TYPES OF MOTION OF THE GYROSCOPE*

BY

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1. **Introduction.** We can display, by a simple graphical representation, the complete history of all the motions of all gyroscopes which are subject to the following restrictions:† each gyroscope is acted upon by gravity, by a constraint which keeps one point on its axis of symmetry fixed in space, and by no other forces. Furthermore, with the aid of Osgood's intrinsic equations,‡ we shall be able to include in this history a discussion of some new intrinsic properties of the cone traced in space by the axis of symmetry of such a gyroscope. This cone intersects the unit sphere whose center is at the fixed point, in a curve which we shall call Γ. We shall describe the properties of this cone in terms of the geodesic curvature, k, of Γ. The advantage in Osgood's equations lies in the fact that they yield a simple expression for k. We shall also be able to obtain additional properties of the motion from an analysis of the curves Γ passing near the poles (the north pole being at the top of the unit sphere). By means of these methods, we shall list the properties of the motion more fully than has hitherto been done, exhibit their dependence upon the initial conditions, and show how one type of motion changes into another as the initial conditions are varied continuously.

2. **Equations of the motion.** In the case of the gyroscope which we are considering, Osgood's equations take the following form:§

$$(1)\qquad \begin{aligned} &\text{(a)} & Avv' &= Mgh(\sin\theta)\theta', \\ &\text{(b)} & Akv^2 - Crv &= -\,Mgh(\sin^2\theta)\psi', \\ &\text{(c)} & C\dot{r} &= 0, \end{aligned}$$

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† For discussions of the motion of such a gyroscope, see, for example Klein und Sommerfeld, *Theorie des Kreisels*, Heft II; Appell, *Traité de la Mécanique Rationnelle*, Tome II, pp. 193–209; Encyklopädie der Mathematischen Wissenschaften, Band IV_1, pp. 619–639; or Lamb, *Higher Mechanics*, Chap. VIII.

‡ Cf. W. F. Osgood, *On the gyroscope*, these Transactions, vol. 23 (1922), pp. 240–264.

§ Cf. Osgood, loc. cit., p. 247, formula (7) and p. 256, formula (1). The term Crv of equation (1b) of this paper is preceded by a minus sign instead of the plus sign in Osgood's article. This change of sign is due to the fact that we have chosen a right-handed instead of a left-handed system of coördinates.

where C is the moment of inertia of the gyroscope about its axis of symmetry; A, the moment of inertia about a perpendicular axis through the fixed point; M, the mass of the gyroscope; h, the distance of the center of mass from the fixed point; g, the acceleration of gravity; and r, the spin of the gyroscope about its axis. Likewise, θ and ψ denote, respectively, the colatitude and longitude of a point, ξ, which is a trace of the axis of the gyroscope on the surface of the unit sphere;* k denotes the geodesic curvature of Γ at the point ξ; and v, the velocity of this point. The primes denote differentiation with respect to the arc length, s, of Γ, and the dot denotes differention with respect to the time, t. Equations (1) are equivalent to the vector equation $\dot{\sigma}=\mu$, where σ is the moment of momentum vector taken with respect to the fixed point and μ is the resultant moment of the external forces with respect to the fixed point. Equation (1c) implies that r is a constant. This constant is positive. Equation (1a) has the following integral:

$$v^2 = v_0^2 + a(u_0 - u), \tag{2}$$

where $a=2Mgh/A$ and $u=\cos\theta$, and where v_0 and u_0 are respectively the values of v and u at some time, t_0.

In addition to equations (1) and (2), we have the following equations:†

$$v^2 = \dot{\theta}^2 + (\sin^2\theta)\dot{\psi}^2, \tag{3}$$

$$(1 - u_0^2)\dot{\psi}_0^2 + \gamma(u_0 - u) = (1 - u^2)\dot{\psi} \tag{4}$$

where $\gamma = Cr/A$, and

$$\dot{u}^2 = (1 - u^2)[v_0^2 + a(u_0 - u)] - [(1 - u_0^2)\dot{\psi}_0 + \gamma(u_0 - u)]^2 = f(u). \tag{5}$$

Equations (4) and (5) define the motion of the gyroscope. If u_0 is a root of $f(u)$, then a solution of these equations is $u\equiv u_0$, $\dot{\psi}\equiv\dot{\psi}_0$. It is a familiar fact that if u_0 is a double root of $f(u)$, then this solution satisfies equations $\dot{\sigma}=\mu$, but that otherwise the solution is extraneous.‡ When $u\equiv u_0$ and $\dot{\psi}\equiv\dot{\psi}_0$,

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* The point ξ is that one of the two traces of the axis on the sphere such that an observer situated at the fixed point and looking at ξ sees the gyroscope rotate in a clockwise sense.

† Equation (3) is a consequence of the fact that the point ξ always lies in the surface of the unit sphere. (Cf. Osgood, loc. cit., p. 247.) Equation (4) is a consequence of the fact that the vector μ is always horizontal and hence the vertical component of the vector σ is a constant. Equation (5) is obtained by combining equations (2), (3), and (4). (For equations (4) and (5), cf. Osgood, loc. cit., p. 257, equation (5); also Appell, loc. cit., Tome II, p. 196, equations 50, 51.)

‡ The intrinsic equations afford the following simple proof of this fact. If $u\equiv u_0$, then by equation (2), v is constant, and hence it follows from equation (1b) that k must be constant. Moreover the value k_0 of k obtained by solving this equation, must be equal to the geodesic curvature k_c of the parallel of latitude, u_0. O. D. Kellogg (*Curvature and the top*, these Transactions, vol. 25 (1923), p. 518, formula 29) has obtained the equation

the gyroscope is said to execute steady precession.* Without loss of generality, we can assume that u_0 is a root of $f(u)$. To see this let us assume that u_0 is not a root of $f(u)$. Then, since $f(u_0)\geqq 0$ and $-1\leqq u_0\leqq +1$ for real motion, it follows that $f(u_0)>0$. Furthermore, since $f(\pm 1)\leqq 0$, then $-1<u_0<+1$ and there exist at least two distinct roots of $f(u)$ in the interval $-1\leqq u\leqq +1$. Let u_1 be one of these roots. Since $f(u)$ is a cubic, u_1 may be so chosen that it is a single root and that no other root of $f(u)$ lies between u_0 and u_1.† Then we may combine the equations $v^2=v_0^2+a(u_0-u)$ and $v_1^2=v_0^2+a(u_0-u_1)$ and obtain the equation

$$v^2 = v_1^2 + a(u_1 - u).$$

Likewise we may combine the equations $(1-u^2)\dot{\psi}=(1-u_0^2)\dot{\psi}_0^2+\gamma(u_0-u)$ and $(1-u_1^2)\dot{\psi}_1=(1-u_0^2)\dot{\psi}_0+\gamma(u_0-u_1)$ and obtain the equation

$$(1 - u^2)\dot{\psi} = (1 - u_1^2)\dot{\psi}_1 + \gamma(u_1 - u).$$

Hence we may write

$$f(u) = (1 - u^2)[v_1^2 + a(u_1 - u)] - [(1 - u_1^2)\dot{\psi}_1 + \gamma(u_1 - u)]^2$$

where $-1\leqq u_1\leqq +1$. Thus we may take the cosine of the colatitude of the initial point to be u_1 instead of u_0. We can now let u_0 play the role of u_1, that is, we can assume that u_0 is a root of $f(u)$.

From this point on we shall assume that $f(u_0)=0$. There is a second root, u_1, of $f(u)$ in the interval $-1\leqq u_1\leqq +1$, and a third root $u_2\geqq +1$. It is a familiar fact that Γ lies completely in the zone bounded by the parallels of latitude $u=u_0$ and $u=u_1$ and that, in general, it oscillates between points of tangency to u_0 and to u_1.

3. **Graphical representation of gyroscopic motion.** Equations (4) and (5) show that the type of motion is completely determined by the values

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$$k_0 - k_c = -\frac{1-u_0}{v_0^3}\dot{\psi}_0\left(u_0\dot{\psi}_0^2 - \gamma\dot{\psi}_0 + \frac{1}{2}a\right).$$

But we have also the equation

$$g(u_0) = 2(1 - u_0^2)[u_0\dot{\psi}_0^2 - \gamma\dot{\psi}_0 + \tfrac{1}{2}a],$$

and hence we may write

$$k_0 - k_c = \frac{\dot{\psi}_0}{2v_0^3}\, g(u_0).$$

It follows that $k_0=k_c$ only when $g(u_0)$ or $\dot{\psi}_0$ vanishes. But $u=u_1$ is the only solution of $g(u)=0$ for which $-1<u<+1$, and hence $g(u_0)=0$ only when $u_0=u_1$. If the factor $\dot{\psi}_0$ vanishes then, since $\dot{u}\equiv 0$, $v_0=0$ and hence $v\equiv 0$. It follows that the moment of momentum vector, σ, is constant and therefore the moment μ is zero. But this is only possible if $u_0=\pm 1$ and if u_0 is a double root of $f(u)$. (Cf. p. 745.)

* Cf. Lamb, loc. cit., p. 130.

† If u_1 is a double root and initially $u=u_1$, then we have seen that the solution is $u\equiv u_1$. This solution cannot be equivalent to one for which $u=u_0$ initially.

of the parameters u_0, ψ_0, a and γ.* We shall see that a given situation which characterizes a type of motion—for example, a change in the sign in the longitudinal velocity—occurs every time Γ crosses a certain parallel of latitude, u. If the parameters u_0, a, and γ are held constant and u and ψ_0 are allowed to vary, then the locus of points for which a given situation occurs, is a curve in the u, ψ_0-plane. These curves are shown in Figs. I to XI. The loci corresponding to points of tangency of Γ to the parallels of latitude, u_0 and u_1, are respectively the straight line $u=u_0$ and the curve $u=u_1$. Since Γ lies entirely within the zone bounded by the parallels of latitude u_0 and u_1, then in a u, ψ_0-diagram, the points which correspond to real motoin must lie in the region between the line $u=u_0$ and the curve $u=u_1$. In Figs. I to XI, this region is indicated by the shading.

The straight line $u=u_L$ and the parabolas $u=u_i$ and $u=u_k$ are loci which correspond respectively to points at which the longitudinal velocity, ψ, the geodesic curvature, k, and the derivative, k', of k with respect to the arc length, s, of Γ, vanish. By studying these loci, we can determine whether Γ has waves, loops, or cusps, whether or not it has points of inflection, and whether or not its curvature is monotone between points of tangency to the parallels of latitude, u_0 and u_1. Only those portions of the curves which lie in the shaded region correspond to real motion. All of these loci are dependent upon the values of u_0, a, and γ. However, their general character remains unchanged throughout large ranges in the values of these parameters, so that the history of gyroscopic motion (which is subject to the restrictions enumerated on p. 737) can be represented by means of eleven u, ψ_0-diagrams shown in Figs. I to XI, pp. 760–763. These diagrams are classified according to whether the quantities D, d, and u_0 are positive, negative, or zero, D and d being defined by the equations

$$D = \gamma^2 - a(1 + u_0), \qquad d = \gamma^2 - 2au_0. \tag{6}$$

Since each of the quantities D, d, and u_0 may be positive, negative, or zero, we should expect twenty-seven different cases. However, we have the inequality $d>D$, since $d-D=(1-u_0)>0$. This inequality eliminates twelve of the twenty-seven possibilities. Of the remaining fifteen, four are eliminated by the fact that $d>0$ when $u_0\leqq 0$.

