THE BEHAVIOR OF A BOUNDARY VALUE PROBLEM AS THE INTERVAL BECOMES INFINITE*

BY

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The boundary value problems for linear self-adjoint differential equations of the second order with homogeneous linear boundary conditions at the ends of a finite interval have been extensively studied and the principal facts are well known. In this connection a problem of some interest arises if the ends of the interval at which the boundary conditions apply are allowed to recede to $-\infty$ and $+\infty$. The aim of this paper is to investigate the behavior of characteristic numbers, characteristic solutions, and oscillation properties, as the interval becomes infinite. A closely related problem for the differential equation

$$(d/dx)(p(x)du/dx) + (\lambda - q(x))u = 0$$

has been solved by Weyl† and further studied by Hilb‡ and Gray.§

In this paper some interesting results are obtained for the equation

$$d^2u/dx^2 + G(x, \lambda)u = 0$$

and are set forth in Theorems I and II.

It is planned to treat the degree of convergence of certain associated expansions for the infinite interval in a subsequent paper.

1. The differential equation under investigation is

$$d^2u/dx^2 + G(x,\lambda) = 0. \tag{1}$$

The function $G(x, \lambda)$ is assumed to be real and continuous and to possess a positive partial derivative with respect to λ for all real values of x and λ. It is further assumed that

$$\lim_{\lambda=-\infty} G(x,\lambda) = -\infty, \quad \lim_{\lambda=+\infty} G(x,\lambda) = +\infty, \tag{2}$$

and that

$$\lim_{x=\pm\infty} G(x,\lambda) = -\infty. \tag{3}$$

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† Mathematische Annalen, vol. 68 (1910), p. 220, and Göttinger Nachrichten, 1910, p. 442.

‡ Mathematische Annalen, vol. 76 (1915), p. 333.

§ American Journal of Mathematics, vol. 50 (1928), p. 431.

These conditions are all satisfied in the important special case

$$G(x,\lambda) = \lambda - q(x)$$

provided that $q(x)$ is real and continuous and $\lim q(x) = +\infty$.

Associated with (1) we consider boundary conditions at $x=a$ and $x=b$,

$$\begin{aligned} &\alpha_1 u(a) + \alpha_2 u'(a) + \alpha_3 u(b) + \alpha_4 u'(b) = 0, \\ &\beta_1 u(a) + \beta_2 u'(a) + \beta_3 u(b) + \beta_4 u'(b) = 0. \end{aligned} \tag{4}$$

It is assumed of course that these conditions are linearly independent. Let the determinant $\alpha_i\beta_j - \alpha_j\beta_i$ be denoted by A_{ij}, and let $A_{12} = A_{34}$, so that the conditions are self-adjoint. Then there exists* an infinite sequence of values of λ, $l_0, l_1, l_2, l_3, \cdots$, with limit point at $+\infty$ only, and furnishing solutions of (1) and (4). If we take these values in increasing order of magnitude, counting each double value twice, the solution $u_n(x)$ corresponding to $\lambda = l_n$ vanishes at least n times and not more than $n+2$ times in the interval $a < x \leqq b$.

Now as a and b recede to $-\infty$ and $+\infty$ respectively, what becomes of the l_n, the $u_n(x)$, and the roots of $u_n(x)$?

2. In order to answer the foregoing question we must recall some facts concerning the solutions of equation (1), their roots and their behavior at infinity. Multiply (1) by $u(x)$ and integrate from x_1 to x_2, integrating the first term by parts. After transposition we have

$$u(x_2)u'(x_2) - u(x_1)u'(x_1) = \int_{x_1}^{x_2} u'^2(x)dx - \int_{x_1}^{x_2} G(x,\lambda)u(x)dx. \tag{5}$$

From the hypothesis (3) it follows that there exist two numbers α and β such that $G(x, \lambda)$ is negative when $x > \beta$ and when $x < \alpha$. Then if $x_2 > x_1 > \beta$ the right hand side of (5) is positive, which shows that the product $u(x)u'(x)$ does not vanish at both x_1 and x_2. Stated otherwise, $u(x)u'(x)$ does not have more than one root greater than β and does not have more than one root less than α. The total number of roots of $u(x)$ is therefore finite since the interval between any two consecutive roots is not less than π/M, where M^2 is a constant such that $G(x, \lambda) \leqq M^2$.

Now it is quite important to show that as x becomes infinite†

$$\lim u'(x)/u(x) = \infty. \tag{6}$$

* See for example Birkhoff, these Transactions, vol. 10 (1909), p. 264.

† For the behavior of solutions at infinity see Wiman, Arkiv för Matematik, Astronomi och Fysik, vol. 12, No. 14.

