

If $\alpha+\beta$ is a singular point of $F(W)$ and if all points such that $|W_0-\alpha_0| < |\beta_0|$ are regular points, the power series expansion of $F(W)$ at the point α will converge where $|W_0-\alpha_0| < |\beta_0|$ and diverge where $|W_0-\alpha_0| > |\beta_0|$. *The region of convergence is thus limited by the nearest singularity in much the same way as in complex variables.*

A singularity of $F(W)$, one point of which is β, is said to be *isolated* if there exists a positive δ such that for all values of W for which $|W_0-\beta_0|<\delta$, $W_0\neq\beta_0$, the function $F(W)$ is analytic.

Suppose that β is a point of an isolated singularity of $F(W)$. Let α be a point such that $|\beta_0-\alpha_0|<\delta/2$, $\beta_0\neq\alpha_0$. If there exists a regular point $\gamma\neq\beta$ such that $\gamma_0=\beta_0$, then the power series expansion of $F(W)$ about the point α will converge at γ. But if the series converges at γ it must also converge at β which is contrary to the hypothesis that β is singular. Hence such a point γ cannot exist, and *every isolated singularity is an ordinary singularity.*

Since the formal laws of manipulation of series are the same as in complex variables, *the formal properties of analytic functions of a hypervariable are the same as for the corresponding function of a complex variable.* This is the principle of permanence of formal laws.

21. **Laurent's expansion.** *If a function $F(W)$ is analytic for all points of a region such that $|W_0-\alpha_0|>\delta$ and $|W_0-\alpha_0|<M$, where α is any point and M and δ are positive numbers, then within this region $F(W)$ can be expanded in a series of the form*

$$(33) \qquad F(W) = \sum_{m=-\infty}^{\infty} a_m(W-\alpha)^m,$$

where

$$a_m = \frac{\tau_0}{2\pi i}\int_C (t-\alpha)^{-m-1}F(t)dt,$$

and C is any curve in the above region which links the flat $W_0=\alpha_0$.

The proof is nearly the same as that in complex variables. Laurent's expansion affords an expression for an analytic function in the neighborhood of an isolated singularity.

22. **Poles and zeros.** If a finite positive integral value of k exists such that

$$\phi(W) = (W - \alpha)^k F(W), \tag{34}$$

where ϕ is analytic in the neighborhood of α and *not a nilfactor* at α, then $F(W)$ is said to have a *pole of order k* at α. If α is a pole of $F(W)$ then

$$\lim_{W \to \alpha} F(W) = \infty .$$

As in complex variables, if $F(W)$ is analytic at α and not identically zero, and if $F(\alpha)=0$, then $F(W)$ can be written in the form

$$F(W) = (W - \alpha)^k \phi(W), \tag{35}$$

where k is a positive integer and ϕ is analytic and not zero at α; and $F(W)$ is said to have a *zero of order k* at α. If α is a pole of order k of $F(W)$, then $\tau_0/F(W)$ has a zero of order k at α. The converse is not always true.

It may happen that the removal of a certain number, m, of zeros and poles from a function leaves it without zeros or poles, even though the function originally had more than m of them. Such a set will be called an *associated set of zeros and poles*. The poles and zeros of an analytic function are isolated, provided they form an associated set.*

For any analytic function, $F(W)$, the ith conjugate function is a function only of the variables $W_0, W_1, \cdots, W_i$. In particular the conjugate function F_0 is a function of W_0 alone. Furthermore, if there is a point α where $F(W)$ can be expressed in a series where the coefficients are ordinary complex numbers, then the conjugate function

$$F_0(W) = F(W_0). \tag{36}$$

As an example, if exp W is defined by the same series as in complex variables, the function $F_0(W) = \exp W_0$.

It is easily shown that the exponential function is an integral function and has no values which are nilfactors. The same is true for the exponential of any rational integral function of W. The method used in complex variables to prove that an integral function not having zero points can be expressed as an exponential function,† can be used in hypercomplex variables to show

* If the set is not associated the zeros and poles may not be isolated. For example, for a two-unit algebra, every one of the infinite set of points where $W_0=0$ is a zero of W^2, but an associated set of zeros would consist of the two points $A\tau_1$, and $-A\tau_1$, the two points coinciding in case A is zero.