Let us see how the quantities d and D are related to the motion of the gyroscope.

We shall show that the curve $u=u_1$ has a single maximum and a single minimum. If $D\geqq 0$, the maximum occurs when $u_1=+1$ (see Figs. I to VI).

* In this section it will be assumed that $-1<u_0<+1$. The case $u_0=\pm 1$ will be treated in §4.

If a gyroscope be given initial conditions such that $D \geqq 0$ and $\dot{\psi}_0$ has the value for which u_1 is a maximum, then for this value of $\dot{\psi}_0$, $u_1 = +1$ and the curve Γ traced by the gyroscope is bounded merely by the parallel of latitude u_0 and the north pole. If D is actually greater than zero, the point ξ, which traces Γ, passes periodically through the north pole. If, however, D is equal to zero, the point ξ approaches but never reaches the north pole. Hence, when $D=0$, t must become infinite as u approaches $+1$. That is, $+1$ must be a double root of $f(u)$. This corresponds in the u, $\dot{\psi}_0$-diagram, to the fact that the maximum of u_1 coincides with the minimum of the curve u_2 when $D=0$ (see Figs. IV, V, VI).

If $D<0$, then the maximum of u_1 is less than $+1$. Hence, for such a value of D, it is impossible for ξ ever to pass through or approach asymptotically the north pole.

The minimum of u_1 is always -1. Therefore, no matter what other initial conditions may be assigned, it is always possible to choose the longitudinal velocity, $\dot{\psi}_0$, so that ξ will pass through the south pole.

A gyroscope executes steady precession on a parallel of latitude, u_0, when and only when u_0 is a double root of $f(u)$. A double root of $f(u)$ corresponds in the u, $\dot{\psi}_0$-diagram, to a point of intersection of the curve $u=u_1$ with the line $u=u_0$. But the curve u_1 has two, one, or zero intersections with the line u_0 according as d is positive, zero, or negative. Hence, there are two, one, or zero values of the longitudinal velocity $\dot{\psi}_0$ for which the gyroscope can execute steady precession on the parallel of latitude, u_0, according as d is positive, zero, or negative.

The equation $\gamma = Cr/A$ shows that the quantities D and d depend upon the spin of the gyroscope. Thus, if the spin is large, D and d are positive. If the spin is small, then D is negative. Furthermore, if u_0 is positive, the spin can be taken so small that d is negative. Hence, by a choice of the spin, we can determine whether the axis of the gyroscope can pass through or approach asymptotically the north pole, and whether or not the gyroscope can execute steady precession on the parallel of latitude, u_0.

Let us see what types of motion are possible when we start a given gyroscope in such a manner that initially its axis passes through a point of a parallel of latitude u_0, the latitudinal velocity, $\dot{\theta}$, is zero, and the spin is so great that both d and D are positive. This situation corresponds to Figs. I, II, or III according as u_0 is positive, zero, or negative. Let us assume for definiteness that $u_0>0$.

We shall first take $\dot{\psi}_0$ equal to zero. In the u, $\dot{\psi}_0$-plane, the points corresponding to this motion constitute the segment of the u-axis included between the line $u=u_0$ and the curve $u=u_1$ (see Fig. I). The point $(0, u_0)$

corresponds to the initial point of the motion. It will be observed that the lines $u=u_0$ and $u=u_L$ intersect in this point. This corresponds to the fact that both the latitudinal and the longitudinal velocities are zero initially. Since these initial velocities are both zero, Γ must be a curve with cusps, the first cusp occurring at the initial point of the motion and the others occurring periodically thereafter. The point $(0, u_0)$ corresponds to all of these cusps.

The fact that the curves $u=u_i$ and $u=u_k$ pass through the point $(0, u_0)$ is without great significance to this motion; this point cannot correspond to a point of inflection of Γ since it corresponds to a cusp. However, the fact that the curves $u=u_i$ and $u=u_k$ do not cut the interior of the segment described above, shows that Γ does not have inflection points and that the geodesic curvature of Γ is monotone between points of contact with the parallels of latitude, u_0 and u_1.

Next let us take ψ_0 small but positive. Since the line $u=u_L$ does not enter the shaded region immediately to the right of the u-axis, it follows that for motions corresponding to such values of ψ_0, the longitudinal velocity never vanishes. Furthermore, since the longitudinal velocity is positive when u is equal to u_0, it is always positive. Thus Γ must be a wavy curve on which the point ξ moves eastward. This fact is indicated in the u, ψ_0-diagram by shading which slants to the right. The curves $u=u_i$ and $u=u_k$ enter the shaded region to the right of the u-axis and therefore a curve Γ corresponding to a small but positive value of ψ_0 has points of inflection and its geodesic curvature is not monotone between points of tangency to the parallels of latitude u_0 and u_1.

As we increase ψ_0, we reach a value corresponding to which the curve $u=u_k$ crosses the curve $u=u_1$ and leaves the shaded region. For this value of ψ_0 (and slightly greater values), the curve Γ is still wavy and still has points of inflection but the geodesic curvature of Γ is now monotone between points of tangency to the limiting parallels of latitude.

As ψ_0 is further increased, a value, $a/(2\gamma)$, is reached corresponding to which the curve $u=u_i$ crosses the line $u=u_0$ and leaves the shaded region. For this value of ψ_0, Γ no longer has inflection points; it is, however, still a wavy curve and the geodesic curvature is monotone between points of tangency to the limiting parallels of latitude. The same is true for slightly greater values of ψ_0.

Next we reach the point at which the curve $u=u_1$ intersects the straight line $u=u_0$. For this value of ψ_0, the limiting parallels of latitude, u_0 and u_1, coincide, Γ itself reduces to a parallel of latitude, u_0, and the gyroscope executes steady precession with positive longitudinal velocity.

If ψ_0 is still further increased, we reach a portion of the shaded region which does not contain the curve $u=u_i$ nor $u=u_k$ nor the straight line $u=u_L$. For such a value of ψ_0, the curve Γ is a wavy curve on which ξ moves eastward, it is without points of inflection and its geodesic curvature is monotone between points of tangency to the limiting parallels of latitude.

As we proceed further to the right, we reach a point at which the curve $u=u_i$ crosses the curve $u=u_1$ and reenters the shaded region. Hence we get again curves Γ with inflection points. As we proceed still further we reach a point at which the curve $u=u_k$ crosses the line $u=u_0$ and reenters the shaded region. Hence we have again curves Γ for which the geodesic curvature fails to be monotone between points of tangency to the limiting parallels of latitude.

When $\psi_0=a/\gamma$, the line $u=u_L$ and the curves $u=u_1$, $u=u_i$, and $u=u_k$ are concurrent. The curves $u=u_i$ and $u=u_k$ leave the shaded region and do not reenter at any point to the right of a/γ. Hence for $\psi_0 \geqq a/\gamma$, the curves Γ are without points of inflection and their geodesic curvatures are monotone between points of tangency to their limiting parallels of latitude. The intersection of the line $u=u_L$ with the curve $u=u_1$ corresponds to the fact that the velocity, v, vanishes. Hence when $\psi_0=a/\gamma$, Γ is a curve with cusps.

The line $u=u_L$ enters the shaded region to the right of the line $\psi_0=a/\gamma$. Hence the longitudinal velocity vanishes between points of tangency of Γ to the limiting parallels of latitude. We shall prove that the longitudinal velocity changes sign whenever it vanishes, provided the value of u at which it vanishes lies in the open interval from u_0 to u_1. Assuming this to be true, it follows that the longitudinal velocity is negative at the point of tangency of Γ to the parallel of latitude u_1 since it is positive at the points of tangency of Γ to the parallel of latitude u_0. Thus Γ is a curve with loops. This type of motion is indicated by vertical shading.

If $\psi_0=\gamma/(1+u_0)$, then $u_1=+1$. Moreover, the curve $u=u_L$ does not pass through the point $(\gamma/(1+u_0), +1)$. Hence $+1$ is a root but not a double root of $f(u)$. It follows that for this value of ψ_0, Γ passes periodically through the north pole. It should be observed that, although $\dot{u}$ vanishes, at the north pole, $\dot{\theta}$ does not (when $D>0$). In fact, v must be different from zero at the north pole, since the contrary assumption leads to the conclusion that $+1$ is a double root of $f(u)$ when $\psi_0=\gamma/(1+u_0)$.

The line $u=u_L$ leaves the shaded region at the point $(\gamma/(1+u_0), +1)$ and does not enter at any point for which $\psi_0>\gamma/(1+u_0)$. It follows that, except when $u_0=u_1$, for such values of ψ_0, Γ is a wavy curve on which the point ξ moves eastward. But $u_0=u_1$ for only one value of ψ_0 greater than

$\gamma/(1+u_0)$. For this value of ψ_0, the gyroscope executes steady precession on the parallel of latitude, u_0.

The curves $u=u_i$ and $u=u_k$ do not enter the shaded region to the left of the u-axis. Hence a curve Γ which corresponds to a negative value of ψ_0 is without inflection points and the geodesic curvature of such a curve is monotone between points of tangency to its limiting parallels of latitude. The straight line $u=u_L$, on the other hand, enters the region to the left of the u-axis. Hence for a small negative value of ψ_0, the corresponding curve Γ is a curve with loops.

When $\psi_0=-\gamma/(1-u_0)$, $u_1=-1$. Thus for this value of ψ_0, the curve Γ passes through the south pole. The line $u=u_L$ leaves the shaded region at the point $(-\gamma/(1-u_0),\ -1)$ and does not reenter at any point for which $\psi_0<-\gamma/(1-u_0)$. It follows that all curves for which $\psi_0<-\gamma/(1-u_0)$, are wavy curves on which ξ moves westward.

Let $\{s_0s_1\}$ be an arc of Γ included between a point of tangency of Γ to the parallel of latitude u_0 and the next subsequent point of tangency to the parallel u_1. It will be recalled that if ψ_0 is such that either $\psi_0<0$ or $\psi_0>a/\gamma$, then on the corresponding arc $\{s_0s_1\}$, neither k nor k' vanishes at an interior point. It will be proved that such an arc lies entirely within its osculating circle at one extremity and entirely without its osculating circle at the other extremity. Within is understood to mean in that one, with the least area, of the two regions into which the given circle divides the spherical surface.

Figs. II to XI can be interpreted in a similar manner. Figs. I to XI are plotted for the following values of the parameters u_0, a, and γ:

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
u_0	.2	0	−.2	.3	0	−.3	.2	0	−.2	.2	.2
a	.8	.8	.8	1.3	1	.7	2	2	2	.1	.1
γ	1	1	1	1.3	1	.7	1	1	1	.2	.1

The general character of any one of these figures will remain unchanged if we make a variation in the values of u_0, a, γ, which does not alter the classification of this figure.

