RIESZ SUMMABILITY FOR DOUBLE SERIES*

BY

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I. Introduction

Among the definitions for summing simple series is that introduced by M. Riesz† in 1909, and later generalized by him to the following form:

Let $\{\lambda_n\}$ be any sequence of positive numbers, increasing and becoming infinite, with $\lambda_1 \geqq 0$; let p be any real number greater than or equal to zero. Consider the series $\sum_{k=1,\cdots,\infty} c_k$. Write

$$y(s) = s^{-p} \sum_{\lambda_k < s} (s - \lambda_k)^p c_k.$$

If

$$\lim_{s \to \infty} y(s) = C,$$

the series $\sum_{k=1,\cdots,\infty} c_k$ is said to be summable (R, λ, p) to sum C.‡

The purpose of this paper is to extend this definition to one for summability of double series, and to develop some of the properties of this extended definition. We consider the double series

$$\begin{aligned} & u_{11} + u_{12} + u_{13} + \cdots \\ (1.1) \qquad & + u_{21} + u_{22} + u_{23} + \cdots \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad . \end{aligned}$$

Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of real numbers, increasing and becoming infinite, with $\lambda_1 \geqq 0$, $\mu_1 \geqq 0$; let p and r be any real numbers. We define

$$(1.2) \qquad C_{\lambda\mu}^{pr}(s,t) = \sum_{\lambda_k < s} \sum_{\mu_l < t} (s - \lambda_k)^p (t - \mu_l)^r u_{kl}.$$

We shall write

$$C_{\lambda\mu}(\sigma,\tau) = C_{\lambda\mu}^{00}(\sigma,\tau) = \sum_{k=1,l=1}^{m,n} u_{kl} \text{ for } \begin{cases} \lambda_m < \sigma \leqq \lambda_{m+1} \\ \mu_n < \tau \leqq \mu_{n+1} \end{cases}.$$

Then for $p>0$, $r>0$, we have

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* Presented to the Society, September 9, 1927; received by the editors in July, 1927.

† Riesz, *Sur la sommation des séries de Dirichlet*, Paris Comptes Rendus, July 5, 1909.

‡ Riesz, *Sur les séries de Dirichlet et les séries entières*, Paris Comptes Rendus, November 22, 1909.

$$C^{pr}_{\lambda\mu}(s,t) = pr\int_0^s\int_0^t C_{\lambda\mu}(\sigma,\tau)(s-\sigma)^{p-1}(t-\tau)^{r-1}\,d\tau d\sigma. \tag{1.3}$$

For $p=0$, $r>0$,

$$C^{0r}_{\lambda\mu}(s,t) = r\int_0^t C_{\lambda\mu}(s,\tau)(t-\tau)^{r-1}\,d\tau, \tag{1.4}$$

and for $r=0$, $p>0$,

$$C^{p0}_{\lambda\mu}(s,t) = p\int_0^s C_{\lambda\mu}(\sigma,t)(s-\sigma)^{p-1}\,d\sigma. \tag{1.5}$$

We shall say that the series $\sum_{k,l=1,\cdots,\infty} u_{kl}$ is summable $(R\colon \lambda,\ p;\ \mu,\ r)$ to sum U, if

$$\lim_{s\to\infty,\, t\to\infty} s^{-p}t^{-r}C^{pr}_{\lambda\mu}(s,t) = U.$$

The present paper includes theorems relative to the regularity and total regularity of the extended definition; it establishes a relation between methods of summation of the same type, λ and μ, when either p or r, or both p and r, are changed; a relation between methods of summation of the same order, p and r, when either λ or μ, or both λ and μ, are changed; certain necessary conditions for the Riesz summability of double series; theorems for the Dirichlet and Cauchy products of double series, corresponding to the theorems of Mertens, Cauchy and Abel for the Cauchy product of simple series; a sufficient condition for the summability of the product of two double series to the correct sum.

The proofs are similar to those of Hardy and Riesz* for the Riesz definition for simple series, with such alterations as might reasonably be expected because of change in dimensionality. Because of this similarity, the proofs are in many cases condensed; where they differ, they are given in more detail.†

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* Hardy and Riesz, *General Theory of Dirichlet's Series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 18. We shall refer to this tract by the letters H. R.

† The theorems of the present paper include as special cases results for the summability of bounded double series. Certain of these results for bounded series, obtained by Dr. G. M. Merriman, have been included in his paper *Concerning the summability of double series of a certain type*, Annals of Mathematics, (2), vol. 28, p. 515. We shall refer to this paper by the letter M. The writer's first information regarding Dr. Merriman's work was obtained on reading an abstract of it which appeared in Bulletin of the American Mathematical Society, vol. 33 (1927), p. 407, after the present paper was completely written for publication.

II. Regularity and total regularity

Denote by $\{x_{mn}\}$ the double sequence

$$\begin{matrix} x_{11}, & x_{12}, & x_{13}, & \cdots \\ x_{21}, & x_{22}, & x_{23}, & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{matrix}$$

associated with the double series (1.1). Then

$$x_{mn} = \sum_{k=1,l=1}^{m,n} u_{kl}.$$

A method for evaluating double series is said to be regular if whenever $\{x_{mn}\}$ is a bounded convergent sequence, the transformed sequence converges to the same value. A regular transformation of real elements is said to be totally regular if when applied to a sequence of real elements $\{x_{mn}\}$ which has the following properties,

(a) x_{mn} bounded for each m,

(b) x_{mn} bounded for each n,

(c) $\lim_{m\to\infty,\, n\to\infty} x_{mn} = +\infty,$

it transforms the sequence into a sequence which has for its limit $+\infty$.

Let

$$s^{-p}t^{-r}C_{\lambda\mu}^{pr}(s,t) = y(s,t) = \sum_{k=1,l=1}^{\infty,\infty} a_{kl}(s,t)x_{kl}. \tag{2.1}$$

In order that a transformation of this form be regular,* it is necessary and sufficient that

(a) $\lim_{s\to\infty,\, t\to\infty} a_{kl}(s,t) = 0,$ for each k and l;

(b) $\lim \sum_{k=1,l=1}^{\infty,\infty} a_{kl}(s,t) = 1;$

(c) $\lim_{s\to\infty,\, t\to\infty} \sum_{k=1}^{\infty} |a_{kl}(s,t)| = 0,$ for each l;

(d) $\lim_{s\to\infty,\, t\to\infty} \sum_{l=1}^{\infty} |a_{kl}(s,t)| = 0,$ for each k;

(e) $\sum_{k=1,l=1}^{\infty,\infty} |a_{kl}(s,t)| < A,$ for every pair of values of s and t.

* G. M. Robison, these Transactions, vol. 28 (1926), p. 67.

In order that a transformation of the form (2.1) be totally regular,* it is necessary and sufficient that there exist integers k_0 and l_0 such that $a_{kl}(s, t) \geqq 0$, when $k > k_0$, $l > l_0$, for all s and t.

We state without proof the following theorems.

THEOREM I. *$(R: \lambda, p; \mu, r)$ is regular for $p \geqq 0$, $r \geqq 0$.*

THEOREM II. *$(R: \lambda, p; \mu, r)$ is totally regular for $p \geqq 0$, $r \geqq 0$.*

III. LEMMAS

We shall derive eight lemmas, which are necessary for the proofs of the subsequent theorems.

We make use of the symbols o, O_x, O_y, and O, which we define as follows.

If ϕ is a positive function, we shall write

$$f(x, y) = o(\phi(x, y)),$$

if $\lim (f/\phi) = 0$, as $x \to \infty$, $y \to \infty$, independently.

If ϕ is a positive function, and if to any pair of constants, α_1, β_1, where $0 \leqq \alpha_1 \leqq \beta_1$, there corresponds a constant M_1, such that

$$\frac{|f|}{\phi} < M_1, \text{ for } \alpha_1 \leqq x \leqq \beta_1, \text{ and for all } y,$$

we write

$$f(x, y) = O_x(\phi(x, y)).$$

If ϕ is a positive function, and if to any pair of constants, α_2, β_2, where $0 \leqq \alpha_2 \leqq \beta_2$, there corresponds a constant M_2, such that

$$\frac{|f|}{\phi} < M_2, \text{ for } \alpha_2 \leqq y \leqq \beta_2, \text{ and for all } x,$$

we write

$$f(x, y) = O_y(\phi(x, y)).$$

If ϕ is a positive function, and if there exists a constant M_3, such that

$$\frac{|f|}{\phi} < M_3, \text{ for all positive values of } x \text{ and } y,$$

we write

$$f(x, y) = O(\phi(x, y)).$$

It is clear that if $f(x, y) = O_x(\phi(x, y))$, $f(x, y) = O_y(\phi(x, y))$ and $f(x,y) = o(\phi(x, y))$, then $f(x, y) = O(\phi(x, y))$.