† Goursat, loc. cit., p. 128.

that *any integral function*, $F(W)$, *which does not assume a nilfactor value at any point is expressible in the form*

$$F(W) = \exp G(W), \tag{37}$$

where $G(W)$ *is an integral function.*

Consider a rational integral function, $F(W)$, of the mth degree, in which the coefficients are ordinary complex multiples of τ_0. Then $F_0(W) = F(W_0)$; and this latter function is a polynomial in a complex variable which has m zeros $\rho_1, \rho_2, \cdots, \rho_m$. Computation shows that

$$F_1(W) = W_1 F'(W_0), \tag{38}$$

so that $F_1(W)$ will vanish if $W_1 = 0$, regardless of the value of W_0. But if $W_1 = 0$,

$$F_2(W) = W_2 F'(W_0)$$

which is satisfied if $W_2 = 0$. Continuing this process we find that $F(W)$ has the zeros $\rho_1\tau_0, \rho_2\tau_0, \cdots, \rho_m\tau_0$. Furthermore, if the ρ's are all distinct, $F'(\rho_i) \neq 0$, and $F_1(W)$ will vanish only if $W_1 = 0$, so that the zeros of $F(W)$ are unique. If, however, the ρ's are not all distinct, the zeros of $F(W)$ may be somewhat arbitrary. *An associated set will always consist of just m zeros, making proper allowance for their multiplicity.*

23. **Mittag-Leffler's and Weierstrass' theorems.** *Let* $\alpha_1, \alpha_2, \alpha_3, \cdots$ *be a sequence of points such that the* $(\alpha_k)_0$*'s are all distinct and*

$$|(\alpha_1)_0| \leqq |(\alpha_2)_0| \leqq |(\alpha_3)_0| \cdots, \quad \lim_{k\to\infty} |(\alpha_k)_0| = \infty.$$

There exists a function of W which is analytic everywhere except for $W = (\alpha_k)_0$, *and which has an arbitrary integral function of* $\tau_0/(W-\alpha_k)$ *namely* $G_k(\tau_0 \div (W-\alpha_k))$ *as the principal part of the expansion at the point* α_k. Furthermore the function is expressible in the form

$$\sum_{k=1}^{\infty}\left\{G_k\left(\frac{\tau_0}{W-\alpha_k}\right) - P_k(W)\right\}, \tag{39}$$

where the P_k's are properly chosen polynomials. The most general function with the desired singularities and principal parts is obtained by adding any integral function. The proof is nearly the same as in complex variables.

In particular, if $G_k(W) = W$, then

$$P_k(W) = -\sum_{m=1}^{\nu} \frac{W^{m-1}}{\alpha_k^m}, \tag{40}$$

where ν in general depends on k. If, however, there is a number p such that $\sum_k |\tau_0/\alpha_k|^p$ converges, it is sufficient to take $\nu = p-1$.

From the above theorem it follows merely by an integration that *if a function $F(W)$ has zeros of the first order at the points $\alpha_1, \alpha_2, \cdots$, it can be written in the form*

$$F(W) = e^{g(W)} \prod_{k=1}^{\infty} \left(\tau_0 - \frac{W}{\alpha_k}\right) e^{Q_\nu(W)}, \tag{41}$$

where $g(W)$ is an integral function, and ν an integer which may depend on k, and

$$Q_\nu(W) = \sum_{m=1}^{\nu} \frac{W^m}{\alpha_k^m}.$$

The proof also holds for multiple zeros, in which case the formal expression is unchanged except that some of the α's become identical.

24. **Restricted variables.** Thus far, all the variables considered have been of the type given in equation (1). The coördinates w_i may be independent functions of another set of n variables x_i without bringing about any change in the general properties of the variable. If, however, one or more relations exist between the coördinates w_i the properties are changed. Such a variable is called a *restricted variable.* If there are p such relations, p coördinates can be expressed in terms of the others so that

$$v = w = \sum_{k=1}^{n-p} w_k e_k + \sum_{k=n-p+1}^{n} h_k(w_1, \cdots, w_{n-p}) e_k. \tag{42}$$

Restricted variables are of importance in the application of hypercomplex variables in solving differential equations. The simplest example of a restricted variable is a real variable considered in the complex field.