Let us investigate whether the data we have selected determines a reasonable physical situation. We may take as our gyroscope a homogeneous cone with altitude H, and radius of base R. For such a gyroscope we have the equations

$$a=\frac{5g}{R}\cdot\frac{\lambda}{1+\lambda^2},\qquad \gamma=\frac{2r}{1+\lambda^2}$$

where $\lambda=2H/R$. We shall choose the altitude to be two and a half times the diameter of the base. Then $\lambda=10$. In Case XI, $a=.1$, and hence for this case, the radius of the base of our gyroscope is about 4850 cm.

We do not, however, need to deal with such an enormous gyroscope. In fact, we can alter the quantities a and γ without changing the classification of the motion provided we keep u_0 and the ratio $\rho=a/\gamma^2$ fixed. Since both a and γ^2 have the dimensions T^{-2}, such an alteration may be brought about by a change in the unit of time. The effect of such a change, upon the u, ψ_0-diagram, is a uniform stretching in the direction of the ψ_0- axis.

We can now choose $\lambda=10$, $a=100$, and $\gamma=10/\rho^{1/2}$. We obtain a conical gyroscope with altitude 24.25 cm. and diameter of base 9.7 cm. In Figs. I to XI, ρ varies from about .78 to 10. Hence for the corresponding motions the gyroscope makes from about 25 to 90 revolutions per second about its axis.

4. Some exceptional cases. In §3, it was assumed that $-1<u_0<+1$. It remains to consider the possibilities $u_0=\pm 1$. In order to study these possibilities, we shall form the function*

$$\frac{f(u)}{u_0-u}=(u_0+u)[v_0^2+a(u_0-u)]-\gamma^2(u_0-u)=g(u). \tag{7}$$

Since $g(u_0)=2u_0v_0^2$ and $g(-u_0)=-2u_0\gamma^2$, it follows that $g(-1)\leqq 0$ and $g(+1)\geqq 0$ and, since $g(+\infty)=-\infty$, there is a root, u_1, of $g(u)$ in the interval $-1\leqq u_1\leqq +1$, and a root $u_2\geqq +1$. Moreover, $u_0=u_1$ or u_2 when and only when $v_0=0$, and $u_0=-u_1$ or $-u_2$ when and only when $\gamma=0$.

When $u_1=u_0=\pm 1$ or $u_2=u_0=+1$, the equation $\dot{\sigma}=\mu$ may be satisfied by setting $v\equiv 0$ and $u\equiv u_0$. For, when $u=\pm 1$, $\mu=0$ and, when $v\equiv 0$ and r is constant, σ is a constant vector. In this case, the axis of the gyroscope remains vertical.

If $\gamma=0$, then $r=0$ and, if $u_0=\pm 1$, then $\psi\equiv 0$. Thus, if $v_0\neq 0$, the gyroscope becomes a plane pendulum.† There are several cases to consider, namely, if $u_0=-1$, then either u_1 or u_2 (or both) is equal to $+1$, and, if $u_0=+1$, then $u_1=-1$ and $u_2>+1$.

First let us take $u_0=-1$ and $u_2=+1>u_1$. In this case, the amplitude of the swing is $2\theta_1$ where $\cos\theta_1=u_1$. Next let us take $u_0=-1$ and $u_1=u_2=+1$. In this case, the angular velocity of the pendulum at the south pole is such that the pendulum approaches but never reaches the north pole.‡ Next

* In equation (7) it is assumed that $u_0=\pm 1$. If $u_0\neq\pm 1$, then the formula for $g(u)=f(u)/(u_0-u)$ is given by equation (8).

† Cf. Love, *Theoretical Mechanics*, p. 129.

‡ Cf. Love, loc. cit., p. 131.

let us take $u_0=-1$ and $u_1=+1<u_2$. In this case, the angular velocity of the pendulum at the south pole is such that the pendulum passes through the north pole and swings completely around.* Finally, let us take $u_0=+1<u_2$ and $u_1=-1$. This case is the same as the case just discussed.

If $u_0=\pm 1$ and neither v_0 nor γ is zero, then $-1<u_1<+1<u_2$, and hence we can interchange the rôles of u_0 and u_1. That is, this case is equivalent to the one in which $-1<u_0<+1$.

5. **The bounding parallels of latitude.** In the u, ψ_0-diagram, the locus of points corresponding to points of tangency of Γ to the parallel of latitude u_0, is the straight line whose equation is $u=u_0$. Since u_1 is also a root of $f(u)$, the equation of the curve $u=u_1$ can be obtained by solving the equation $g(u)=0$ where $g(u)$ is defined by the equation

$$
\begin{aligned}
(8)\qquad g(u) &= f(u)/(u_0-u)\\
&= -au^2+[(1-u_0^2)\psi_0^2+\gamma^2]u+[(1-u_0^2)\psi_0^2u_0\\
&\qquad\qquad + a - 2(1-u_0^2)\gamma\psi_0-\gamma^2u_0].
\end{aligned}
$$

Likewise, u_2 is a root of $g(u)$. Solving equation (8), we get

$$
(9)\quad u_1=\frac{(1-u_0^2)\psi_0+\gamma^2-(R(\psi_0))^{1/2}}{2a},\quad u_2=\frac{(1-u_0^2)\psi_0+\gamma^2+(R(\psi_0))^{1/2}}{2a},
$$

where

$$
\begin{aligned}
R(\psi_0) &= (1-u_0^2)^2\psi_0^4+2(1-u_0^2)(\gamma^2+2au_0)\psi_0^2-8a\gamma(1-u_0^2)\psi_0\\
&\quad +(\gamma^4-4a\gamma^2u_0+4a^2).
\end{aligned}
$$

In order to study the nature of the curve $u=u_1$, we shall compute its maxima and minima, treating, for convenience, u_1 as an implicit function defined by the equation $g(u)=0$ instead of making use of its explicit expression. Thus

$$
\frac{du_\nu}{d\psi_0}=-\left.\frac{\dfrac{\partial g}{\partial\psi_0}}{\dfrac{\partial g}{\partial u}}\right|_{u=u_\nu}=\frac{-2(1-u_0^2)[(u_0+u_\nu)\psi_0-\gamma]}{(1-u_0^2)\psi_0^2-2au_\nu+\gamma^2}\qquad\text{where } \nu=1,2.
$$

Setting the numerator equal to zero, we obtain the equation $u_\nu=\gamma/\psi_0-u_0$, and hence we obtain the following value for $g(u_\nu)$:

$$
(10)\quad g(\gamma/\psi_0-u_0)=\frac{-\gamma(1-u_0^2)}{\psi_0^2}\left(\psi_0-\frac{\gamma}{1+u_0}\right)\left(\psi_0+\frac{\gamma}{1-u_0}\right)\left(\psi_0-\frac{a}{\gamma}\right).
$$

* Cf. Love, loc. cit., p. 131.

The roots of this last expression give the values of ψ_0 at the extrema of u_1 and u_2. The extrema are thus seen to occur at the points

$$(11) \quad P_1: \left(\frac{-\gamma}{1-u_0}, -1\right), \quad P_2: \left(\frac{\gamma}{1+u_0}, +1\right), \quad P_3: \left(\frac{a}{\gamma}, \frac{\gamma^2}{a} - u_0\right).$$

In order to compare the points P_2 and P_3, we shall write the coördinates of P_3 in the form

$$P_3: \left(\frac{\gamma}{1+u_0} - \frac{D}{\gamma(1+u_0)}, \ 1 + \frac{D}{a}\right),$$

where D is defined by equation (6).

When $du_\nu/d\psi_0 = 0$,

$$\frac{d^2u_\nu}{d\psi_0^2} = -\frac{\partial^2 g}{\partial\psi_0^2} \Big/ \frac{\partial g}{\partial u}\bigg|_{u=u_\nu}.$$

Hence,

$$\text{at } P_1, \quad \frac{d^2u_\nu}{d\psi_0^2} = \frac{(1-u_0^2)(1-u_0)^2}{\gamma^2 + a(1-u_0)} > 0;$$

$$\text{at } P_2, \quad \frac{d^2u_\nu}{d\psi_0^2} = \frac{-(1-u_0^2)(1+u_0)^2}{D},$$

$$\text{at } P_3, \quad \frac{d^2u_\nu}{d\psi_0^2} = \frac{2(1-u_0^2)\gamma^4}{a[\gamma^2 + a(1-u_0)]D}.$$

P_1 is always a minimum but P_2 is a maximum and P_3 is a minimum if D is positive, and P_2 is a minimum and P_3 is a maximum if D is negative. It should be observed that $\partial g/\partial u$ never vanishes at P_1 and that it vanishes at P_2 and P_3 only when $D=0$. But, when $D=0$,

$$(12) \quad u_1 = \begin{cases} 1 + \left(\psi_0 - \dfrac{a}{\gamma}\right)\lambda_1(\psi_0) & \text{if } \psi_0 < a/\gamma, \\ 1 + \left(\psi_0 - \dfrac{a}{\gamma}\right)\lambda_2(\psi_0) & \text{if } \psi_0 > a/\gamma, \end{cases}$$

$$u_2 = \begin{cases} 1 + \left(\psi_0 - \dfrac{a}{\gamma}\right)\lambda_2(\psi_0) & \text{if } \psi_0 < a/\gamma, \\ 1 + \left(\psi_0 - \dfrac{a}{\gamma}\right)\lambda_1(\psi_0) & \text{if } \psi_0 > a/\gamma, \end{cases}$$

where

$$(13) \quad \lambda_1(\psi_0) = \frac{1}{a}\left[(1-u_0)\gamma + \frac{1}{2}(1-u_0^2)(\psi_0 - a/\gamma) + r(\psi_0)\right],$$

$$\lambda_2(\psi_0) = \frac{1}{a}\left[(1-u_0)\gamma + \frac{1}{2}(1-u_0^2)(\psi_0 - a/\gamma) - r(\psi_0)\right],$$

and

$$(14)\qquad r(\psi_0) = \left(\left[\gamma(1 - u_0) + \frac{1}{2}(1 - u_0^2)(\psi_0 - a/\gamma)\right]^2 + (1 - u_0^2)\gamma^2\right)^{1/2}.$$

Equations (12) may be derived from equation (9) by repeated use of the condition $D=0$. When this condition is fulfilled, the points P_2 and P_3 coincide, and, if we rewrite this condition in the form $a/\gamma=\gamma/(1+u_0)$, equation (12) shows that this point lies on both u_1 and u_2. This point is a maximum of u_1 and a minimum of u_2 (as we have defined these functions).