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* Robison, loc. cit., p. 70.

Lemma 1. *Let $\psi(x)$ be a continuous function such that*

$$\psi(x) \sim Ax^{\alpha} \text{ as } \quad x \to \infty \quad , \text{ where } \alpha > -1 \, ;$$

let $\phi(x, y)$ be a function of x and y, and let

$$\chi(x,y) = \int_0^x \phi(u,y)\psi(x-u)du.$$

Then

(a) $$\text{if } \phi(x,y) = O_y(x^{\beta}), \text{ and } \beta > -1,$$
$$\chi(x,y) = O_y(x^{\alpha+\beta+1}) \, ;$$

(b) $$\text{if } \phi(x,y) = O_x(y^{\gamma}),$$
$$\chi(x,y) = O_x(y^{\gamma}) \, ;$$

(c) *if $\phi(x,y) \sim Bx^{\beta}y^{\gamma}$ as $x \to \infty$, $y \to \infty$, and if* (b) *is satisfied,*

$$\chi(x,y) \sim AB \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} x^{\alpha+\beta+1}y^{\gamma}.$$

(a) We can write

$$\psi(u) = Au^{\alpha} + \psi_1(u), \text{ where } \psi_1(u) = o(u^{\alpha}).$$

By hypothesis, constants y_1 and M_1 exist such that for $0 \leqq y \leqq y_1$,

$$| \chi(x,y) | \leqq \int_0^x \frac{| \phi(u,y) |}{u^{\beta}} | A(x-u)^{\alpha} + \psi_1(x-u) | u^{\beta}du$$
$$< M_1 \int_0^x | A(x-u)^{\alpha} + \psi_1(x-u) | u^{\beta}du$$
$$= M_1\left\{ | A | \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} x^{\alpha+\beta+1} + \int_0^x |\psi_1(u')(x-u')^{\beta}|du' \right\}.$$

Given $\epsilon > 0$, we can choose ξ so that

$$\int_0^x |\psi_1(u')(x-u')^{\beta}| du' < M_2 \int_0^{\xi} (u')^{\alpha}(x-u')^{\beta}du'$$
$$+ \epsilon \int_{\xi}^x (u')^{\alpha}(x-u')^{\beta}du' = J_1 + J_2,$$

where M_2 is a number depending only on ξ. Since $(x-u')^{\beta}$ is increasing throughout the interval $(0, \xi)$, or decreasing throughout this interval, by the second law of the mean for integrals, and for $0 < \xi_1 < \xi$,

$$J_1 = M_2\left[(x-\xi)^\beta \int_{\xi_1}^{\xi} (u')^\alpha du' + (x)^\beta \int_0^{\xi_1} (u')^\alpha du'\right] = O(x^\beta) = o(x^{\alpha+\beta+1}).$$

We have

$$J_2 \leqq \epsilon \int_0^x (u')^\alpha (x-u')^\beta du' = o(x^{\alpha+\beta+1}).$$

Therefore $|\chi(x, y)|$ is of the desired form.

(b) For $0 \leqq x \leqq x_1$,

$$|\chi(x,y)| \leqq \int_0^x \left|\frac{\phi(u,y)\psi(x-u)}{y^\gamma (x-u)^\alpha}\right| (x-u)^\alpha y^\gamma du$$

$$= y^\gamma M_3 \int_0^x (x-u)^\alpha du = O_x(y^\gamma),$$

where x_1 and M_3 are constants.

(c) Write $\psi(u) = Au^\alpha + \psi_1(u)$, where $\psi_1(u) = o(u^\alpha)$, and $\phi(u, y) = Bu^\beta y^\gamma + \phi_1(u, y)$, where $\phi_1 = o(u^\beta y^\gamma)$. Substituting in $\chi(x, y)$ we obtain the sum of four integrals. We have

$$(3.11) \qquad AB^\gamma \int_0^x u^\beta (x-u)^\alpha du = AB\, y^\gamma \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} x^{\alpha+\beta+1}.$$

It is necessary to prove that the other three integrals are of the form $o(x^{\alpha+\beta+1}y^\gamma)$.

Given $\epsilon > 0$, we can choose ξ and η so that

$$|\phi_1(u,y)| < \epsilon u^\beta y^\gamma \quad \text{for } u \geqq \xi, \ y \geqq \eta.$$

Assume $y \geqq \eta$.

We have

$$\int_0^x |A\phi_1(u,y)(x-u)^\alpha| du$$

$$= \int_0^\xi |A\phi_1(u,y)(x-u)^\alpha| du + \int_\xi^x |A\phi_1(u,y)(x-u)^\alpha| du$$

$$< |A| \left\{ M y^\gamma \int_0^\xi (x-u)^\alpha du + \epsilon y^\gamma \int_\xi^x u^\beta (x-u)^\alpha du \right\}$$

where M is a constant.

Applying the second law of the mean for integrals to $\int_0^\xi y^\gamma (x-u)^\alpha du$, we have

$$(3.12)\quad \int_0^x |A\phi_1(u,y)(x-u)^\alpha|\,du = O(x^\alpha y^\gamma) + o(x^{\alpha+\beta+1}y^\gamma) = o(x^{\alpha+\beta+1}y^\gamma),$$

which is the desired form.

For the same ξ and η,

$$\begin{aligned}
&\int_0^x |\phi_1(u,y)\psi_1(x-u)|\,du \\
&= \int_0^\xi |\phi_1(u,y)\psi_1(x-u)|\,du + \int_\xi^x |\phi_1(u,y)\psi_1(x-u)|\,du \\
&< My^\gamma \int_0^\xi |\psi_1(x-u)|\,du + \epsilon y^\gamma \int_\xi^x u^\beta |\psi_1(x-u)|\,du \\
&= My^\gamma M_4\xi + \epsilon y^\gamma \int_0^{x-\xi} |(x-u')^\beta \psi_1(u')|\,du',
\end{aligned}$$

where M_4 depends only on ξ.

By the proof for J_1 and J_2 in (a),

$$\epsilon y^\gamma \int_0^{x-\xi} |(x-u')^\beta\psi_1(u')|\,du' = o(x^{\alpha+\beta+1}y^\gamma)$$

and therefore

$$(3.13)\quad \int_0^x |\phi_1(u,y)\psi_1(x-u)|\,du = o(x^{\alpha+\beta+1}y^\gamma).$$

From the proof used for J_1 and J_2,

$$(3.14)\quad \int_0^x |Bu^\beta y^\gamma\psi_1(x-u)|\,du = o(x^{\alpha+\beta+1}y^\gamma).$$

From the results for (3.11), (3.12), (3.13) and (3.14) it follows that $\chi(x, y)$ is of the desired form.

LEMMA 2. *Let $\phi(x, y)$ and $\psi(x, y)$ be functions of x and y, and let*

$$\chi(x,y) = \int_0^x \int_0^y \psi(u,v)\phi(x-u, y-v)\,dv\,du.$$

Then

(a) *if $\psi(x, y)=O_y(x^\alpha)$ and $\phi(x, y)=O_y(x^\gamma)$ where α and γ are >-1,*

$$\chi(x,y) = O_y(x^{\alpha+\gamma+1});$$

(b) *if $\psi(x, y)=O_x(y^\beta)$ and $\phi(x, y)=O_x(y^\delta)$ where β and δ are >-1,*

$$\chi(x,y) = O_x(x^{\beta+\delta+1});$$

(c)* *if* $\psi(x, y)\sim Ax^{\alpha}y^{\beta}$† *and* $\phi(x, y)\sim Bx^{\gamma}y^{\delta}$ *as* $x\to\infty$, $y\to\infty$, *and if* (a) *and* (b) *are satisfied,*

$$\chi(x,y)\sim AB\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\delta+1)}{\Gamma(\alpha+\gamma+2)\Gamma(\beta+\delta+2)}\,x^{\alpha+\gamma+1}y^{\beta+\delta+1}.$$

(a) We can find a constant M, such that when $0\leqq v\leqq y_1$,

$$\left|\frac{\psi(u,v)}{u^{\alpha}}\right|<M \text{ and } \left|\frac{\phi(x-u,y-v)}{(x-u)^{\gamma}}\right|<M.$$

Assuming $0\leqq y\leqq y_1$, we have

$$|\chi(x,y)|<M^2\int_0^x\int_0^y u^{\alpha}(x-u)^{\gamma}dv\,du$$

$$\leqq M^2\frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+2)}x^{\alpha+\gamma+1}y_1=O_y(x^{\alpha+\gamma+1}).$$

(b) The proof of (b) is similar to that of (a).