Since a restricted variable ranges over a certain limited range of the more general variable, the properties of the restricted variable can be obtained by a consideration of the properties of the general variable within the region of restriction. *Any property of the general variable which holds for all regions alike is evidently a property of the restricted variable.* Restricted variables may, however, have properties not possessed by the general variable, as will now be shown.

We shall confine our attention to restricted variables in which the functions h_k are constants, a_k. In that case

$$v = \sum_{k=1}^{n-p} w_k e_k + \sum_{k=n-p+1}^{n} a_k e_k, \quad \text{and} \quad dv = \sum_{k=1}^{n-p} dw_k e_k.$$

It has been shown that w^2 is monogenic where and only where w is commutative with all numbers of the algebra. v^{2}* is therefore monogenic for all values of v if and only if every value of v is commutative with all numbers of the algebra. Hence *if the constants a_k are all zero, the units $e_{n-p+1}, \cdots, e_n$ need not be commutative with each other.*

For the restricted variable, v, for which the modulus is a combination of the first $n-p$ units, *the necessary and sufficient conditions that a function be monogenic are that the conjugate functions be analytic and satisfy the equations*

$$\frac{\partial f_s}{\partial w_k} = \sum_{i,j=1}^{n,n-p} \gamma_{iks}\epsilon_j \frac{\partial f_i}{\partial w_j} \qquad \begin{pmatrix} s = 1, \cdots, n \\ k = 1, \cdots, n-p \end{pmatrix}.$$

For simplicity we now restrict ourselves to commutative and associative algebras. With the same restrictions on the modulus as above, equations (13) with j and k ranging only from 1 to $n-p$, together with the analyticity of the conjugate functions, are necessary and sufficient conditions for monogenicity. Cauchy's integral theorem and his integral formula and Mittag-Leffler's theorem, as well as most of the other properties of w hold for v under restrictions that are evident in each case.

It is possible by means of restricted variables to construct a virtual exception to Weierstrass' (loc. cit.) theorem, that every algebra which allows but a finite number of zeros of polynomials is reducible to a direct sum of complex algebras. Thus, consider an algebra having the units τ_0 and τ_1. Let v be the restricted variable $v_0\tau_0$. No polynomial in v whose coefficients are not nilfactors, can have an infinity of zeros; but the algebra cannot be broken down into complex subalgebras.

25. Applications to differential equations. It is desired to obtain at least some real solutions of the equation

$$\Delta h(x_1, \cdots, x_p) \equiv \sum_{k=1}^{p} \alpha_k(x_1, \cdots, x_p)\frac{\partial^2 h}{\partial x_k^2} = 0, \tag{43}$$

of which Laplace's equation and the wave equation are special cases. Consider the corresponding equation

$$\Delta f(w) = 0, \qquad \text{where} \qquad w = \sum w_i(x_1, x_2, \cdots, x_p)e_i. \tag{44}$$

If an analytic function $f(w)$ satisfies (44), then, due to the distributive nature of Δ, the conjugate functions of $f(w)$ must satisfy (43). If, therefore, solutions of (44) are obtained, these immediately furnish real solutions of (43).†

* It should be noted that functions of a restricted variable are by no means confined to the same range of values as the variable.

† From this point on, the coördinate variables will be considered to be real.

We wish now to determine what hypercomplex variables w (of any arbitrary order n), if any, exist such that every analytic function of w satisfies (44). In such a case the conjugate functions of every analytic function of w satisfy (43). In order to determine the variable it is necessary to find values for the units (i.e., the multiplication constants, γ_{iks}) and the functions w_i. This will now be done.