We have proved that u_1 and u_2 are real and that $-1 \leqq u_1 \leqq +1 \leqq u_2$ for all values of ψ_0. Furthermore, when $D>0$, u_1 has a maximum at P_2, and P_3 lies above and to the left of P_2; when $D<0$, u_1 has a maximum at P_3 and P_3 lies below and to the right of P_2; when $D=0$, the points P_2 and P_3 coincide at a maximum of u_1 and a minimum of u_2.

The point P_1 corresponds to a passage of the curve Γ through the south pole. When $D>0$, P_2 corresponds to a passage of Γ through the north pole. When $D=0$, u_1 and u_2 coincide at P_2 which corresponds to an asymptotic (with respect to t) approach to the north pole.* If $D<0$, P_2 does not correspond to real motion.

Another property of the function $u_1(\psi_0)$ is that it approaches $-u_0$ from above as ψ_0 becomes positively infinite and from below as ψ_0 becomes negatively infinite. This is proved as follows. Let $x=1/\psi_0$, $h(u, x)=x^2g(u)$. If $x\neq 0$, the roots of g and h are the same. But $\partial h(-u_0, 0)/\partial u=1-u_0^2>0$. Hence $h(u, x)=0$ defines a function $u=u(x)$ which is continuous at the point $(0, -u_0)$. Therefore

$$\lim_{x\to 0} u(x) = \lim_{\psi_0\to\pm\infty} u_1 = -u_0,$$

and, since the derivative is positive at $x=0$, u_1 takes on values slightly greater than or slightly less than $-u_0$ according as x takes on values slightly greater than or slightly less than zero or according as ψ_0 is large and positive or large and negative.

This situation is in accordance with what should be expected from dynamical considerations. For, as ψ_0 becomes large, v also becomes large and the gyroscopic force and the force of gravity become relatively unimportant. Hence the path of the gyroscope becomes approximately a great circle. But, if a great circle is drawn tangent to a parallel of latitude $u=u_0$, then it must also be tangent to the parallel of latitude $u=-u_0$, the two points of tangency being at opposite ends of a diameter of the great circle. Thus we would

* Cf. Routh, *Advanced Rigid Dynamics*, 1905, pp. 136, 137.

expect that, as ψ_0 becomes infinite, the geodesic curvature, k, must approach zero, and the difference in longitude, $\Delta\psi$, between a point of tangency of Γ with u_0 and the next subsequent point of tangency of Γ with u_1 must approach π.

An expression for k may be obtained by solving equation (1b) and combining equations (2) and (4). Thus*

$$k = \frac{\gamma}{v^3}\left[(1-u_0^2)\psi_0^2 - \frac{a}{2\gamma}(1-u_0^2)\psi_0 + \frac{a}{2}(u_0-u)\right] \tag{15}$$

and, since $v^2=(1-u_0^2)\psi_0^2+a(u_0-u)$, k must approach zero as ψ_0 becomes positively or negatively infinite.

The difference in longitude, $\Delta\psi$, may be obtained from equations (4) and (5). Thus

$$\Delta\psi = \int_{u_0}^{u_1} \frac{(1-u_0^2)\psi_0 + \gamma(u_0-u)}{(1-u^2)(f(u))^{1/2}}du.$$

This expression may be written in the form

$$\Delta\psi = \frac{(1-u_0^2)+\gamma(u_0+\bar{u})/\psi_0}{\left((1-u_0^2)+\epsilon\left(\frac{1}{\psi_0},\bar{u}\right)\right)^{1/2}}\int_{u_0}^{u_1}\frac{du}{(1-u^2)(u_0^2-u^2)^{1/2}},$$

where $u_0<\bar{u}<u_1$ and

$$\epsilon\left(\frac{1}{\psi_0},\bar{u}\right) = \frac{(1/\psi_0)\left[a(1-u^2)-\gamma^2(u_0-\bar{u})\right]-2\gamma(u_0-u)}{\psi_0(u_0+\bar{u})}.$$

Since

$$\lim_{\psi_0\to\pm\infty}\frac{(1-u_0^2)+\gamma(u_0+\bar{u})/\psi_0}{((1-u_0^2)+\epsilon(1/\psi_0,\bar{u}))^{1/2}} = (1-u_0)^2 \quad\text{and}\quad \lim_{\psi_0\to\pm\infty} u_1 = -u_0>0,$$

and since

$$\int_{+u_0}^{-u_0}\frac{du}{(1-u^2)(u_0^2-u^2)^{1/2}} = \pi/(1-u_0^2)^{1/2} \quad\text{when}\quad u_0<0,$$

it follows that

$$\lim_{\psi_0\to\pm\infty}\Delta\psi = \pi.$$

We proceed similarly when $u_0>0$.

The curve $u=u_1$ is further characterized by its intersections with the line $u=u_0$. These points of intersection are of special dynamical interest

* Cf. Osgood, loc. cit., p. 258, equation IV.

since the condition $u_1 = u_0$ is necessary and sufficient for steady precession.* The values of $\dot\psi_0$ at which the curve $u = u_1$ crosses the straight line $u = u_0$ are given by solving the equation $g(u_0) = 0$ for $\dot\psi_0$. Thus

$$\dot\psi_0 = \frac{\gamma \pm d^{1/2}}{2u_0} \tag{16}$$

where d is defined by equation (7) and where $u_0 \neq 0$. These values of $\dot\psi_0$ are real and distinct, conjugate imaginary or coincident according as d is positive, negative, or zero. If $u_0 = 0$ the solution of $g(u_0) = 0$ is $\dot\psi_0 = a/(2\gamma)$.

6. **Paths for which the longitudinal velocity vanishes.** If the path, Γ, of the gyroscope is a curve with loops, then the longitudinal velocity, $\dot\psi_0$, when $u = u_0$, must be opposite in sign to the longitudinal velocity, $\dot\psi_1$, when when $u = u_1$, and hence $\dot\psi$ must vanish for some value of u between u_0 and u_1. If, at certain points, Γ has cusps, then, at such points, v must vanish, since by equation (16), k can be discontinuous only when $v = 0$. But, when v vanishes, $\dot u$ and $\dot\psi$ must both be equal to zero. Hence the character of Γ is related to the vanishing or non-vanishing of $\dot\psi$.

If $\dot\psi = 0$, we may solve equation (4) for u and get

$$u = u_L(\dot\psi_0) = u_0 + (1 - u_0^2)\frac{\dot\psi_0}{\gamma} \cdot \tag{17}$$

The equation $u = u_L$ represents a straight line intersecting the line $u = u_0$ in the point P_0: $(0, u_0)$. The intersections of $u = u_L$ with the curve $u = u_1$ are given by solving $g(u_L) = 0$ for $\dot\psi_0$. Substituting $u = u_L$ and factoring, we get

$$g(u_L) = \frac{1}{\gamma}(1 - u_0^2)^2\left[\dot\psi_0 - \frac{\gamma}{1 + u_0}\right]\left[\dot\psi_0 + \frac{\gamma}{1 - u_0}\right]\left[\dot\psi_0 - \frac{a}{\gamma}\right].$$

Hence the points of intersection of $u_L(\dot\psi_0)$ with $u_1(\dot\psi_0)$ are P_1: $(-\gamma/(1-u_0), -1)$, P_2: $(\gamma/(1+u_0), +1)$ and P_4: $(a/\gamma, u_0+(1-u_0^2)a/\gamma^2)$. We may write the coördinates of P_4 in the form P_4: $(\gamma/(1+u_0)-D/[\gamma(1+u_0)], 1-D(1-u_0)/\gamma^2)$. Hence if $D \neq 0, P_4$ lies on the curve $u_1(\dot\psi_0)$ when P_3 lies on the curve $u_2(\dot\psi_0)$ and P_4 lies on the curve $u_2(\dot\psi_0)$ when P_3 lies on the curve $u_1(\dot\psi_0)$, and, if $D = 0$, P_2, P_3, and P_4 coincide. The line $u = u_L$ is shown in Figs. I to XI.

When $0 < \dot\psi_0 < a/\gamma$, the line $u_L(\dot\psi_0)$ lies outside of the region between u_0 and u_1, since u_L lies outside of this region for $\dot\psi_0$ small and positive, and does not cross u_0 or u_1 between P_0 and P_4. The line u_L lies inside the region between u_0 and u_1 between the points P_0 and P_1 and, when $D < 0$, between P_2

* Cf. p. 738.

and P_4. All other parts of u_L lie outside of this region. Hence the values of ψ_0 corresponding to the points of u_L which lie in this region, are given by the inequalities*

$$\frac{-\gamma}{1-u_0} \leqq \psi_0 \leqq 0 \quad \text{or} \quad \frac{a}{\gamma} \leqq \psi_0 \leqq \frac{\gamma}{1+u_0}\cdot$$

If ψ_0 is such that Γ is a curve with loops, then we have seen that, for this value of ψ_0, the point $(\psi_0, u_L(\psi_0))$ must lie in the open region included between u_0 and $u_1(\psi_0)$. The converse is also true; namely, if $(\psi_0, u_L(\psi_0))$ lies in the open region between u_0 and u_1, then ψ_0 and ψ_1 have opposite signs and hence Γ is a curve with loops. For, by equation (4),

$$\ddot{\psi}_0 = \frac{-\gamma\dot{u}}{1-u^2} \quad \text{when } \dot{\psi} = 0,$$

and thus $\dot{\psi}$ and $\ddot{\psi}$ cannot vanish simultaneously when u lies in the open interval between u_0 and u_1. Therefore $\dot{\psi}$ changes sign whenever it vanishes. But $\dot{\psi}$ vanishes for, at most, one value of u between u_0 and u_1, since $u=u_L(\psi_0)$ is a single-valued function. Hence ψ_0 and ψ_1 have opposite signs when and only when $(\psi_0, u_L(\psi_0)$ lies between $u=u_0$ and $u=u_1$, and consequently when and only when $-\gamma/(1-u_0)<\psi_0<0$ or $a/\gamma<\psi_0<\gamma/(1+u_0)$. It follows that ψ_1 is negative when $\psi_0 <-\gamma/(1-u_0)$ and positive when $-\gamma/(1-u_0)<\psi_0\leqq 0$. If $D>0$, ψ_1 is positive when $0\leqq\psi_0<a/\gamma$, negative when $a/\gamma<\psi_0<\gamma/(1+u_0)$, and positive when $\gamma/(1+u_0)<\psi_0$. If $D=0$, ψ_1 is positive when $0\leqq\psi_0<\gamma/(1+u_0)$ and when $\gamma/(1+u_0)<\psi_0$. If $D<0$, ψ_1 is positive when $0\leqq\psi_0$. Consequently Γ is a wavy curve for which the drift of the motion is westward when $\psi_0<-\gamma/(1-u_0)$, and Γ is a wavy curve for which the drift of the motion is eastward when $D>0$ and $0<\psi_0<a/\gamma$ or $\gamma/(1+u_0)<\psi_0$, when $D=0$ and $0<\psi_0<\gamma/(1+u_0)$ or $\gamma/(1+u_0)<\psi_0$, or when $D<0$ and $\psi_0>0$. Γ is a curve with loops when $-\gamma/(1-u_0)<\psi_0<0$ or $a/\gamma<\psi_0<\gamma/(1+u_0)$.