(c) Using the method used in M., p. 519, we write $\chi(x, y)$ as the sum of four integrals, the first of which is of the desired form.

We write

$$(3.21)\qquad \int_0^x\int_0^y Au^{\alpha}v^{\beta}\phi_1(x-u,y-v)dv\,du$$

$$=\int_0^x\int_0^y A(x-u')^{\alpha}(y-v')^{\beta}\phi_1(u',v')dv'du'$$

in the form

$$\left|\int_0^x\int_0^y\right|\leqq\int_0^{\xi}\int_0^y+\int_0^x\int_0^{\eta}+\int_{\xi}^x\int_{\eta}^y,$$

where the integrands on the right hand side of the inequality are in absolute value. Given $\epsilon>0$, we choose ξ and η so that

$$\phi_1(u',v')<\epsilon(u')^{\gamma}(v')^{\delta},\ \text{for } \xi\leqq u',\ \eta\leqq v',$$

and by methods similar to those used in Lemma 1, we can show that the integral of (3.21) is of the desired form.

Similarly, we can show that

$$(3.22)\qquad \int_0^x\int_0^y B(x-u)^{\gamma}(y-v)^{\delta}\psi_1(u,v)dv\,du=o(x^{\alpha+\gamma+1}y^{\beta+\delta+1}).$$

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* Case (c) corresponds to M., p. 519, Lemma 4. The conditions (a) and (c) are replaced here by the less stringent conditions (a) and (b).

† By $\psi(x,y)\sim Ax^{\alpha}y^{\beta}$ we mean that $\psi/(x^{\alpha}y^{\beta})\to A$ as $x\to\infty$, $y\to\infty$, independently.

It remains to prove that

$$(3.23)\qquad \int_0^x \int_0^y \psi_1(u,v)\phi_1(x-u,y-v)dv\,du = o(x^{\alpha+\gamma+1}y^{\beta+\delta+1}).$$

Given $\epsilon > 0$, we can choose ξ and η so that

$$\psi_1(u,v) < \epsilon u^\alpha v^\beta, \text{ for } \xi \leqq u,\ \eta \leqq v.$$

We express the absolute value of the integral as the sum of three integrals, J_1', J_2', and J_3', as in (3.21).

We have

$$J_3' < \epsilon \int_\xi^x \int_\eta^y u^\alpha v^\beta \phi_1(x-u,y-v)dv\,du$$

$$= \epsilon \int_0^{x-\xi} \int_0^{y-\eta} (x-u')^\alpha (y-v')^\beta \phi_1(u',v')dv'du'.$$

We can choose ξ_1 and η_1 such that

$$\phi_1(u',v') < \epsilon (u')^\gamma (v')^\delta, \text{ for } \xi_1 \leqq u',\ \eta_1 \leqq v'.$$

We express J_3' as the sum of three integrals as in (3.21) and by methods similar to those used in Lemma 1, we can show that J_1', J_2', and J_3', and hence (3.23), are of the form $o(x^{\alpha+\gamma+1}\,y^{\beta+\delta+1})$.

From the results for (3.21), (3.22), and (3.23), it follows that $\chi(x, y)$ is of the desired form.

Lemma 3. *If $p \geqq 0$, $p' > 0$, $r \geqq 0$,*

$$C_{\lambda\mu}^{p+p',r}(s,t) = \frac{\Gamma(p+p'+1)}{\Gamma(p+1)\Gamma(p')}\int_0^s C_{\lambda\mu}^{pr}(u,t)(s-u)^{p'-1}du.$$

We have a similar result if $r \geqq 0$, $r' > 0$, $p \geqq 0$.

We omit the proof, because of its similarity to that of H. R., Lemma 6.

Lemma 4. *If $p \geqq 0$, $p' < 1$, $p' \leqq p$ and $r \geqq 0$,*

$$C_{\lambda\mu}^{p-p',r}(s,t) = \frac{\Gamma(p-p'+1)}{\Gamma(p+1)\Gamma(1-p')}\int_0^s \frac{\partial C_{\lambda\mu}^{pr}(u,t)}{\partial u}(s-u)^{-p'}du.$$

We have a similar result if $r \geqq 0$, $r' \leqq r$, $r' < 1$ and $p \geqq 0$. The proof is obtained from the preceding lemma by a method analogous to that of M., Lemma 1, part 4.

LEMMA 5. *If u_{kl} is real, $0\leqq\eta\leqq t$, $p'\geqq 0$, $r'\geqq 0$, $0<r\leqq 1$, then*

$$\frac{\Gamma(r+r'+1)}{\Gamma(r'+1)\Gamma(r)}\left|\int_0^\eta C_{\lambda\mu}^{p'r'}(\sigma,\tau)(t-\tau)^{r-1}d\tau\right| \leqq \max_{0\leqq\tau\leqq\eta}\left|C_{\lambda\mu}^{p',r+r'}(\sigma,\tau)\right|.$$

We have a similar result for $0\leqq\xi\leqq s$, $r'\geqq 0$, $p'\geqq 0$, $0<p\leqq 1$. The proof is analogous to that of H.R., Lemma 8, and M., Lemma 2.

LEMMA 6. *If $p\geqq 0$, $r\geqq 0$, $p'<1$, $r'<1$, $p'\leqq p$ and $r'\leqq r$, then*

$$C_{\lambda\mu}^{p-p',r-r'}(s,t) = \frac{\Gamma(p-p'+1)\Gamma(r-r'+1)}{\Gamma(p+1)\Gamma(1-p')\Gamma(r+1)\Gamma(1-r')}$$
$$\cdot\int_0^s\int_0^t \frac{\partial^2 C_{\lambda\mu}^{pr}(u,v)}{\partial v\partial u}(s-u)^{-p'}(t-v)^{-r'}dv\,du.$$

The proof is obtained by applying Lemma 4 twice.

LEMMA 7. *If u_{kl} is real, $0\leqq\xi\leqq s$, $0\leqq\eta\leqq t$, $p'\geqq 0$, $r'\geqq 0$, $0<p\leqq 1$ and $0<r\leqq 1$, then*

$$\frac{\Gamma(p+p'+1)\Gamma(r+r'+1)}{\Gamma(p'+1)\Gamma(p)\Gamma(r'+1)\Gamma(r)}\left|\int_0^\xi\int_0^\eta C_{\lambda\mu}^{p'r'}(\sigma,\tau)(s-\sigma)^{p-1}(t-\tau)^{r-1}d\tau\,d\sigma\right|$$
$$\leqq \max_{\substack{0\leqq\sigma\leqq\xi\\0\leqq\tau\leqq\eta}}\left|C_{\lambda\mu}^{p+p',r+r'}(\sigma,\tau)\right|.$$

The proof is analogous to the proofs referred to in Lemma 5.

LEMMA 8. *If ϕ is a positive function of x such that*

$$\int^x \phi dx \to \infty\ ,\quad as\quad x\to\infty\ ,$$

and if f is a function of x and y, such that

$$f = o(\phi),\quad and\quad f = O_x(1),$$

then

$$\int^x o(\phi)dx = o\left\{\int^x \phi\, dx\right\}.$$

Since $f=o(\phi)$, for any $\epsilon>0$ we can find ξ and η such that

$$|f| < \epsilon\phi,\quad\text{for } x\geqq\xi \text{ and } y\geqq\eta.$$

Since $f=O_x(1)$, there exists some constant M, such that

$$|f| < M,\quad\text{for } x<\xi.$$

Assuming $y \geqq \eta$, we have

$$\left|\int_0^x f(x,y)dx\right| \leqq \left|\int_a^\xi f(x,y)dx\right| + \left|\int_\xi^x f(x,y)dx\right|$$

$$< M(\xi - a) + \epsilon \int_\xi^x \phi dx \leqq M(\xi - a) + \epsilon \int_a^x \phi dx.$$

Therefore

$$\frac{\left|\int_a^x f(x,y)dx\right|}{\int_a^x \phi(x)dx} < \frac{M(\xi - a)}{\int_a^x \phi(x)dx} + \epsilon.$$

Since $\int_a^x \phi dx$ diverges, we have the desired result.