From the definition of a derivative it follows that

$$\frac{\partial f}{\partial x_k} = \frac{\partial w}{\partial x_k} f'. \tag{45}$$

Hence

$$\frac{\partial^2 f}{\partial x_k^2} = \frac{\partial^2 w}{\partial x_k^2} f' + \left(\frac{\partial w}{\partial x_k}\right)^2 f'',$$

and

$$\Delta f = \sum_{k=1}^{p} \alpha_k \left[\frac{\partial^2 w}{\partial x_k^2} f' + \left(\frac{\partial w}{\partial x_k}\right)^2 f''\right] = 0. \tag{46}$$

The terms in the bracket are linearly independent for at least one analytic function f, so (46) is satisfied if and only if

$$\sum_{k=1}^{p} \alpha_k \frac{\partial^2 w}{\partial x_k^2} = 0, \text{ and } \sum_{k=1}^{p} \alpha_k \left(\frac{\partial w}{\partial x_k}\right)^2 = 0. \tag{47}$$

The first condition (47) merely requires that w be a particular solution of (44). The second condition can be written in the form

$$\sum_{ij,k=1}^{n,p} \alpha_k \frac{\partial w_i}{\partial x_k} \frac{\partial w_j}{\partial x_k} \gamma_{ijs} = 0. \tag{48}$$

From these relations and the first condition (47) (together with (2), (3), and (4)) the γ_{iks}'s and w_i's can be found. These conditions are usually not sufficient to determine uniquely a particular hypercomplex variable. In other words there is usually an infinity of variables with the desired properties. Particular variables satisfying equations (47) and (48) can be found by making certain additional assumptions. Thus if the functions w_i are assumed to be linear they automatically satisfy the first condition (47).

The particular conditions for Laplace's equation are obtained by putting $p=3$ and $\alpha_k=1$ in equations (48). An algebra whose analytic functions satisfy Laplace's equation will be called a *harmonic algebra*. If the additional assumption is made that $w_1=x_1$, equations (48) become $\sum_{k=1}^{3}\gamma_{kks}=0$. But these are the conditions that the sum of the squares of the first three units shall vanish. Hence, under the above assumptions, *any algebra is harmonic if the sum of the squares of any three units is zero.*

For the analytic functions of a hypercomplex variable to satisfy the wave equation, when $w_1 = x_1$, *the units must satisfy the relation*

$$e_1^2 + e_2^2 + e_3^2 - \frac{1}{c^2}e_4^2 = 0$$

(w_4 is here taken as the time variable t).

The above solution is illustrative of the general method which can be applied to homogeneous or non-homogeneous equations of any order. The method may also be applied to simultaneous equations with one or more dependent variables. In the latter case more than one hypercomplex variable must be determined. For a linear equation with real or complex coefficients, if the w's are linear functions of the x's, the method fails in case the differential equation contains any term with derivatives of less than the second order.

26. **Solution of differential equations by the use of Scheffer's equations.** The above method of solution gives conditions which do not contain the associativity conditions. Bechk-Widmanstetter (loc. cit.) has given a method of solution of Laplace's equation, for the case where $w_1 = x_1$, which is easily generalized, and which gives equations corresponding to (48), that contain part of the associativity conditions. These equations can in fact be obtained directly from (48) by combining with (3). As will be seen, Bechk-Widmanstetter's method may also be used to determine variables for which only one or a limited number of the conjugate functions satisfy the given differential equation. This is not possible by the other method of solution.

Bechk-Widmanstetter's method will now be generalized by taking the w's to be linear functions of the x's. We assume the modulus to be e_1. This is no restriction since the modulus can always be made to satisfy this condition by a linear transformation of the units as given in (17). In any linear transformation $f(w) = f(w')$. Hence if a variable is such that its analytic functions satisfy a given differential equation, then it possesses the same property after undergoing any linear transformation. Scheffer's equations (12) now become

(49) $$\frac{\partial f_s}{\partial w_k} = \sum_i \gamma_{iks} \frac{\partial f_i}{\partial w_1}.$$

But

(50) $$\frac{\partial f_s}{\partial x_j} = \sum_k \frac{\partial f_s}{\partial w_k} \frac{\partial w_k}{\partial x_j}.$$

Substituting (49) into (50),

$$\frac{\partial f_s}{\partial x_j} = \sum_{ik} \gamma_{iks} \frac{\partial w_k}{\partial x_j} \frac{\partial f_i}{\partial w_1}. \tag{51}$$

Applying this relation twice, the differential equation (44) becomes equal to

$$\Delta f_s = \sum_{j=1}^{p} \sum \alpha_j \gamma_{iks} \gamma_{tqi} \frac{\partial w_k}{\partial x_j} \frac{\partial w_q}{\partial x_j} \frac{\partial^2 f_t}{\partial w_1^2}. \tag{52}$$

But in §9 it was shown that the derivatives $f_t^{(2)}$ are linearly independent. Hence, *for Δf_s to vanish it is necessary and sufficient that*

$$\sum_{j=1}^{p} \sum_{ikq} \alpha_j \gamma_{iks} \gamma_{tqi} \frac{\partial w_k}{\partial x_j} \frac{\partial w_q}{\partial x_j} = 0 \qquad (t = 1, \cdots, n). \tag{53}$$

In order for Δf to vanish it would be necessary and sufficient that equations (53) hold for $s = 1, \cdots, n$.