7. **Paths having points of inflection.** The purpose of this paragraph is to determine what initial conditions are necessary in order that Γ may have points of inflection. At such points, the geodesic curvature, k, must vanish. But, when k is zero, we may solve equation (15) for u and get

$$u = u_0 + (1-u_0^2)\frac{2\psi_0}{a}(\psi_0 - a/(2\gamma)) = u_i(\psi_0). \tag{18}$$

The curve $u=u_i$ is a parabola with its concave side upward. Only those portions of this parabola which lie in the region between $u=u_0$ and $u=u_1$ correspond to real motion. The intersections of the curve u_i with the bound-

* The inequality $a/\gamma\leqq\gamma/(1-u_0)$ implies $D\geqq 0$.

ary of this region are obtained by solving the equation $g(u_i)=0$ for ψ_0 where

$$
(19)\qquad \begin{aligned} g(u_i) &= \frac{-2}{a}(1-u_0^2)(\psi_0 - a/\gamma)h(\psi_0), \\ h(\psi_0) &= (1-u_0^2)\psi_0^3 - \frac{a}{2\gamma}(1-u_0^2)\psi_0^2 - (\gamma^2 - au_0)\psi_0 + \frac{a\gamma}{2}\cdot \end{aligned}
$$

We shall study the roots of $h(\psi_0)$. The signs of the coefficients of the powers of ψ_0 in $h(\psi_0)$ are given in the following scheme:

$$
\begin{aligned} + - - + &\text{ when } \gamma^2 - au_0 > 0, \\ + - + &\text{ when } \gamma^2 - au_0 = 0, \\ + - + + &\text{ when } \gamma^2 - au_0 < 0. \end{aligned}
$$

In all three cases the number of sign changes is two and therefore (by Descartes' rule of signs) the number of positive roots is at most two. If we replace ψ_0 by $-\psi_0$, the signs of the coefficients are given by the following scheme:

$$
\begin{aligned} - - + + &\text{ when } \gamma^2 - au_0 > 0, \\ - - + &\text{ when } \gamma^2 - au_0 = 0, \\ - - - + &\text{ when } \gamma^2 - au_0 < 0. \end{aligned}
$$

Thus in all cases, the number of negative roots is at most one. Since $h(0)=\frac{1}{2}a\gamma>0$ and $h(-\infty)=-\infty$, $h(\psi_0)$ must have precisely one negative root and therefore the other two roots must be either both positive or else conjugate imaginary. If the roots are real, the question which concerns us is whether these roots correspond to intersections of u_i with u_1 or with u_2. In order to investigate this question we form the function

$$
(20)\qquad \phi(\psi_0) = u_i(\psi_0) - u_1(\psi_0).
$$

We shall show that $\phi(0)=u_0-u_1(0)$ is positive. The condition $\phi(0)>0$ is equivalent to the statement that *if the longitudinal velocity on a limiting circle vanishes, then that circle is the upper limiting circle.* Let us denote by u_0 the limiting circle upon which the longitudinal velocity vanishes. Then $v^2=v_0^2+a(u_0-u)$. But the latitudinal velocity also vanishes when $u=u_0$ and therefore $v_0=0$. Thus, if u takes on greater values than u_0, v becomes imaginary, which is impossible. Therefore u_0 must be the upper limiting circle and $\phi(0)$ must be positive.*

Since $u_1(+\infty)=-u_0$, it follows that $\phi(+\infty)=+\infty$. But, since ϕ is everywhere continuous, if it has one positive root, it must have a second.

* This fact may also be proved directly from the explicit formula for $u_1(0)$.

The same holds true for the negative roots of ϕ. But the roots of ϕ are also roots of $g(u_i)$ and since

$$\frac{d}{d\psi_0}g[u_i(\psi_0)] = \frac{\partial g}{\partial u}\frac{du_i}{d\psi_0} + \frac{\partial g}{\partial\psi_0} = \frac{\partial g}{\partial u}\left(\frac{du_i}{d\psi_0} - \frac{du_i}{d\psi_0}\right) = \frac{\partial g}{\partial u}\frac{d\phi}{d\psi_0} \text{ when } u_i = u_1 = u,$$

the double roots of ϕ are also the double roots of $g(u_i)$. Therefore ϕ cannot have a negative root. Hence the negative root of $h(\psi_0)$ corresponds to an intersection of $u_i(\psi_0)$ with $u_2(\psi_0)$.

As we proceed to the left, starting at $\psi_0 = 0$, the curve u_i does not enter the region between u_0 and u_1, since initially u_0 is greater than u_1 and, as ψ_0 decreases, u_i increases from the value u_0. Also, since the curve u_i does not cross the line u_0 or the curve u_1 at a negative value of ψ_0, it lies completely without the region bounded by u_0 and u_1 to the left of the ψ_0-axis. But, since our choice of u_0 was arbitrary, we have proved that Γ *cannot have a point of inflection if the longitudinal velocity on either limiting circle is negative.*

Formula (19) shows that

$$\psi_0 = a/\gamma = \frac{\gamma}{1 + u_0} - \frac{D}{\gamma(1 + u_0)}$$

is a root of $g(u_i)$; the corresponding value of u is

$$u_0 + (1 - u_0^2)a/\gamma^2 = 1 - \frac{(1 - u_0)D}{\gamma^2}\cdot$$

Hence the point

$$P_4\colon \left(\frac{\gamma}{1 + u_0} - \frac{D}{\gamma(1 + u_0)},\ 1 - \frac{(1 - u_0)D}{\gamma^2}\right)$$

lies on the curve u_2, u_1, or both, according as D is less than, greater than, or equal to zero. Thus, if $D \leqq 0$, then $u_i(a/\gamma) \geqq +1$, and since u_i is an increasing function to the right of the line $\psi_0 = a/\gamma$, the curve u_i lies completely withou the region bounded by u_1 and u_0 to the right of a/γ.

If $D > 0$, ψ_1 is negative for ψ_0 slightly greater than a/γ and thus the curve u_i cannot enter the region between u_0 and u_1 at a/γ as we proceed to the right, since Γ cannot have a point of inflection if the longitudinal velocity at $u = u_1$ is negative. But, since u_i and u_1 have definite slopes at a/γ, the curve u_i must enter the region between u_0 and u_1 as we proceed to the left at a/γ, or else u_i and u_1 must have the same slope at this point. The latter possibility is excluded since it implies a double root of ϕ at a/γ which is contrary to the fact that

$$\begin{aligned} h(a/\gamma) &= (1 - u_0^2)a^3/(2\gamma^3) - (\gamma^2 - au_0)a/\gamma + \tfrac{1}{2}a\gamma \\ &= -D[\gamma^2 + a(1 - u_0)]a/(2\gamma^3) \neq 0 \quad \text{when } D > 0. \end{aligned}$$

Thus, when D is positive, ϕ must have a root between 0 and a/γ. Furthermore, ϕ must be positive for values of ψ_0 slightly greater than a/γ and therefore, if ϕ has one root greater than a/γ, it must have two such roots (since $\phi(+\infty)=+\infty$), and hence must have four positive roots. But ϕ can have at most three positive roots (the root a/γ and the two positive roots of h). Therefore ϕ cannot have a root greater than a/γ even when D is positive. We have thus proved that *a necessary condition that* Γ *have a point of inflection is that* ψ_0 (*and incidentally* ψ_1) *lie in the closed interval from* 0 *to* a/γ.

The parabola u_i goes below the line u_0 at $\psi_0=0$ and comes above the line again at $a/(2\gamma)$. The curve u_1 may cut this parabola in one, two, or no points in the interval from 0 to $a/(2\gamma)$. The first possibility always occurs when D is positive. The second possibility may be seen by observing that the minimum of u_i is at $a/(4\gamma)$ and $u_i(a/(4\gamma))=u_0-(1-u_0^2)a/(8\gamma^2)$. This may be made less than -1 by choosing γ sufficiently small, and if such a choice of γ is made, u_1 must certainly cross u_i in the interval from 0 to $a/(2\gamma)$. The third possibility may be shown by a numerical example. Thus, let $u_0=\frac{1}{2}$, $a=4$, and $\gamma=1$. Then $h(\psi_0)=\frac{3}{4}\psi_0^3-\frac{3}{2}\psi_0^2+\psi_0+2$ and this expression has no positive roots.

We have proved that when u lies in the closed interval from u_0 to u_1, k can vanish only when $0\leqq\psi_0\leqq a/\gamma$. Hence for all other values of ψ_0, k has the same sign as k_0 (where k_0 is the value of k when $u=u_0$). But $k_0=(\gamma/v_0^3)\cdot(1-u_0^2\psi)_0(\psi_0-a/(2\gamma))$ and this expression is positive whenever $\psi_0<0$ or $\psi_0>a/\gamma$. *Hence k is always positive except when* $0\leqq\psi_0\leqq a/\gamma$. *Therefore, only wavy curves for which the drift of the motion is eastward can have points of inflection.*

8. **Paths having monotone geodesic curvatures.** If the derivative of the geodesic curvature with respect to the arc length of the path does not vanish in any given interval, then the geodesic curvature is monotone in that interval. An expression for k' may be obtained by differentiating equation (15) and combining terms by means of equation (3). Thus*

$$
\begin{aligned}
k' &= \frac{+\gamma a\dot{u}}{2v^6}L(u,\psi_0),\\
L(u,\psi_0) &= 2(1-u_0^2)\psi_0^2-\frac{3a}{2\gamma}(1-u_0^2)\psi_0+\frac{a}{2}(u_0-u).
\end{aligned}
\tag{21}
$$

Setting $L(u,\psi_0)$ equal to zero, we obtain the equation

$$u=u_0+\frac{4}{a}(1-u^2)\psi_0\left(\psi_0-\frac{3a}{4\gamma}\right)=u_k(\psi_0).$$

* Cf. Kellogg, loc. cit., p. 519.

Hence k' vanishes when and only when $u=u_k(\psi_0)$, $u=u_0$, or $u=u_1(\psi_0)$.

The curve $u_k(\psi_0)$ is a parabola which intersects the parabola $u_i(\psi_0)$ in the points P_0: $(0, u_0)$ and P_4: $(a/\gamma,\ u_0+a(1-u_0^2)/\gamma^2)$, and only at these points. But P_0 lies on the line $u=u_0$ and P_4 lies on the curve $u=u_1$ or $u=u_2$. Hence k and k' vanish simultaneously (at points corresponding to real motion) only at the intersections of u_i with u_0 and u_1. These points of intersection cannot correspond to points of inflection of the path, since Γ is symmetric in any meridian circle through a point of contact with the parallel of latitude u_0 or u_1 and hence k cannot change sign at such a point. Therefore, in the u, ψ_0-plane, the points of the parabola $u=u_i$ which lie in the open region included between u_0 and u_1, and only these points, correspond to points of inflection of the path Γ.