IV. Theorems of consistency

Theorem III. *If the series $\sum u_{mn}$ is bounded and summable $(R: \lambda, p; \mu, r)$ to sum U, then $\sum u_{mn}$ is summable $(R: \lambda, p'; \mu, r)$ to sum U, provided $p' \geqq p \geqq 0$, $r \geqq 0$. Both $s^{-p}t^{-r}C_{\lambda\mu}^{pr}(s, t)$ and $s^{-p'}t^{-r}C_{\lambda\mu}^{p'r}(s, t)$ are bounded.*

It has been proved that a bounded sequence is transformed by a regular transformation into a bounded sequence;* therefore Theorem III is a special case of the following more general theorem.

Theorem III′. *If $\sum u_{mn}$ is summable $(R: \lambda, p; \mu, r)$ to sum U, and if $C_{\lambda\mu}^{pr}(s, t) = O_s(t^r)$, then $\sum u_{mn}$ is summable $(R: \lambda, p'; \mu, r)$ to sum U, and $C_{\lambda\mu}^{p'r}(s, t) = O_s(t^r)$, provided $p' \geqq p \geqq 0$, and $r \geqq 0$. If $C_{\lambda\mu}^{pr}(s, t) = O(s^p t^r)$, then $C_{\lambda\mu}^{p'r}(s, t) = O(s^{p'}t^r)$.*

The theorem is obviously true for $p' = p \geqq 0$. We assume that $p' = p + p''$, where $p'' > 0$; from Lemma 3, we have

$$C_{\lambda\mu}^{p+p'',r}(s,t) = \frac{\Gamma(p + p'' + 1)}{\Gamma(p + 1)\Gamma(p'')} \int_0^s C_{\lambda\mu}^{pr}(u,t)(s - u)^{p''-1}du. \tag{4.1}$$

Since by hypothesis $C_{\lambda\mu}^{pr}(s, t) \sim U s^p t^r$ as $s \to \infty$, $t \to \infty$, and since $C_{\lambda\mu}^{pr}(s, t) = O_s(t^r)$, we have, by Lemma 1(c),

$$\int_0^s C_{\lambda\mu}^{pr}(u,t)(s - u)^{p''-1}du \sim U \frac{\Gamma(p'')\Gamma(p + 1)}{\Gamma(p + p'' + 1)} s^{p+p''}t^r,$$

and by Lemma 1(b)

* Robison, loc. cit., p. 67.

$$\int_0^s C_{\lambda\mu}^{pr}(u,t)(s-u)^{p''-1}\,du = O_s(t^r).$$

If $C_{\lambda\mu}^{pr}(s,t)=O(s^p t^r)$, we have, by Lemma 1(a),

$$\int_0^s C_{\lambda\mu}^{pr}(u,t)(s-u)^{p''-1}\,du = O_t(s^{p+p''}).$$

Using these results in (4.1), we have the desired theorem.

The analogous theorem for $p \geqq 0$ and $r' \geqq r \geqq 0$ may be proved in the same way. If we change p to p' under the condition of Theorem III, and then change r to r' under the conditions of the analogous theorem, we obtain the following theorem.

THEOREM IV. *If the series $\sum u_{mn}$ is bounded and summable $(R\colon \lambda, p; \mu, r)$ to sum U, then $\sum u_{mn}$ is summable $(R\colon \lambda, p'; \mu, r')$ to sum U, provided $p' \geqq p \geqq 0$ and $r' \geqq r \geqq 0$. Both $s^{-p}t^{-r}C_{\lambda\mu}^{pr}(s,t)$ and $s^{-p'}t^{-r'}C_{\lambda\mu}^{p'r'}(s,t)$ are bounded.**

This theorem is a special case of the following theorem.

THEOREM IV′. *If $\sum u_{mn}$ is summable $(R\colon \lambda, p; \mu, r)$ to sum U, and if $C_{\lambda\mu}^{pr}(s,t)=O(s^p t^r)$, then $\sum u_{mn}$ is summable $(R\colon \lambda, p'; \mu, r')$ to sum U, and $C_{\lambda\mu}^{p'r'}(s,t)=O(s^{p'}t^{r'})$, provided $p' \geqq p \geqq 0$, $r' \geqq r \geqq 0$.*

This group of theorems is a generalization for double series of the first theorem of consistency, established by Hardy and Riesz† for simple series. For the generalization for double series of the second theorem of consistency‡ we require the following lemmas,§ which we state without proof.

LEMMA 9. *Any logarithmico-exponential function (or, in short, L-function) $\mu(\lambda)$ is continuous, of constant sign, and monotonic from a certain value of λ onwards; and the same is true of any of its derivatives.*

---

* Cf. M., p. 521, Theorem I. This theorem is true for bounded series. That the proof, even with this restriction, does not follow from the statement $C^{r,s}(\sigma,\tau_1) \approx C_1\sigma^r$, $C^{r,s}(\sigma_1,\tau) \approx C_2\tau^s$, is shown by the series

$$c_{1n} = (-1)^{n-1}\,; \quad c_{mn} = (-1)^n \frac{1}{2^{m-1}}, \text{ for } m \neq 1,$$

which is bounded and convergent to zero. If $\lambda_n = 2^{n-1}$, $\lambda_m = 2^{m-1}$, the series is C^{11} summable, but $C^{11}(\sigma_1,\tau) \approx C_2\tau$ is not true.

† H. R., p. 29.

‡ H. R., p. 30; Hardy, *The second theorem of consistency for summable series*, Proceedings of the London Mathematical Society, (2), vol. 15 (1915), pp. 72–88.

§ Hardy, *The second theorem* etc., p. 75.

LEMMA 10. *If $\mu\to\infty$, and a number Δ exists such that*

$$\mu = O(\lambda^{\Delta}),$$

then

$$\mu^{(r)} = O\left(\frac{\mu}{\lambda^r}\right),$$

$\mu^{(r)}$ denoting the rth derivative of $\mu(\lambda)$.

LEMMA 11. *If μ satisfies the conditions of Lemma 10, and if ν lies between two positive numbers g and G, then positive numbers h and H exist, such that*

$$h \leqq \frac{\mu(\lambda\nu)}{\mu(\lambda)} \leqq H.$$

The same result holds for decreasing functions which decrease less rapidly than $\lambda^{-\Delta}$ for some value of Δ.

The first theorem of the group corresponding to the second theorem of consistency is as follows.

THEOREM V. *If the series $\sum u_{kl}$ is bounded and summable $(R\colon \lambda, p; \mu, r)$ to sum U, and if η is an L-function of λ, such that*

$$\eta = O(\lambda^{\Delta}),$$

where Δ is a constant, then $\sum u_{kl}$ is summable $(R\colon \eta, p; \mu, r)$ to sum U, provided $p\geqq 0$, $r\geqq 0$. Both $s^{-p}t^{-r}C^{pr}_{\lambda\mu}(s, t)$ and $s^{-p}t^{-r}C^{pr}_{\eta\mu}(s, t)$ are bounded.

This is a special case of the following theorem.

THEOREM V′. *If the series $\sum u_{kl}$ is summable $(R\colon \lambda, p; \mu, r)$ to sum U and if $C^{pr}_{\lambda\mu}(s, t) = O_s(t^r)$; if η is an L-function of λ, such that*

$$\eta = O(\lambda^{\Delta}),$$

where Δ is a constant, then $\sum u_{kl}$ is summable $(R\colon \eta, p; \mu, r)$ to sum U, and $C^{pr}_{\eta\mu}(s, t) = O_s(t^r)$, provided $p\geqq 0$ and $r\geqq 0$. If $C^{pr}_{\lambda\mu}(s, t) = O(s^pt^r)$, then $C^{pr}_{\eta\mu}(s, t) = O(s^pt^r)$.