This method is again illustrative of a general process applicable to much more complicated differential equations. The method fails for differential equations which contain non-derivative terms.

For Laplace's equation, if we take $w_k = x_k$, equations (53) reduce to

$$\sum_{i,j=1}^{n,3} \gamma_{ijs} \gamma_{tji} = 0. \tag{54}$$

If $n = 3$ these are Bechk-Widmanstetter's conditions (loc. cit.). He has shown that they have no solution unless some of the γ's become complex. This, however, in no way impairs the usefulness of the algebra, but only indicates that n must be greater than 3.

27. **Example of a harmonic algebra.** Consider an algebra with the four units 1, i, j, ij, which have the following multiplication table:

1	i	j	ij
i	-1	ij	$-j$
j	ij	-1	$-i$
ij	$-j$	$-i$	1

The table is easily remembered by noticing that the units i and j obey the same rules of multiplication as does the imaginary unit of complex numbers, except in respect to each other.

Now consider the restricted variable

$$w = x + aiy + bjz,$$

where i and j are units in the above table and x, y and z are the coördinates of the units. It is easily shown by the method of the sections on differential equations that every analytic function of this variable satisfies Laplace's equation provided $a^2+b^2=1$. The corresponding conjugate functions therefore represent potentials which satisfy certain particular boundary conditions. We proceed to discuss some of the simple functions of this variable. If $f(w)=U+iV+jW+ijZ$, the Scheffer's differential equations for this variable are

$$\begin{aligned}
\frac{\partial U}{\partial x} &= \frac{1}{a}\frac{\partial V}{\partial y} = \frac{1}{b}\frac{\partial W}{\partial z},\\
\frac{\partial V}{\partial x} &= -\frac{1}{a}\frac{\partial U}{\partial y} = \frac{1}{b}\frac{\partial Z}{\partial z},\\
\frac{\partial W}{\partial x} &= \frac{1}{a}\frac{\partial Z}{\partial y} = -\frac{1}{b}\frac{\partial U}{\partial z},\\
\frac{\partial Z}{\partial x} &= -\frac{1}{a}\frac{\partial W}{\partial y} = -\frac{1}{b}\frac{\partial V}{\partial z}.
\end{aligned} \tag{55}$$

28. **The function w^n.** This function may be calculated by direct multiplication. For example, for the function w^2,

$$U = x^2 - a^2y^2 - b^2z^2, \qquad W = 2bxz,$$

$$V = 2axy, \qquad Z = 2abyz.$$

The last three parts represent potentials between planes at right angles. The U part gives the potential inside and outside of an elliptic cone. If $a^2=b^2=\frac{1}{2}$ (since $a^2+b^2=1$) the cone becomes circular. To find the angle of the cone it is convenient to express U in spherical coördinates, measuring the angle θ from the X axis and ϕ from the Z axis in the YZ plane. Then

$$U = r^2\left[\cos^2\theta - \sin^2\theta(a^2\sin^2\phi + b^2\cos^2\phi)\right].$$

An electrostatic potential is constant over metallic surfaces. The function U vanishes, and is therefore constant, wherever the bracket expression vanishes, so that the metallic boundary is given by

$$\tan^2\theta = [a^2\sin^2\phi + b^2\cos^2\phi]^{-1}.$$

For $a^2=b^2=\frac{1}{2}$ this reduces to $\tan\theta=2^{1/2}$. This represents a circular cone having an angle of approximately 54°44′.

Direct calculation also shows that $\partial U/\partial\theta$ vanishes for $\theta=0$, and $\pi/2$ so that U represents the velocity potential of an elliptic stream flowing at right angles against a solid wall. If $a^2=b^2$ the stream becomes circular.