We have proved that the parabola $u=u_i(\psi_0)$ lies inside the shaded region* between u_0 and u_1 only when $0\leqq\psi_0\leqq a/\gamma$. The same is true for the parabola u_k. This follows from the fact that the portion of u_k for which $\psi_0<0$ lies in the region bounded on the left by the parabola u_i and on the right by the ψ_0-axis. This region, we have seen, does not overlap the region between u_0 and u_1. Likewise, the portion of u_k for which $a/\gamma<\psi_0$, lies in the region bounded on the left by the line $\psi_0=a/\gamma$ and on the right by the parabola u_i. This region does not overlap the region between u_0 and u_1. Consequently, the path Γ has monotone geodesic curvature between a point of tangency with u_0 and the first subsequent point of tangency with u_1 (or vice versa), whenever $\psi_0<0$ or $a/\gamma<\psi_0$. The same is true whenever $\psi_1<0$ or $a/\gamma<\psi_1$.

Since $L(u, \psi_0)$ can vanish only when $0\leqq\psi_0\leqq a/\gamma$, and since $L(u_0, \psi_0) =2(1-u_0^2)\psi_0(\psi_0-3a/(4\gamma))$ is positive whenever $\psi_0<0$ or $\psi_0>a/\gamma$, it follows that $\operatorname{sgn} k'=\operatorname{sgn} \dot{u}$ whenever $\psi_0<0$ or $\psi_0>a/\gamma$.

O. D. Kellogg has proved† that if the arc derivative K' of the curvature‡, K, of an arc $\{s_1s_2\}$ (where $s_1<s_2$) of a spherical curve is always positive, then the arc $\{s_1s_2\}$ lies entirely within the osculating circle at s_1, and entirely without the osculating circle at s_2. The curvature, K, is related to the geodesic curvature, k, by means of the equation

$$1 + k^2 = K^2.$$

Hence $kk'=KK'$. But we have proved that, whenever $\psi_0<0$ or $\psi_0>a/\gamma$, $k>0$ and $\operatorname{sgn} k'=\operatorname{sgn} \dot{u}$. Therefore, when $\psi_0<0$ or $\psi_0>a/\gamma$, an arc of Γ

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* Cf. Figs. I to XI.

† Cf. O. D. Kellogg, loc. cit., pp. 509, 521.

‡ The arc $\{S_1S_2\}$ is regarded as an arc of a space curve and K is the curvature at a given point of this arc.

consisting of a half wave lies entirely within the osculating circle at its point of tangency with u_0, and entirely without the osculating circle at its point of tangency with u_1, or entirely within the osculating circle at u_1 and entirely without the osculating circle at u_0, according as u_1 is north or south of u_0.

9. **The spherical pendulum.** When $\gamma=0$, the gyroscope becomes a spherical pendulum.* Thus we can discuss the behavior of the spherical pendulum if we know the nature of the curves u_1, u_L, u_i, and u_k.†

As γ approaches zero, the point P_1: $(-\gamma/(1-u_0), -1)$ approaches the point $(0, -1)$, the point P_2: $(\gamma/(1+u_0), +1)$ approaches the point $(0, +1)$, and the point $P_3(a/\gamma, \gamma^2/(a-u_0))$ goes out to $+\infty$. Hence the line u_L turns into coincidence with the u-axis and the curves u_i and u_k degenerate into the u-axis. Thus we have again the familiar facts that the paths of the spherical pendulum are always wavy curves and are without inflection points.‡ We have in addition the fact that the curvature of a path of a spherical pendulum is always monotone between points of tangency to the bounding parallels of latitude.

The curve u_i crosses the straight line u_0 at the points $\dot\psi_0=\pm(-a/(2u_0))^{1/2}$. These points are real when and only when $u_0<0$. For these values of $\dot\psi_0$, the phenomenon of the conical pendulum occurs. When $\dot\psi_0=0$, we have a plane pendulum.

The curve u_i approaches the straight line $-u_0$ when $\dot\psi_0$ becomes positively or negatively infinite. Thus, as in the case of the gyroscope, the force of gravity is relatively unimportant when $\dot\psi_0$ is large.

10. **Curves near the north or south pole.** Let us consider a curve passing through the north pole. For this curve the longitudinal velocity at u_0 is $\dot\psi_0=\gamma/(1+u_0)$. Hence equation (4) reduces to

$$\dot\psi = \frac{\gamma}{1+u}\cdot$$

Thus the change in longitude, $\Delta\psi$, between u_0 and the north pole is given by the integral

$$\Delta\psi = \int_{u_0}^{1} \frac{\gamma du}{(1+u)(a(u-u_0)(1-u)(u_2-u))^{1/2}}.$$

This integral may be written in the form

$$\Delta\psi = \frac{\gamma}{(a(u_2-\bar u))^{1/2}} \int_{u_0}^{1} \frac{du}{(1+u)((u-u_0)(u_2-u))^{1/2}}$$

* Cf. Appell, loc. cit., Tome I, pp. 513–524.

† These curves are shown in Fig. XII.

‡ Cf. the photographs of the paths of spherical pendulums (Webster, *Dynamics*, pp. 50, 51).

where $u_0 < \bar{u} < 1$. Therefore

$$\Delta\psi = \frac{\pi\gamma}{(2a(1+u_0)(u_2-\bar{u}))^{1/2}}\cdot$$

Next let us find an expression for u_2. This is easily done since, when $\dot\psi_0=\gamma/(1+u_0)$, $f(u)=(1-u)\ (u_0-u)\ [a(1+u)-2\gamma^2/(1+u_0)]$. Thus we obtain the expression

$$u_2 = -1 + \frac{2\gamma^2}{a(1+u_0)}\cdot$$

Substituting this value in the formula for $\Delta\psi$, we obtain the equation

$$\Delta\psi = \frac{\pi}{2}[1 - a(1+u_0)(1+\bar{u})/(2\gamma^2)]^{-1/2}.$$

Furthermore if we set $\gamma^2=2a$, then as u_0 approaches 1, $\Delta\psi$ becomes infinite. Hence, by a proper choice of a, γ and u_0, we can make $\Delta\psi$ as large as we please.

If we set $\dot\psi_0=\gamma/(1+u_0)+h$, then, corresponding to any value of h, we obtain a path $\Gamma(h)$. Let us denote by $\Delta\psi(h)$ the change in longitude along $\Gamma(h)$ between u_0 and u_1. As h approaches zero either through positive or through negative values, the circle u_1 approaches the north pole and the curve $\Gamma(h)$ approaches the limiting curve $\Gamma(0)$. If $|h|$ is small and $h<0$, $\Gamma(h)$ is a curve with loops, and if $h>0$, $\Gamma(h)$ is a wavy curve. This situation is shown in Fig. XIII.

If $h>0$, then the change in longitude along the curve $\Gamma(h)$ from P_0 to P_1 is $2\Delta\psi(h)$. It is easily seen geometrically that $2\Delta\psi(h)-2\Delta\psi(0)$ differs from π by an amount which is numerically equal to the difference in longitude between P_1 and P_2. But, as h approaches zero, P_1 approaches P_2. Therefore,*

$$\lim_{h\to 0^+}\Delta\psi(h) = \Delta\psi(0) + \frac{\pi}{2}\cdot$$

In a like manner, it may be proved that*

$$\lim_{h\to 0^-}\Delta\psi(h) = \Delta\psi(0) - \frac{\pi}{2}\cdot$$

* These equations can also be obtained analytically. For we have the equation

$$\Delta\psi(h) = \int_{u_0}^{u_1}\frac{\gamma du}{(1+u)(a(u-u_0)(u_1-u)(u_2-u))^{1/2}} + \int_{u_0}^{u_1}\frac{(1-u_0^2)hdu}{(1-u^2)(a(u-u_0)(u_1-u)(u_2-u))^{1/2}}.$$

As h approaches zero, u_1 approaches $+1$. Hence

$$\lim_{h\to 0}\int_{u_0}^{u_1}\frac{\gamma du}{(1+u)(a(u-u_0)(u_1-u)(u_2-u))^{1/2}} = \int_{u_0}^{1}\frac{\gamma du}{(1+u)(a(u-u_0)(1-u)(u_2-u))^{1/2}}$$
$$= \Delta\Psi(0).$$

It remains to prove that

But we have proved that $\Delta\psi(0)$ may be made as large as we please and hence by choosing $|h|$ sufficiently small, $\Delta\psi(h)$ may be made as large as we please.*

$$\lim_{h\to 0}\int_{u_0}^{u_1}\frac{(1-u_0^2)hdu}{(1-u^2)(a(u-u_0)(u_1-u)(u_2-u))^{1/2}}=\pm\frac{\pi}{2}.$$

By means of the substitution $u=1-(1-u_1)x$, we obtain the equation

$$\int_{u_0}^{u_1}\frac{(1-u_0^2)hdu}{(1-u^2)(a(u-u_0)(u_1-u)(u_2-u))^{1/2}}$$
$$=\frac{h}{(1-u_1)^{1/2}}\int_1^{(1-u_0)/(1-u_1)}\frac{(1-u_0^2)dx}{x[2-(1-u_1)x](a[(1-u_0)-(1-u_1)x][(u_2-1)+(1-u_1)x][x-1])^{1/2}}.$$

Furthermore

$$\lim_{h\to 0}\int_1^{(1-u_0)/(1-u_1)}\frac{(1-u_0^2)dx}{x[2-(1-u_1)x](a[(1-u_0)-(1-u_1)x][(u_2-1)+(1-u_1)x][x-1])^{1/2}}$$
$$=\int_1^{\infty}\frac{(1-u_0^2)dx}{2x[(x-1)a(1-u_0)(u_2-1)]^{1/2}}=\frac{\pi}{2}\frac{1-u_0^2}{[a(1-u_0)(u_2-1)]^{1/2}}.$$

Finally we have to evaluate $\lim_{h\to\infty}[h/(1-u_1)^{1/2}]$. In order to obtain a value for this limit, we shall expand the function $u_1=u_1(\gamma/(1+u_0)+h)$ about the point $h=0$. But when $h=0$ (cf. p. 747)

$$\frac{du_1}{d\psi_0}=0\text{ and }\frac{d^2u_1}{d\psi_0^2}=-\frac{(1-u_0^2)(1+u_0)^2}{D}.$$

Hence

$$u_1=1-\frac{(1-u_0^2)(1+u_0)^2}{D}h^2+O_3(h)$$

where $O_3(h)$ is an infinitesimal of at least the third order in h. Therefore

$$\lim_{h\to 0+}\frac{h}{(1-u_1)^{1/2}}=-\lim_{h\to 0-}\frac{h}{(1-u_1)^{1/2}}=\frac{(2D)^{1/2}}{(1+u_0)(1-u_0^2)^{1/2}}.$$

It follows that

$$\lim_{h\to 0+}\int_{u_0}^{u_1}\frac{(1-u_0^2)hdu}{(1-u_0^2)[a(u-u_0)(u_1-u)(u_2-u)]^{1/2}}=-\lim_{h\to 0-}\int_{u_0}^{u_1}\frac{(1-u_0^2)hdu}{(1-u_0^2)[a(u-u_0)(u_1-u)(u_2-u)]^{1/2}}$$
$$=\frac{\pi}{2}\left(\frac{2D}{a(1+u_0)(u_2-1)}\right)^{1/2}.$$

But $u_2=-1+2\gamma^2/[a(1+u_0)]$ and $D=\gamma^2-a(1+u_0)$ and thus

$$\left(\frac{2D}{a(1+u_0)(u_2-1)}\right)^{1/2}=1.$$

Consequently

$$\lim_{h\to 0^+}\Delta\psi(h)=\Delta\psi(0)+\pi/2$$

and

$$\lim_{h\to 0^-}\Delta\psi(h)=\Delta\psi(0)-\pi/2\cdot$$

* In the case of the spherical pendulum, G. H. Halphen proved (cf. *Traité des Fonctions Elliptiques*, vol. 2, 1888, p. 128) that the difference in longitude between successive points of tangency to limiting circles,is always less than π; and V. Puisseux proved (cf. Journal de Mathématiques, vol. 7 (1842), p. 517) that this difference in longitude is always greater than $\pi/2$. Simple proofs of both theorems are given by A. de Saint Germain in the Bulletin des Sciences Mathématiques, 1896, 1898, 1901, and in the Mémoires de l'Académie de Caen, 1901. Cf. also the remark of Kellogg (loc. cit., p. 521) concerning this change in longitude in the case of the gyroscope.