To see this we note first of all that when x is greater than the greatest root of $u(x)$ the function $R=u'(x)/u(x)$ is continuous. Let N be any positive number, as large as we please, and let x be chosen so large that $-G(x, \lambda)$ remains greater than $2N^2$. By differentiation and substitution from (1) we get

$$dR/dx = -G(x,\lambda) - R^2,$$

so that if R ever comes within the interval $-N<R<N$, the derivative dR/dx will be greater than N^2. Consequently R will increase beyond N, and will not return, since R is continuous and dR/dx is positive for $R=N$. Thus (6) is established.

Let the principal solutions of equation (1) at the origin be denoted by $u_1(x)$ and $u_2(x)$ so that

$$u_1(0) = u_2'(0) = 1, \quad u_2(0) = u_1'(0) = 0. \tag{7}$$

These solutions satisfy the well known identity

$$u_1(x)u_2'(x) - u_2(x)u_1'(x) = 1. \tag{8}$$

Then the general solution of equation (1) can be written

$$u(x) = C(u_1^2 + u_2^2)^{1/2} \sin [\phi(x) - \theta], \tag{9}$$

in which C and θ are arbitrary constants and $\phi(x)$ is defined by the equation

$$\phi(x) = \tan^{-1} u_2(x)/u_1(x). \tag{10}$$

By differentiating (10) and using (8) we get

$$d\phi/dx = [u_1^2 + u_2^2]^{-1},$$

so that $d\phi/dx$ is positive and ϕ is an increasing function of x. In view of the fact that $u(x)$ has a finite number of roots, $\phi(x)$ cannot increase indefinitely, and therefore must approach a limit. The same conclusion applies as x becomes negatively infinite. We therefore may define ϕ_1 and ϕ_2 as follows:

$$\phi_1 = \lim_{x=-\infty} \phi(x), \quad \phi_2 = \lim_{x=\infty} \phi(x).$$

It is now convenient to define two pairs of independent particular solutions of equation (1) as follows:

$$\begin{aligned} V_1(x) &= (u_1^2 + u_2^2)^{1/2} \sin [\phi(x) - \phi_1], \\ W_1(x) &= (u_1^2 + u_2^2)^{1/2} \cos [\phi(x) - \phi_1], \end{aligned} \tag{11}$$

and another pair V_2 and W_2 similarly defined with ϕ_2 in place of ϕ_1. As x approaches $-\infty$ it can be shown that

$$\text{(12)} \qquad \lim V_1(x) = \lim V_1'(x) = 0, \quad \lim W_1(x) = \lim W_1'(x) = \infty,$$

and as x approaches $+\infty$

$$\text{(13)} \qquad \lim V_2(x) = \lim V_2'(x) = 0, \quad \lim W_2(x) = \lim W_2'(x) = \infty.$$

3. It is now necessary to consider how ϕ_1 and ϕ_2 vary with λ. If $u(x)$ is a solution expressed in the form (9) in which C and θ are independent of λ we may derive in the usual manner* the equation

$$\text{(14)} \qquad u'\partial u/\partial\lambda - u\partial u'/\partial\lambda = \int_0^x (\partial G/\partial\lambda)u^2 dx,$$

since at the origin $\partial u/\partial\lambda = \partial u'/\partial\lambda = 0$ in view of (7). Let r be a root of $u(x)$, so that $-u_1(r)\sin\theta + u_2(r)\cos\theta = 0$. By differentiating this equation with respect to λ and making some simplifications by means of (8) and (14) we obtain

$$\text{(15)} \qquad \partial r/\partial\lambda = -\left[u_1^2(r) + u_2^2(r)\right]\int_0^x (\partial G/\partial\lambda)u^2 dx.$$

This shows that as λ increases all roots of $u(x)$ move toward the origin. In the same manner if λ is constant and θ varies

$$\text{(16)} \qquad \partial r/\partial\theta = u_1^2(r) + u_2^2(r),$$

from which we see that as θ increases all roots of $u(x)$ move to the right. Finally if λ and θ vary in such a manner as to keep the root r fixed, we get from (15) and (16)

$$\text{(17)} \qquad \partial\theta/\partial\lambda = \int_0^r (\partial G/\partial\lambda)u^2 dx.$$

We conclude that as λ increases θ also increases when r is positive, but decreases when r is negative. Now at a root of $u(x)$ we have $\theta = \phi(x) + k\pi$ $(k = 0, \pm 1, \pm 2, \cdots)$ so that

$$\text{(18)} \qquad \partial\phi/\partial\lambda = \int_0^r (\partial G/\partial\lambda)u^2 dx.$$

From this we see that ϕ_2 increases as λ increases while ϕ_1 decreases as λ increases.