We shall assume that u_{kl} is real and that $U=0$. If u_{kl} is complex, we can consider the real and imaginary parts of the series separately. If $U\neq 0$, we prove the theorem for the double series $\sum u'_{kl}$, where

$$u'_{kl} = \begin{cases} u_{11} - U, & \text{for } k = l = 1, \\ u_{kl}, & \text{for all other } k \text{ and } l. \end{cases}$$

We complete the proof by adding the convergent double series $\sum u''_{kl}$, where

$$u''_{kl} = \begin{cases} U, & \text{for } k = l = 1, \\ 0, & \text{for all other } k \text{ and } l. \end{cases}$$

Case 1. $p=0,\ r\geqq 0$.

In $C^{0r}_{\eta\mu}(s, w)$, let $s=\eta(v)$. Since $\eta_m=\eta(\lambda_m)<\eta(v)\leqq\eta_{m+1}$ when $\lambda_m<v\leqq\lambda_{m+1}$, it follows immediately from the definition of $C^{00}_{\lambda\mu}(\sigma, \tau)$ that

$$C^{00}_{\eta\mu}[\eta(v), w] = C^{00}_{\lambda\mu}(v, w)$$

and that therefore

$$C^{0r}_{\eta\mu}[\eta(v), w] = C^{0r}_{\lambda\mu}(v, w).$$

Hence the theorem is true under the conditions of Case 1.

From Lemma 3, for $p>0,\ r\geqq 0$, we have

$$C^{pr}_{\lambda\mu}(v, w) = p\int_{\lambda_1}^{v} C^{0r}_{\lambda\mu}(\nu, w)(v-\nu)^{p-1}\,d\nu = o(v^p w^r). \tag{4.21}$$

We wish to prove that

$$C^{pr}_{\eta\mu}(s, w) = p\int_{\eta_1}^{s} C^{0r}_{\eta\mu}(\sigma, w)(s-\sigma)^{p-1}d\sigma = o(s^p w^r). \tag{4.22}$$

Let $s=\eta(v)$ and $\sigma=\eta(\nu)=\eta$.

Since

$$C^{0r}_{\eta\mu}[\eta(\nu), w] = C^{0r}_{\lambda\mu}(\nu, w),$$

we may write (4.22) in the form

$$p\int_{\lambda_1}^{v} C^{0r}_{\lambda\mu}(\nu, w)(s-\eta)^{p-1}\eta' d\nu = o(s^p w^r). \tag{4.23}$$

Case 2. Assume p a positive integer and $r\geqq 0$.

By repeated applications of (3.4), we have

$$C^{0r}_{\lambda\mu}(\nu, w) = \frac{1}{p!}\left(\frac{\partial}{\partial\nu}\right)^p C^{pr}_{\lambda\mu}(\nu, w).$$

Substituting this expression in (4.23), we have a constant multiple of

$$J = \int_{\lambda_1}^{v}\left(\frac{\partial}{\partial\nu}\right)^p C^{pr}_{\lambda\mu}(\nu, w)(s-\eta)^{p-1}\eta' d\nu.$$

Integrating J by parts p times, we obtain

$$J = (-1)^{p-1}\left[C_{\lambda\mu}^{pr}(\nu,\ w)\left(\frac{\partial}{\partial\nu}\right)^{p}(s-\eta)^{p}\right]_{\nu=\lambda_1}^{\nu=v}$$

$$+(-1)^{p}\int_{\lambda_1}^{v}C_{\lambda\mu}^{pr}(\nu,w)\left(\frac{\partial}{\partial\nu}\right)^{p+1}(s-\eta)^{p}\,d\nu$$

$$=(-1)^{p-1}C_{\lambda\mu}^{pr}(v,w)\left[\left(\frac{\partial}{\partial\nu}\right)^{p}(s-\eta)^{p}\right]_{\nu=v}$$

$$+(-1)^{p}\int_{\lambda_1}^{v}C_{\lambda\mu}^{pr}(\nu,w)\left(\frac{\partial}{\partial\nu}\right)^{p+1}(s-\eta)^{p}\,d\nu$$

$$=J_1+J_2.$$

Since

$$\left[\left(\frac{\partial}{\partial\nu}\right)^{p}\right]_{\nu=v}(-1)^{p}p!(s')^{p},$$

where

$$s'=\frac{\partial}{\partial\nu}\eta(\nu),\ \text{when}\ \nu=v,$$

we have

$$J_1=-p!C_{\lambda\mu}^{pr}(v,w)(s')^{p}=-p!\frac{C_{\lambda\mu}^{pr}(v,w)}{v^{p}w^{r}}\left(\frac{vs'}{s}\right)^{p}s^{p}w^{r} \tag{4.31}$$

$$=o(1)O(1)s^{p}w^{r}=o(s^{p}w^{r}),$$

by Lemma 10.

We can easily prove that

$$\left(\frac{\partial}{\partial\nu}\right)^{p+1}(s-\eta)^{p}=\sum A s^{p-\alpha}\eta^{\beta_0}(\eta')^{\beta_1}(\eta'')^{\beta_2}\cdots,$$

where the A's are constants, and

$$\begin{aligned}0<\beta_0+\beta_1+\beta_2+\cdots&=\alpha\leqq p,\\ \beta_1+2\beta_2+3\beta_3+\cdots&=p+1.\end{aligned} \tag{4.32}$$

Therefore J_2 is the sum of constant multiples of integrals of the form

$$s^{p-\alpha}\int_{\lambda_1}^{v}C_{\lambda\mu}^{pr}(\nu,w)\eta^{\beta_0}(\eta')^{\beta_1}(\eta'')^{\beta_2}\cdots d\nu. \tag{4.33}$$

Since $C_{\lambda\mu}^{pr}(\nu,w)=o(\nu^{p}w^{r})$, and since, by Lemma 10,

$$\eta^{(m)}=O\left(\frac{\eta}{\nu^{m}}\right),$$

(4.33) is of the form

$$s^{p-\alpha}\int_{\lambda_1}^{v} o(\nu^p w^r)\eta^{\beta_0}\eta'\left[O\left(\frac{\eta}{\nu}\right)\right]^{\beta_1-1}\left[O\left(\frac{\eta}{\nu^2}\right)\right]^{\beta_2}\cdots$$

$$(4.34)\qquad = s^{p-\alpha}\int_{\lambda_1}^{v} o\left\{w^r\nu^{p-\beta_1-2\beta_2+\cdots+1}\eta^{\beta_0+\beta_1+\cdots-1}\eta'\right\}d\nu$$

$$= s^{p-\alpha}w^r\int_{\lambda_1}^{v} o\left\{\eta^{\alpha-1}\eta'\right\}d\nu.$$

The conditions of Lemma 8 are satisfied, since

$$\int^{\nu}\eta^{\alpha-1}\eta' d\nu = \frac{\eta^\alpha}{\alpha}\to\infty,\ \text{as } \nu\to\infty,$$

and since

$$\frac{C_{\lambda\mu}^{pr}(\nu,w)}{w^r}\eta^{\beta_0}(\eta')^{\beta_1}(\eta'')^{\beta_2}\cdots = O_\nu(1).$$

Therefore we may write (4.34) as

$$(4.35)\qquad s^{p-\alpha}w^r o\left\{\int_{\lambda_1}^{v}\eta^{\alpha-1}\eta' d\nu\right\} = (s^p w^r).$$

From (4.31) and (4.35) we have

$$J = o(s^p w^r),$$

which proves that the series is summable (R: $\eta, p; \mu, r$) under the conditions of Case 2. To complete the proof for Case 2, we must show that under the same conditions, $C_{\eta\mu}^{pr}(s, t) = O_s(t^r)$, $[C_{\eta\mu}^{pr} = O(s^p t^r)]$, provided $C_{\lambda\mu}^{pr} = O_s(t^r)$, $[C_{\lambda\mu}^{pr} = O(s^p t^r)]$.