Furthermore, by choosing a sufficiently small (or γ sufficiently large) we may make $\Delta\psi(0)$ differ from $\pi/2$ by an amount which can be made as small as desired. Thus by first choosing a small and then choosing $h<0$ and $|h|$ small, we can make $\Delta\psi(h)$ differ from zero by an amount which can be made arbitrarily small. When $h<0$, we recall that Γ is a curve with loops.

At the south pole, $\psi_0=-\gamma/(1-u_0)$, and the change in longitude, $\Delta\psi$, is given by the equation

$$\Delta\psi = \int_{-1}^{u_0} \frac{-\gamma du}{(1-u)[a(u_0-u)(u+1)(u_2-u)]^{1/2}}.$$

The evaluation of this integral gives the expression

$$\Delta\psi = \frac{-\gamma\pi}{[2a(1-u_0)(u_2-\bar{u})]^{1/2}} \quad \text{where } -1<\bar{u}<u_0.$$

But, since $u_2=1+2\gamma^2/[a(1-u_0)]$, we obtain the equation

$$\Delta\psi = -\frac{\pi}{2}\,\frac{1}{\left(1+\dfrac{a(1-u_0)(1-\bar{u})}{2\gamma}\right)^{1/2}}.$$

Hence $-\pi/2<\Delta\psi<0$, and by a proper choice of a (or γ) $\Delta\psi$ may be made to differ from 0 or $-\pi/2$ by an arbitrarily small amount.

If we let $\psi_0=-\gamma/(1-u_0)+h$, then corresponding to any value of h, we obtain a path $\Gamma(h)$. If $|h|$ is small and $h>0$, $\Gamma(h)$ is a curve with loops, and if $h<0$, $\Gamma(h)$ is a wavy curve westward. This situation is shown in Fig. XIV.

It is easily seen that at the south pole*

$$\lim_{h\to 0^+} \Delta\psi(h) = \Delta\psi(0)+\pi/2$$

and

$$\lim_{h\to 0} \Delta\psi(h) = \Delta\psi(0)-\pi/2\,\cdot$$

But, since $-\pi/2<\Delta\psi(0)$, it follows that when $|h|$ is sufficiently small

$$0<\Delta\psi(h)<\pi/2 \quad \text{when } h>0$$

and

$$-\pi<\Delta\psi(h)<-\pi/2 \quad \text{when } h<0.$$

When $h>0$, $\Delta\psi(h)$ can be made arbitrarily near to either 0 or $\pi/2$, and when $h<0$, $\Delta\psi(h)$ can be made arbitrarily near to either $-\pi$ or $-\pi/2$.

* These equations may also be obtained analytically. Cf. footnote p. 757.