For w^3 the conjugate functions are

$$U = r^3\cos\theta[\cos^2\theta - 3\sin^2\theta(a^2\sin^2\phi + b^2\cos^2\phi)],$$
$$V = ar^3\sin\theta\sin\phi[3\cos^2\theta - \sin^2\theta(a^2\sin^2\phi + 3b^2\cos^2\phi)],$$
$$Z = 6abxyz.$$

U vanishes for $\theta=\pi/2$ and for values of θ and ϕ which make the bracket expression vanish. It represents the potential inside an elliptic cone, or the potential between an elliptic cone and a plane intersecting the cone at its vertex and at right angles to its axis. When $a^2=b^2$ the cone becomes circular and has an angle of approximately 39°14′. It also represents the velocity potential of an elliptic stream flowing into an elliptic cone along its axis and out around the sides. If $a^2=b^2$ the stream and cone are circular and the cone has an angle of approximately 63°26′.

The function V vanishes for $\phi=0$ and therefore represents the potential inside and outside of an elliptic cone which is cut in half by an infinite plane which passes through the axis of the cone. If $a^2=3b^2$ the cone is circular and has an angle of approximately 63°26′. For all functions the W part is obtained from the V part by merely interchanging ay and bz. Hence only the V part will be considered.

The Z part is the potential in a square corner.

In general for w^m the U part is the potential due to a system of concentric cones, but for $m\geqq 4$ the cones are no longer elliptical, but are of higher degree, and the constants a and b cannot be chosen so as to make them circular. The V and W parts are the potentials of a system of coaxial cones which are cut in half by a plane passing through the axis. The Z part is the same except that the cones are cut by two planes at right angles to each other, both passing through the axis of the cone.

29. **The inverse function.** A direct computation shows that

$$\Delta_w = (x^2 + a^2y^2 + b^2z^2 + 2abyz)(x^2 + a^2y^2 + b^2z^2 - 2abyz).$$

The points where $\Delta_w=0$ are nilfactors of w. In that case

$$x = 0, \text{ and } ay \pm bz = 0. \tag{56}$$

Every point of these two lines will be a point where $1/w$ becomes infinite and hence a singular point of that function. If A_{ij} is the cofactor of the term

α_{ij} in Δ_w, then the conjugate functions of $1/w$ are $U=A_{11}/\Delta_w$, $V=A_{12}/\Delta_w$ etc.

The *residue* of the second line of (56) is $\frac{1}{2}(1+ij)$, if it is linked in the direction $1, i, -1, j, 1$. The residue of the first line is $\frac{1}{2}(1-ij)$ if it is linked in the direction $1, i, -1, -j, 1$. The residue of both lines is 1 when they are linked by a curve in the x, iy plane in the direction $1, i, -1, -i, 1$ or by any curve which can be deformed into such a curve without cutting any singularities. Similarly the residue of both lines is $-ij$ when they are linked by a curve in the x, jz plane in the direction $1, j$ etc., or its equivalent.

The U part of the inverse function represents the potential due to two intersecting linear doublets. Thus, a positively charged wire lying directly above an equal negatively charged wire is situated along the first axis given in (56), and there is a similar arrangement along the other axis.

The V part differs from the U part only in that the wires lie side by side instead of one above the other, with the positive wire in both cases on the $+y$ side of the negative wire.

The Z part is the same as the U part except that the positive and negative wires are interchanged for one axis.

The function w^{-p} differs from w^{-1} only in the number of wires situated along the two axes of (56).

The exponential function is defined as in complex variables and obeys the same formal laws as in complex variables. The conjugate functions can easily be calculated directly and are found to be

$$U = e^x \cos bz \cos ay,$$
$$V = e^x \cos bz \sin ay,$$
$$Z = e^x \sin bz \sin ay.$$

These functions have two periods, one of length $2\pi/b$ in the z direction and the other of length $2\pi/a$ in the y direction. If $a=b$, the entire function has too independent periods of length $2^{3/2}\pi$. In the x direction the function behaves as in complex variables.