Assume s bounded; then v is bounded. Let $w\to\infty$. From (4.31) we have

$$(4.41)\qquad J_1 = -p!\frac{C_{\lambda\mu}^{pr}(v,w)}{v^p w^r}O(1)s^p w^r$$

$$= O_s(1)O(1)w^r = O_s(w^r).$$

If we assume w bounded, while $s\to\infty$, and hence $\nu\to\infty$, we have

$$(4.42)\qquad J_1 = O_w(1)O(1)s^p = O_w(s^p).$$

The integral of (4.33) may be written

$$s^{p-\alpha}\int_{\lambda_1}^{v} C_{\lambda\mu}^{pr}(\nu,w)\eta^{\beta_0}\eta'\left[O\left(\frac{\eta}{\nu}\right)\right]^{\beta_1-1}\left[O\left(\frac{\eta}{\nu^2}\right)\right]^{\beta_2}\cdots.$$

If $w \to \infty$, while s and v are bounded, this integral is of the form

$$s^{p-\alpha} O_s(w^r) \int_{\lambda_1}^{v} \nu^{-p} \eta^{\alpha-1} \eta' d\nu = O_s(w^r). \tag{4.43}$$

If we assume w bounded, while s and $v \to \infty$, we have

$$\begin{aligned} s^{p-\alpha} \int_{\lambda_1}^{v} O_w(1) \nu^{p-\beta_1-2\beta_2-\cdots+1} \eta^{\alpha-1} \eta' d\nu \\ = s^{p-\alpha} O_w(1) \int_{\lambda_1}^{v} \eta^{\alpha-1} \eta' d\nu = O_w(s^p). \end{aligned} \tag{4.44}$$

From (4.41), (4.42), (4.43), and (4.44) we see that the theorem is true under the conditions of Case 2.

To complete the theorem, we must consider the cases $0 < p < 1$, $r \geqq 0$, and $p > 1$ and not integral, $r \geqq 0$. The proof that the series is summable $(R: \eta, p; \mu, r)$ under the conditions of Case 2 is the same as the proof of the corresponding case of the second theorem of consistency* for simple series, if, in the latter proof, we replace $C_\lambda{}^\kappa(\sigma)$ by $C_{\lambda\mu}^{pr}(\nu, w)$ and $o(\zeta^\kappa)$ by $o(s^p w^r)$, letting $p = \kappa$, $\lambda = \lambda$, and making appropriate changes in ν and s for changes in σ and ζ. The proof that the series is summable under the conditions of the remaining cases may be obtained in the same way from the corresponding cases of the theorem for simple series; the extended proof requires Lemmas 5 and 8, together with the group of lemmas 9, 10 and 11. As in Case 2, the fact that $C_{\eta\mu}^{pr}(s, t) = O_s(t^r)$, $[C_{\eta\mu}^{pr} = O(s^p t^r)]$, may be deduced from the equations used in proving the summability.

Theorem VI. *If the series $\sum u_{kl}$ is summable $(R: \lambda, 0; \mu, r)$ to sum U, and if η is an increasing function of λ, $r \geqq 0$, then $\sum u_{kl}$ is summable $(R: \eta, 0; \mu, r)$. If $C_{\lambda\mu}^{0r}(s, t)$ is bounded for s bounded, or t bounded, or both bounded, $C_{\eta\mu}^{0r}(s, t)$ is bounded similarly.*

This theorem follows immediately from the proof of Case 1, Theorem V′.

If we hold λ fixed, and let $\theta = O(\mu^\Delta)$, and if $p \geqq 0$, $r \geqq 0$, we have a theorem analogous to Theorem V. The theorems corresponding to Theorems V′ and VI are also true. If we now change λ to η under the conditions of Theorem V, and change μ to θ under the conditions of the analogous theorem, we have the following theorem of consistency.

Theorem VII. *If the series $\sum u_{kl}$ is bounded and summable $(R: \lambda, p; \mu, r)$ to sum U, and if η is an L-function of λ, and θ an L-function of μ, such that*

$$\eta = O(\lambda^{\Delta_1}) \text{ and } \theta = O(\mu^{\Delta_2}).$$

* Hardy, loc. cit., pp. 77, 78.

where Δ_1 *and* Δ_2 *are constants, then* $\sum u_{kl}$ *is summable* $(R\colon \eta, p;\ \theta, r)$ *to sum* U, *provided* $p\geqq 0$, $r\geqq 0$. *Both* $s^{-p}t^{-r}C_{\lambda\mu}^{pr}(s,t)$ *and* $s^{-p}t^{-r}C_{\eta\theta}^{pr}(s,t)$ *are bounded.*

This is a special case of the following theorem.

THEOREM VII′. *If the series* $\sum u_{kl}$ *is summable* $(R\colon \lambda, p;\ \mu, r)$ *to sum* U, *and if* $C_{\lambda\mu}^{pr}(s,t)=O(s^pt^r)$; *if* η *is an L-function of* λ, *and* θ *an L-function of* μ, *such that*

$$\eta = O(\lambda^{\Delta_1}) \text{ and } \theta = O(\mu^{\Delta_2}),$$

where Δ_1 *and* Δ_2 *are constants, then* $\sum u_{kl}$ *is summable* $(R\colon \eta, p;\ \theta, r)$ *to sum* U, *and* $C_{\eta\theta}^{pr}(s,t)=O(s^pt^r)$, *provided* $p\geqq 0$, $r\geqq 0$.

V. NECESSARY CONDITIONS FOR SUMMABILITY

THEOREM VIII. *If* $\sum_{k,l=1,\cdots,\infty} u_{kl}$ *is summable* $(R\colon \lambda, p;\ \mu, r)$ *to sum* U, *then*

$$U_{kl} - U = o\left[\left(\frac{\lambda_{k+1}}{\lambda_{k+1}-\lambda_k}\right)^p\left(\frac{\mu_{l+1}}{\mu_{l+1}-\mu_l}\right)^r\right].$$

If, in addition, $C_{\lambda\mu}^{pr}(s,t)=O_s(t^r)$, *then*

$$U_{kl} - U = O_k\left[\left(\frac{\mu_{l+1}}{\mu_{l+1}-\mu_l}\right)^r\right];$$

if $C_{\lambda\mu}^{pr}(s,t)=O_t(s^p)$, *then*

$$U_{kl} - U = O_l\left[\left(\frac{\lambda_{k+1}}{\lambda_{k+1}-\lambda_k}\right)^p\right].$$

This theorem is a special case of the following more general theorem.

THEOREM IX. *If* $\sum_{k,l=1,\cdots,\infty} u_{kl}$ *is summable* $(R\colon \lambda, p;\ \mu, r)$ *to sum* U, *and if* p' *and* r' *are integers such that* $0\leqq p'\leqq p$, $0\leqq r'\leqq r$, *where* $\lambda_k<s$ *for* $p'=0$, $\mu_l<t$ *for* $r'=0$, $\lambda_k\leqq s\leqq\lambda_{k+1}$, *for* $p'>0$, *and* $\mu_l\leqq t\leqq\mu_{l+1}$ *for* $r'>0$, *then*

$$C_{\lambda\mu}^{p'r'}(s,t) - Us^{p'}t^{r'} = o\left[\frac{\lambda_{k+1}^p}{(\lambda_{k+1}-\lambda_k)^{p-p'}}\cdot\frac{\mu_{l+1}^r}{(\mu_{l+1}-\mu_l)^{r-r'}}\right].$$

If, in addition, $C_{\lambda\mu}^{pr}(s,t)=O_s(t^r)$, *then*

$$C_{\lambda\mu}^{p'r'}(s,t) - Us^{p'}t^{r'} = O_s\left[\frac{\mu_{l+1}^r}{(\mu_{l+1}-\mu_l)^{r-r'}}\right];$$

if $C_{\lambda\mu}^{pr}(s,t)=O_t(s^p)$, *then*

$$C_{\lambda\mu}^{p'r'}(s,t) - Us^{p'}t^{r'} = O_t\left[\frac{\lambda_{k+1}^p}{(\lambda_{k+1}-\lambda_k)^{p-p'}}\right].*$$

* Cf. M., p. 521, Theorem II (i). The proof is incorrect when p and r (or either p or r) are integral, since under these conditions the formula for $C^{r,s}(X,\Omega)$ does not follow from Lemma 1.

Case 1. Assume p and r integral; $p \geqq 1, r \geqq 1$.

Assume $U=0$, and u_{kl} real.

Let $\Omega_x-\lambda_k=xh$, where $x=1, 2, \cdots, p$ and $h=(\lambda_{k+1}-\lambda_k)/p$, and let $\psi_y-\mu_l=yj$, where $y=1, 2, \cdots, r$ and $j=(\mu_{l+1}-\mu_l)/r$.

Let Ω and ψ denote any of these numbers.