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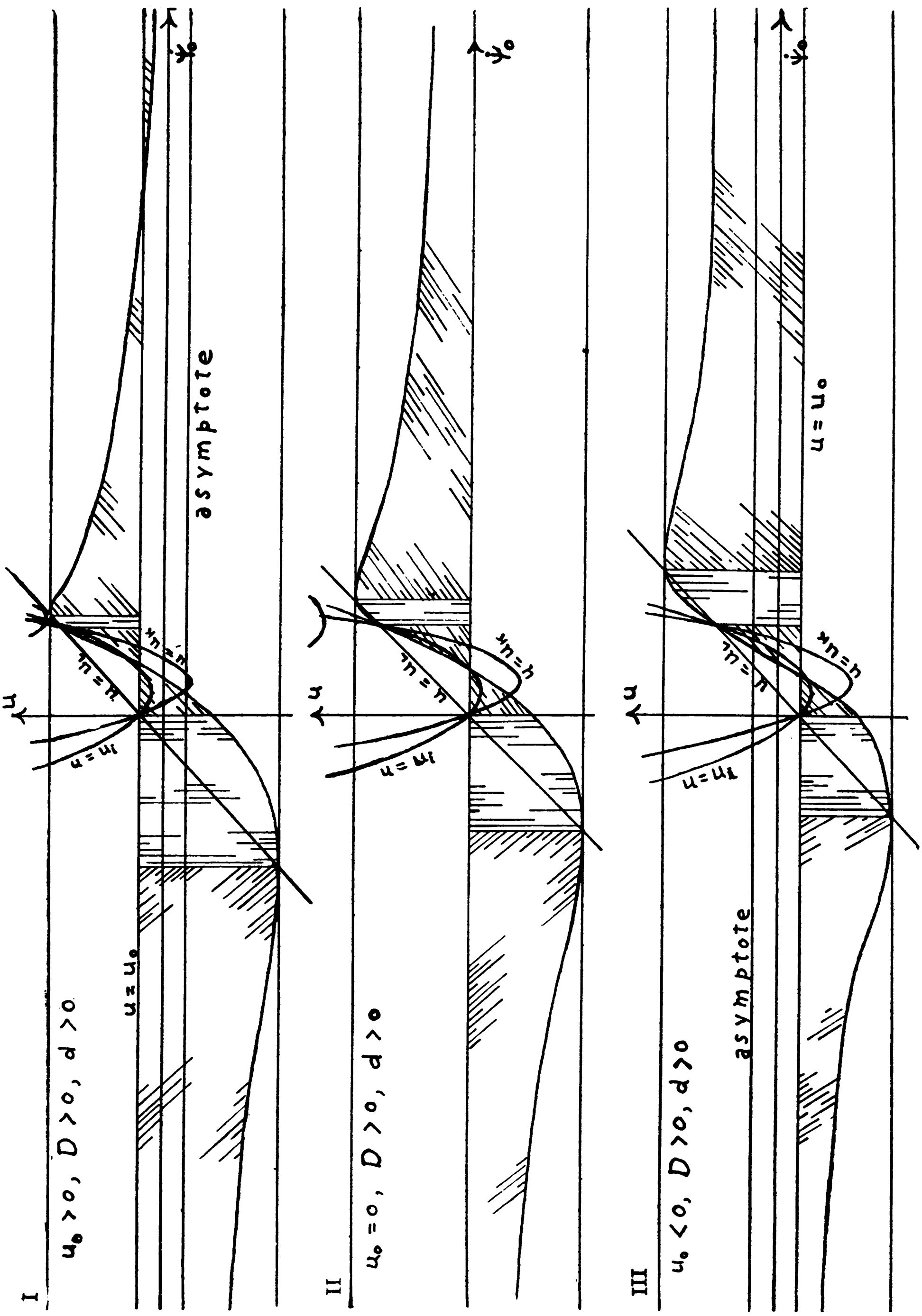
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$u=u_1$
$u=u_k$
$u=u_k'$
ψ_0
u
II
$u_0=0,\ D>0,\ d>0$
$u=u_1$
$u=u_k$
III
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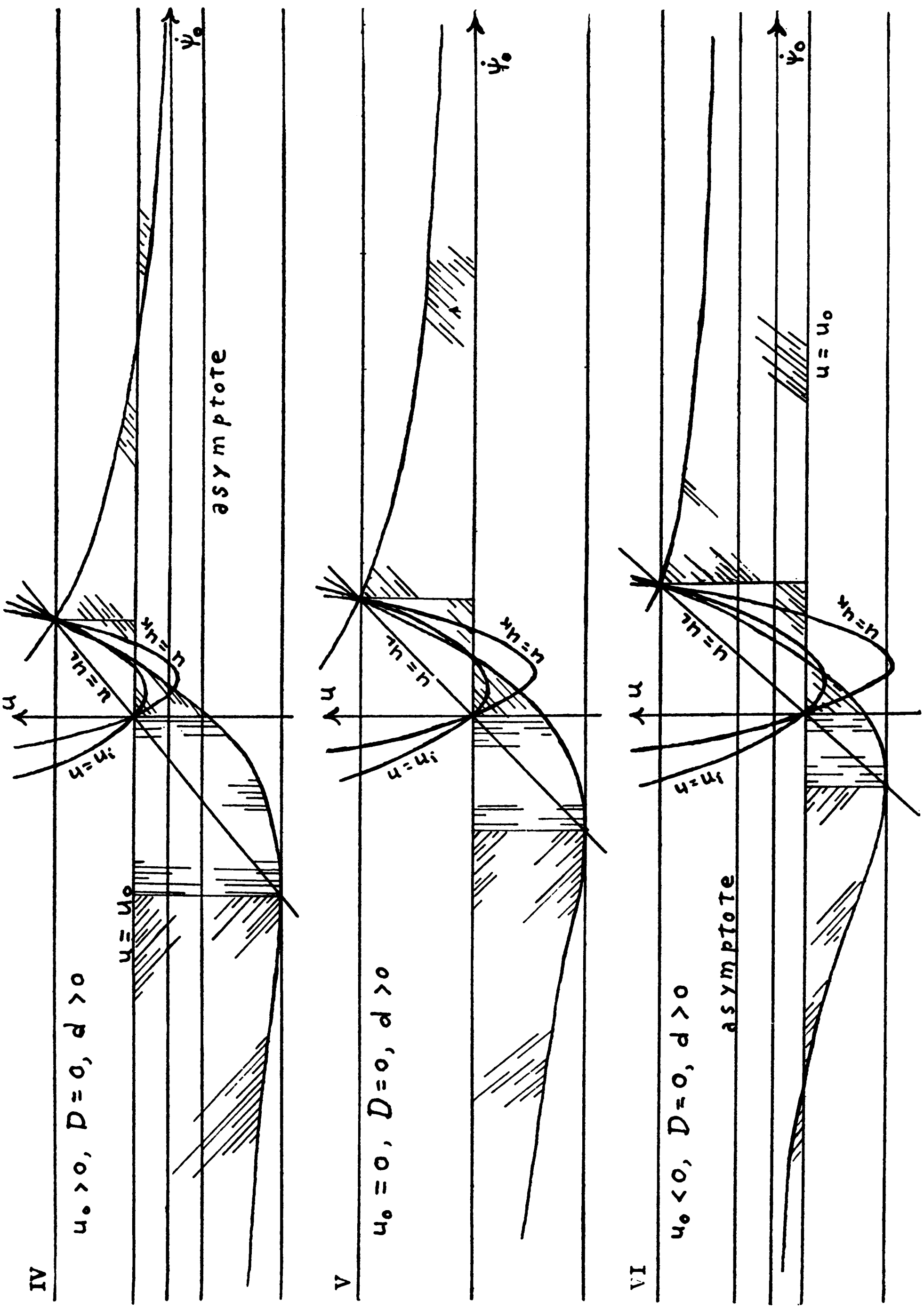
asymptote
$u=u_0$
$u=u_1$
$u=u_k$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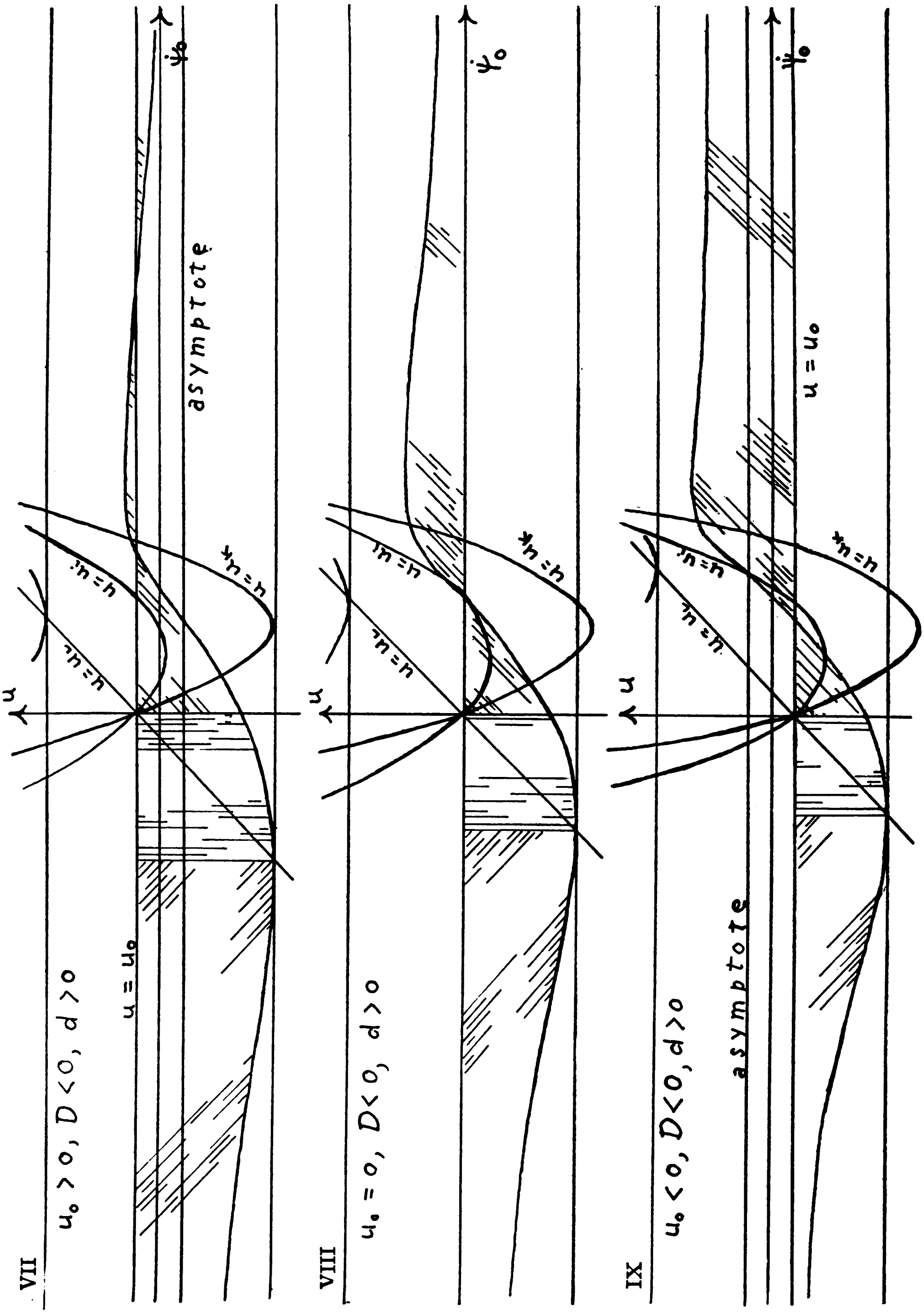
$u = u_i$
$u = u_k$
$\dot\psi_0$
u
V
$u_0 = 0$, $D = 0$, $d > 0$
$u = u_i$
$u = u_k$
$\dot\psi_0$
u
VI
$u_0 < 0$, $D = 0$, $d > 0$
asymptote
$u = u_0$
$u = u_i$
$u = u_k$
$\dot\psi_0$
u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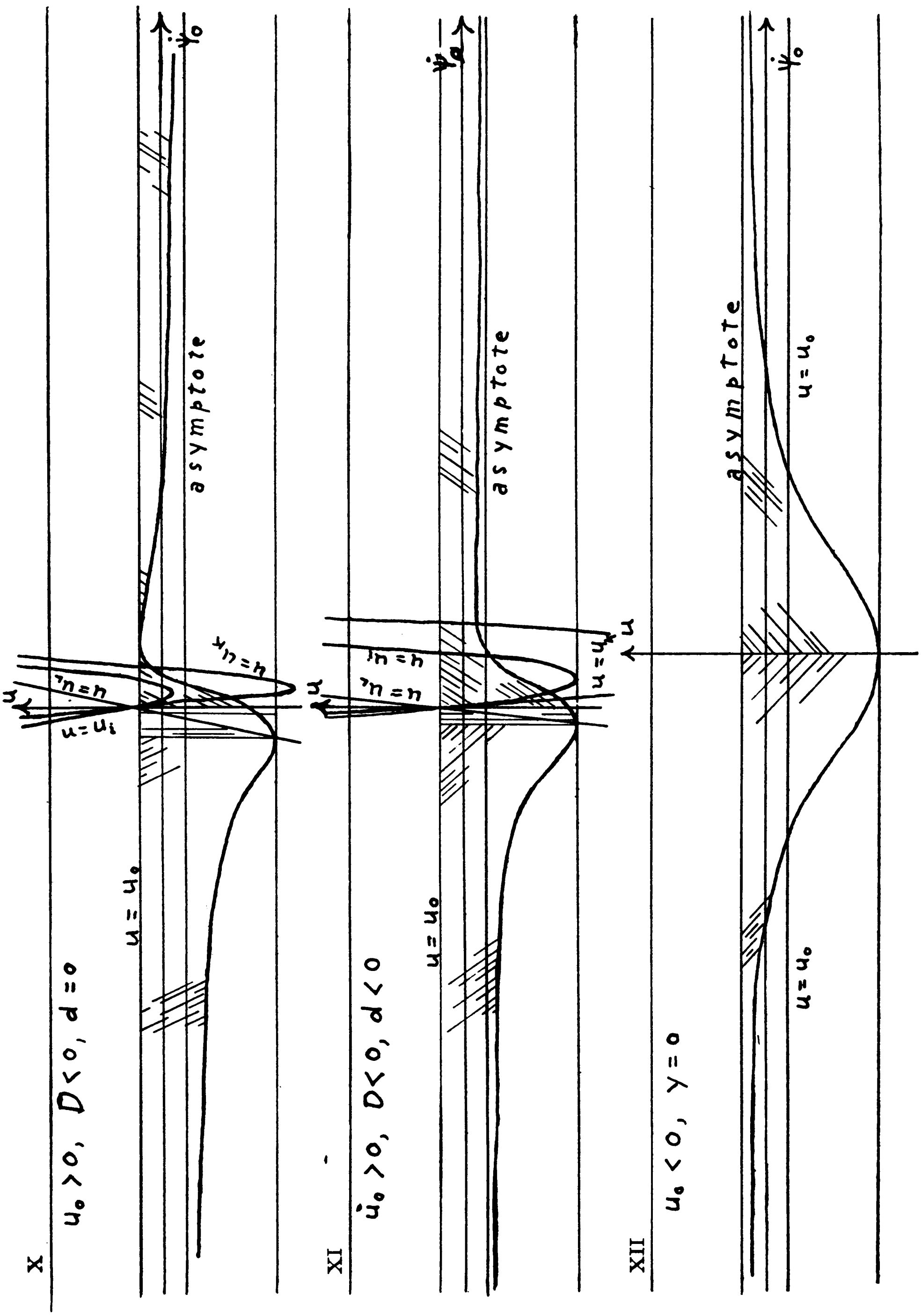
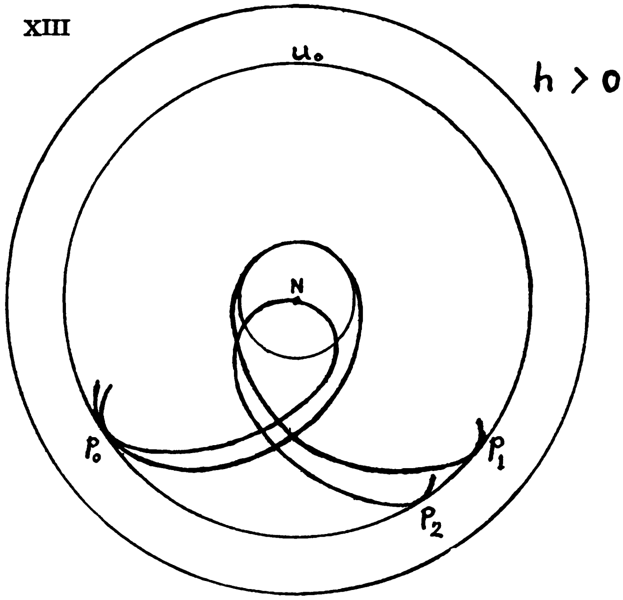
$u = u_0$
asymptote
$u = u_i$
$u = u_k$
$\dot{\psi}_0$
VIII
$u_0 = 0, D < 0, d > 0$
$u = u_i$
$u = u_k$
$\dot{\psi}_0$
IX
$u_0 < 0, D < 0, d > 0$
asymptote
$u = u_0$
$u = u_i$
$u = u_k$
$\dot{\psi}_0$


X
u₀ > 0, D < 0, d = 0
asymptote
u = u₀
u = u₁
u = u_k
u = u_i
XI
u̇₀ > 0, D < 0, d < 0
asymptote
u = u₀
XII
u₀ < 0, Y = 0
asymptote
u = u₀
u = u₀

XIII



XIV