30. **Idempotent units.** For the more complicated functions it is convenient to employ idempotent units. For this we take $\tau_1=\frac{1}{2}(1+ij)$ and $\tau_2=\frac{1}{2}(1-ij)$. The variable then becomes

$$\begin{aligned} w &= x + aiy + bjz, \\ &= x\tau_1 + (ay - bz)i\tau_1 + x\tau_2 + (ay + bz)i\tau_2. \end{aligned}$$

Now suppose we have given any function

$$f(\omega) = R(u,v) + iI(u,v),$$

of the ordinary complex variable $\omega=u+iv$. Then for the function $f(w)$,

$$U = R(x,ay - bz) + R(x,ay + bz),$$
$$V = I(x,ay - bz) + I(x,ay + bz),$$
$$Z = R(x,ay - bz) - R(x,ay + bz) .$$

Applied to the function log w these formulas give for the conjugate functions

$$U = \tfrac{1}{2}\log\,[x^2 + (ay - bz)^2][x^2 + (ay + bz)^2] = \tfrac{1}{2}\log \Delta_w,$$
$$V = \arc\tan\frac{ay - bz}{x} + \arc\tan\frac{ay + bz}{x},$$
$$Z = \tfrac{1}{2}\log\frac{x^2 + (ay - bz)^2}{x^2 + (ay + bz)^2}\cdot$$

The U part represents the potential of two negatively charged wires, one lying on the first axis of (56), and the other on the second axis. The Z part is the same except that one wire has a positive charge.

It is evident that any conjugate function of a function of w is always the sum of two corresponding two-dimensional harmonic functions placed at an angle with each other.

31. **A harmonic nilpotent variable.** A few functions of a simple nilpotent harmonic algebra, which was given by Bechk-Widmanstetter (loc. cit.), will now be discussed. Considered as a four-unit algebra it can be written in the form

1	i	k	ik
i	-1	ik	$-k$
k	ik	0	0
ik	$-k$	0	0

The variable is

$$w = x + iy + kz,$$

and

$$f(w) = U + iV + kW + ikZ.$$

For the function w^m the U and V parts are always two-dimensional functions since they never contain the variable z.

For w^3, $\quad W = 3z(x^2 - y^2), \quad Z = 6xyz$;

for w^4, $\quad W = 4z(x^3 - 3xy^2), \quad Z = 4z(3x^2y - y^3)$.

In general for w^m,

$$W = mz(\text{real part of } \omega^{m-1}),$$
$$Z = mz(\text{imag. part of } \omega^{m-1}) \qquad (m = 2,3, \cdots).$$

Hence W and Z represent the potentials of a configuration of $m-1$ planes intersecting on the z axis, with an angle of $\pi/(m-1)$ between consecutive planes; and another plane coinciding with the XY coördinate plane.

Also,

$$\Delta_w = (x^2 + y^2)^2.$$

Hence the nilpotent numbers are the only nilfactors. The singularity of $1/w$ is the z axis and the residue is 1. Hence Cauchy's integral formula can be applied at all points within a cylinder whose sides are parallel to the z axis and whose directrix is the given curve γ.

32. **Completeness of solution.** In neither of the above algebras can every harmonic function be expressed as a conjugate function of an analytic function of w. In fact, the simple function $1/r$ does not occur among the conjugate functions. The conjugate functions satisfy other differential equations besides Laplace's equation. For example, in the case of the first algebra, from (55)

$$\frac{\partial^2 U}{\partial x^2} = -\frac{1}{a^2}\frac{\partial^2 U}{\partial y^2} = -\frac{1}{b^2}\frac{\partial^2 U}{\partial z^2}$$

and similar equations are readily obtained for the V, W, and Z parts.

Since the singularities of functions of hypercomplex variables never reduce to mere points in any three-dimensional flat, it seems that it would be impossible to obtain every harmonic function from any single variable.* By combining the solutions obtained from different variables, however, it is possible to obtain every harmonic function. This is the method that Whittaker has employed in obtaining a complete solution of Laplace's equation and of the wave equation.

Acknowledgement: The present paper was prepared as a doctor's thesis in physics at the University of Illinois, under the direction of Professor Jakob Kunz. The writer wishes to thank the Department of Physics for affording the opportunity to carry out this investigation, and Professor Kunz for the interest and help that he has rendered.

* Recently, however, the writer has discovered an algebra with an infinity of units which does furnish all the harmonic functions.

University of Illinois,
Urbana, Ill.