By Lemma 3, applied twice, we have

$$(5.11)\quad pr\int_0^\Omega\int_0^\psi C_{\lambda\mu}^{p-1,r-1}(\sigma,\tau)d\tau d\sigma = C_{\lambda\mu}^{pr}(\Omega,\psi) = o(\Omega^p\psi^r) = o(\lambda_{k+1}^p\mu_{l+1}^r).$$

Using Lemma 7, and the fact that

$$C_{\lambda\mu}^{pr}(\sigma,\tau) = o(\sigma^p\tau^r) = o(\lambda_{k+1}^p\mu_{l+1}^r),$$

we have

$$\int_{\lambda_k}^\Omega\int_{\mu_l}^\psi C_{\lambda\mu}^{p-1,r-1}(\sigma,\tau)d\tau d\sigma$$

$$= \int_0^\Omega\int_0^\psi - \int_0^{\lambda_k}\int_0^\psi - \int_0^\Omega\int_0^{\mu_l} + \int_0^{\lambda_k}\int_0^{\mu_l} = o(\nu_{k+1}^p\mu_{l+1}^r).$$

Writing

$$\int_{\lambda_k}^\Omega\int_{\mu_l}^\psi C_{\lambda\mu}^{p-1,r-1}(\sigma,\tau)d\tau d\sigma = \int_{\lambda_k}^\Omega\int_{\mu_l}^\psi C_{\lambda\mu}^{p-1,r-1}(\sigma,\tau)d(\psi-\tau)d(\Omega-\sigma)$$

and integrating $(r-1)$ times by parts with respect to τ, and $(p-1)$ times with respect to σ, we have

$$(5.2)\qquad (\psi-\mu_l)\Big[(\Omega-\lambda_k)\,C_{\lambda\mu}^{p-1,r-1}(\lambda_k,\mu_l) + \frac{p-1}{2}(\Omega-\lambda_k)^2C_{\lambda\mu}^{p-2,r-1}(\lambda_k,\mu_l)$$

$$+ \cdots + \frac{1}{p}(\Omega-\lambda_k)^pC_{\lambda\mu}^{0,r-1}(\lambda_k+0,\mu_l)\Big]$$

$$+ \frac{r-1}{2}(\psi-\mu_l)^2\Big[(\Omega-\lambda_k)\,C_{\lambda\mu}^{p-1,r-2}(\lambda_k,\mu_l)$$

$$+ \cdots + \frac{1}{p}(\Omega-\lambda_k)^pC_{\lambda\mu}^{0,r-2}(\lambda_k+0,\mu_l)\Big] + \cdots$$

$$+ \frac{1}{r}(\psi-\mu_l^r)\Big[(\Omega-\lambda_k)C_{\lambda\mu}^{p-1,0}(\lambda_k,\mu_l+0)$$

$$+ \cdots + \frac{1}{p}(\Omega-\lambda_k)^pC_{\lambda\mu}^{00}(\lambda_k+0,\mu_l+0)\Big].$$

For the p different values of x, and the r different values of y, we get pr different linear combinations of the form (5.2). The proof of the first part of the theorem may be completed as in M., p. 523.

The proof of the second part of the theorem is entirely similar if o is replaced by O_s or by O_t.

Case 2. Assume p and r not integral. The proof is that of M., p. 522.

Case 3. Assume p integral, and r not integral.

We define Ω_x as in Case 1, and ψ_y as in Case 2; we apply Lemma 3 twice, and then Lemma 7, and integrate the result by parts, $[r]$ times with respect to τ and $(p-1)$ times with respect to σ. The rest of the proof is similar to that for Case 1.

By a similar proof, we can show that the theorem is true for p not integral, and r integral.

By means of Theorem VIII, we can prove the following theorems.

THEOREM X. *If* $\lambda_m = O(\lambda_m - \lambda_{m-1})$, *then no double series can be summable* $(R: \lambda, p; \mu, 0)$ *unless it is convergent.*

THEOREM XI. *If* $\mu_n = O(\mu_n - \mu_{n-1})$, *then no double series can be summable* $(R: \lambda, 0; \mu, r)$ *unless it is convergent.*

THEOREM XII. *If* $\lambda_m = O(\lambda_m - \lambda_{m-1})$ *and* $\mu_n = O(\mu_n - \mu_{n-1})$, *then no double series can be summable* $(R: \lambda, p; \mu, r)$ *unless it is convergent.*

VI. MULTIPLICATION

We consider the double series $\sum_{m,n=1,\cdots,\infty} a_{mn}$ and $\sum_{m,n=1,\cdots,\infty} b_{mn}$, and form the Dirichlet double series

$$\sum_{1,1}^{\infty,\infty} a_{mn}\, e^{-\lambda_m' s - \mu_n' t} \text{ and } \sum b_{mn}\, e^{-\lambda_m'' s - \mu_n'' t}.$$

Let λ_i be the ascending sequence formed by all the values of $\lambda_p' + \lambda_q''$, and μ_j the ascending sequence formed by all the values of $\mu_{p'}' + \mu_{q'}''$. If $\lambda_{p_1}' + \lambda_{q_1}'' = \lambda_{p_2}' + \lambda_{q_2}''$, so that two values of λ are the same, the order of these two values is indifferent. The same thing is true for two equal values of μ.

The series $\sum_{i,j=1,\cdots,\infty} c_{ij}$, where

$$c_{ij} = \sum_{\lambda_n' + \lambda_k'' = \lambda_i} \quad \sum_{\mu_n' + \mu_l'' = \mu_j} a_{mn} b_{kl},$$

is analogous to the Dirichlet product for simple series. We shall call it the Dirichlet product of the double series $\sum a_{mn}$, $\sum b_{mn}$, of type $(\lambda', \mu'; \lambda'', \mu'')$.

We shall call the double series $\sum_{m,n=1,\cdots,\infty} c_{mn}$, where

$$
\begin{aligned}
c_{mn} = \quad & a_{11} b_{mn} \quad + \cdots + a_{1n} b_{m1} \\
& + a_{21} b_{m-1,n} + \cdots + a_{2n} b_{m-1,1} + \cdots \\
& + a_{m1} b_{1n} \quad + \cdots + a_{mn} b_{11},
\end{aligned}
$$

the Cauchy product of the double series $\sum a_{mn}$ and $\sum b_{mn}$. It is analogous to the Cauchy product for simple series.

The analogues of the theorems of Mertens, Abel and Cauchy for the Cauchy product of simple series have been proved for the Dirichlet product of simple series.* We shall prove corresponding theorems for the Dirichlet product of double series. Since the Cauchy product is a special case of the Dirichlet product, it follows that these theorems must be true also for the Cauchy product of double series.

THEOREM XIII. *If $\sum a_{mn}$ is absolutely convergent to sum A, and $\sum b_{mn}$ convergent to sum B and bounded, then $\sum c_{ij}$, the Dirichlet product of the type $(\lambda', \mu'; \lambda'', \mu'')$, is convergent to sum C and bounded, and $C = AB$.*

We assume that $B = 0$.

Let $C_{\mu\nu}$ be a partial sum of the product series, and let a_{mn} be the a of highest rank in it. Then

$$
c_{\mu\nu} = \sum_{i=1,j=1}^{m,n} a_{ij} B_{kl}, \quad \text{where } B_{kl} = \sum_{i=1,j=1}^{k,l} b_{ij}
$$

and where k is a function of i and m, and l a function of j and n.

Suppose $C_{\mu\nu}$ contains a term $a_{\gamma\delta} b_{\gamma\delta}$. Then it contains all the terms $a_{pq} b_{rs}$ $(p \leqq \gamma, r \leqq \gamma, q \leqq \delta, s \leqq \delta)$. Therefore

$$
\begin{aligned}
& m \geqq \gamma, \qquad n \geqq \delta ; \\
& k \geqq \gamma, \quad \text{for } i = 1, 2, \cdots, \gamma, \\
& l \geqq \delta, \quad \text{for } j = 1, 2, \cdots, \delta.
\end{aligned}
$$

Since $\sum b_{mn} \to 0$, and since $\sum a_{mn}$ is absolutely convergent, it is possible to choose γ and δ so that

$$
| B_{kl} | < \epsilon, \quad k \geqq \gamma, \; l \geqq \delta
$$

and

$$
\sum_{1,1}^{\infty,\infty} | a_{ij} | - \sum_{1,1}^{\gamma,\delta} | a_{ij} | < \epsilon .
$$

* H. R., pp. 63, 64.

By hypothesis, B_{kl} is bounded. Let $\bar{B}$ be an upper bound of $|B_{kl}|$. Then

$$|C_{\mu\nu}| \leqq \bar{B}\left[\sum_{1,1}^{m,n}|a_{ij}| - \sum_{1,1}^{\gamma,\delta}|a_{ij}|\right] + \epsilon\sum_{1,1}^{\gamma,\delta}|a_{ij}|$$
$$< \epsilon(\bar{B} + A).$$

Therefore $C_{\mu\nu}$ approaches zero as γ and δ approach ∞, that is, as m and n approach ∞.

If $B \neq 0$, we form a new series, $\sum b'_{mn}$, such that

$$b'_{11} = b_{11} - B$$

and

$$b'_{mn} = b_{mn}, \quad \text{for } m \neq 1 \text{ and } n \neq 1.$$

Then the Dirichlet product of $\sum a_{mn}$ and $\sum b'_{mn} \to 0$: therefore the Dirichlet product of $\sum a_{mn}$ and $\sum b_{mn} \to AB$.

To prove that the product series is bounded, we prove first that since $\sum a_{mn}$ is absolutely convergent, $\sum |a_{mn}|$ is bounded. For there exists an M, such that when $m > M$, $n > M$,

$$\sum_{1,1}^{m,n}|a_{ij}| < A + \epsilon.$$

But for any p and q we can find an m and an n, such that $p < m$, $q < n$, $m > M$, $n > M$. Therefore

$$\sum_{1,1}^{p,q}|a_{ij}| < \sum_{1,1}^{m,n}|a_{ij}| < A + \epsilon, \text{ for all } p \text{ and } q,$$

and $\sum |a_{mn}|$ is bounded.

Let A be an upper bound of $\sum |a_{mn}|$. We have

$$C_{\mu\nu} = \sum_{i=1,j=1}^{m,n} a_{ij}B_{kl}.$$

Therefore

$$\sum_{i=1,j=1}^{m,n} |a_{ij}||B_k| \leqq B\cdot\bar{A}$$

Theorem XIV. *If $\sum a_{mn}$, $\sum b_{mn}$ and $\sum c_{mn}$ are all convergent and bounded, where $\sum c_{mn}$ is the Dirichlet product, then $AB = C$.*

By means of Theorem IV, we deduce this theorem from the following more general theorem.

THEOREM XV. *If $\sum a_{mn}$ is summable $(R: \lambda', p'; \mu', r')$ to sum A, and if $\sum b_{mn}$ is summable $(R: \lambda'', p''; \mu'', r'')$ to sum B, where p', r', p'' and $r'' > -1$, and if $\sigma^{p'}\tau^{r'}A^{p'r'}_{\lambda'\mu'}(s,t)$ and $\sigma^{p''}\tau^{r''}B^{p''r''}_{\lambda''\mu''}(s,t)$ are bounded, then the Dirichlet product series, $\sum c_{mn}$, is summable $(R: \lambda, p; \mu, r)$ to sum C, where $p = p'+p''+1$, $r = r'+r''+1$, and $AB = C$, and $C^{pr}_{\lambda\mu}(s, t)$ is bounded.*

We have

$$A^{p'r'}_{\lambda'\mu'}(s,t) = \sum_{\lambda'_m<s} \sum_{\mu'_n<t} (s - \lambda'_m)^{p'}(t - \mu_n)^{r'}a_{mn},$$

$$B^{p''r''}_{\lambda''\mu''}(s,t) = \sum_{\lambda_k''<s} \sum_{\mu_l''<t} (s - \lambda_k'')^{p''}(t - \mu_l'')^{r''}b_{kl},$$

and

$$C^{pr}_{\lambda\mu}(s,t) = \sum_{\lambda_i<s} \sum_{\mu_j<t} (s - \lambda_i)^p(t - \mu_j)^r c_{ij}.$$

We shall prove that

$$(6.1) \qquad C^{pr}_{\lambda\mu}(s,t) = \frac{\Gamma(p+1)}{\Gamma(p'+1)\Gamma(p''+1)} \frac{\Gamma(r+1)}{\Gamma(r'+1)\Gamma(r''+1)} \cdot \int_0^s \int_0^t A^{p'r'}_{\lambda'\mu'}(\sigma, \tau)B^{p''r''}_{\lambda''\mu''}(s - \sigma, t - \tau)d\tau d\sigma.$$

For consider the term $a_{mn}b_{kl}$. It occurs in $C^{pr}_{\lambda\mu}(s, t)$ if $\lambda_m'+\lambda_k''<s$ and $\mu_n'+\mu_l''<t$, with the coefficient

$$(s - \lambda_i)^p(t - \mu_j)^r = (s - \lambda'_m - \lambda_k'')^p(t - \mu'_n - \mu_l'')^r.$$

The term a_{mn} occurs in $A^{p'r'}_{\lambda'\mu'}(\sigma, \tau)$ if $\lambda_m'<\sigma$, and $\mu_n'<\tau$, with the coefficient $(\sigma-\lambda_m')^{p'}(\tau-\mu_n')^{r'}$; b_{kl} occurs in $B^{p''r''}_{\lambda''\mu''}(s-\sigma, t-\tau)$ if $\lambda_k''<s-\sigma$, and $\mu_l''<t-\tau$, with the coefficient $(s-\sigma-\lambda_k'')^{p''}$ $(t-\tau-\mu_l'')^{r''}$. Therefore $a_{mn}b_{kl}$ occurs in the right hand member of (6.1) if $\lambda_m'+\lambda_k''<s$ and $\mu_n'+\mu_l''<t$, and it scoefficient is

$$\frac{\Gamma(p+1)}{\Gamma(p'+1)\Gamma(p''+1)} \frac{\Gamma(r+1)}{\Gamma(r'+1)\Gamma(r''+1)}$$

$$\cdot \int_{\lambda'_m}^{s-\lambda''_k} \int_{\mu'_n}^{t-\mu''_l} (\sigma - \lambda'_m)^{p'}(\tau - \mu_n)^{r'}(s - \sigma - \lambda_k'')^{p''}(t - \tau - \mu_l'')^{r''}d\tau d\sigma$$

$$= \frac{\Gamma(p+1)}{\Gamma(p'+1)\Gamma(p''+1)} \frac{\Gamma(r+1)}{\Gamma(r'+1)\Gamma(r''+1)} \int_0^x \int_0^y u^{p'}v^{r'}(x - u)^{p''}(y - v)^{r''}dvdu$$

$= x^{p'+p''+1}y^{r'+r''+1} = (s - \lambda_m' - \lambda_k'')^p(t - \mu_n' - \mu_l'')^r.$

Therefore (6.1) is established.

But $A_{\lambda'\mu'}^{p'r'}(\sigma, \tau) \sim A\sigma^{p'}\tau^{r'}$ and $B_{\lambda''\mu''}^{p''r''}(\sigma, \tau) \sim B\sigma^{p''}\tau^{r''}$, and $A_{\lambda'\mu'}^{p'r'}(\sigma, \tau)/(\sigma^{p'}\tau^{r'})$ and $B_{\lambda''\mu''}^{p''r''}(\sigma, \tau)/(\sigma^{p''}\tau^{r''})$ are bounded.

Therefore by Lemma 2, $C_{\lambda\mu}^{pr}(s, t) \sim ABs^pt^r$, and the transformed sequence is bounded, which proves the theorem.

THEOREM XVI. *If $\sum a_{mn}$ and $\sum b_{mn}$ are absolutely convergent with sums A and B, then the Dirichlet product series of type $(\lambda', \mu'; \lambda'', \mu'')$ is absolutely convergent with sum AB.*

Since $\sum a_{mn}$ and $\sum b_{mn}$ are absolutely convergent, the series $\sum |a_{mn}|$ and $\sum |b_{mn}|$ are convergent and bounded. Therefore, by Theorem XIII, their product series is convergent and bounded.

Suppose

$$c_{ij} = \sum_{\lambda_m'+\lambda_k''=\lambda_i} \sum_{\mu_n'+\mu_l''=\mu_j} |a_{mn}b_{kl}|$$

and

$$d_{ij} = \sum_{\lambda_m'+\lambda_k''=\lambda_i} \sum_{\mu_n'+\mu_l''=\mu_j} a_{mn}b_{kl}.$$

Then $|d_{ij}| \leqq c_{ij}$, and by the comparison test for double series, $\sum |d_{ij}|$ is convergent. Hence $\sum d_{ij}$ is absolutely convergent and by Theorem XIII its sum is AB.

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