4. Now let λ increase from $-\infty$ to $+\infty$ and note the change in the quantity $\phi_2 - \phi_1$. Since there cannot be two roots of $u(x)u'(x)$ when λ is

* See Sturm, Journal de Mathématiques Pures et Appliquées, vol. 1 (1836), pp. 106–186, especially p. 113.

large and negative because $G(x, \lambda)$ is negative, it follows that, for such values of λ, $\phi_2-\phi_1<\pi$. But as λ increases there will be an infinite sequence of values of λ for which the increasing quantity $\phi_2-\phi_1=\pi, 2\pi, 3\pi, \cdots$, since $\phi_2-\phi_1$ must increase without limit. We may denote these values of λ in order of increasing magnitude by $\lambda_0, \lambda_1, \lambda_2, \cdots$. For $\lambda=\lambda_n$, it is at once apparent that the solutions $V_1(x)$ and $V_2(x)$ are identical, except perhaps for sign, and the same is true of $W_1(x)$ and $W_2(x)$. We may therefore define a solution $U_n(x)$ corresponding to λ_n as follows:

$$U_n(x) = V_1(x) = \pm V_2(x), \text{ when } \lambda = \lambda_n. \tag{19}$$

We have therefore

$$\lim_{x=\pm\infty} U_n(x) = \lim_{x=\pm\infty} U_n'(x) = 0. \tag{20}$$

If we form equation (5) for the solution $U_n(x)$ and let x_2 approach $+\infty$ and x_1 approach $-\infty$, we see at once from (20) that the integrals

$$\int_{-\infty}^{+\infty} U_n'^2(x)dx \quad \text{and} \quad \int_{-\infty}^{+\infty} G(x,\lambda)U_n^2(x)dx$$

both converge. From the second of these we may conclude that

$$\int_{-\infty}^{+\infty} U_n^2(x)dx$$

also converges.

Since the number of roots of $U_n(x)$ is equal to $k-1$ when $\phi_2-\phi_1=k\pi$, we see that $U_n(x)$ has exactly n roots.

The foregoing conclusions may be summarized as follows:

THEOREM I. *There exists an infinite set of critical values of* λ, $\lambda_0, \lambda_1, \lambda_2, \cdots$, *with limit point at* $+\infty$ *only, corresponding to which equation* (1) *has solutions* (*unique except for a constant factor*) *satisfying the conditions*

$$\lim_{x=\pm\infty} U_n(x) = \lim_{x=\pm\infty} U_n'(x) = 0.$$

The solution $U_n(x)$ *vanishes* n *times in the interval* $-\infty<x<\infty$, *and the integrals*

$$\int_{-\infty}^{\infty} U_n^2(x)dx, \quad \int_{-\infty}^{\infty} U_n'^2(x)dx, \quad \textit{and} \int_{-\infty}^{\infty} G(x,\lambda)U_n^2(x)dx$$

all exist.

5. We return to the consideration of the conditions (4). The general solution of (1) may be written

$$u(x) = V_1(x)h + W_1(x)k,$$

so that the conditions (4) are equivalent to

$$
(21)\quad
\begin{aligned}
&[\alpha_1 V_1(a) + \alpha_2 V_1'(a) + \alpha_3 V_1(b) + \alpha_4 V_1'(b)]h \\
&\qquad + [\alpha_1 W_1(a) + \alpha_2 W_1'(a) + \alpha_3 W_1(b) + \alpha_4 W_1'(b)]k = 0, \\
&[\beta_1 V_1(a) + \beta_2 V_1'(a) + \beta_3 V_1(b) + \beta_4 V_1'(b)]h \\
&\qquad + [\beta_1 W_1(a) + \beta_2 W_1'(a) + \beta_3 W_1(b) + \beta_4 W_1'(b)]k = 0.
\end{aligned}
$$

The determinant of these equations is

$$
\Delta = \begin{vmatrix}
[\alpha_1 V_1(a) + \alpha_2 V_1'(a) + \alpha_3 V_1(b) + \alpha_4 V_1'(b)] & [\alpha_1 W_1(a) + \alpha_2 W_1'(a) + \alpha_3 W_1(b) + \alpha_4 W_1'(b)] \\
[\beta_1 V_1(a) + \beta_2 V_1'(a) + \beta_3 V_1(b) + \beta_4 V_1'(b)] & [\beta_1 W_1(a) + \beta_2 W_1'(a) + \beta_3 W_1(b) + \beta_4 W_1'(b)]
\end{vmatrix}.
$$

We shall treat only the case for which A_{42} is not zero, as the modifications to be made when $A_{42}=0$ are sufficiently obvious. Let ϵ be a positive constant, as small as we please. We may choose b so large that $|V_1(b)/V_1'(b)|<\epsilon$, and then choose a so large (and negative) that $|V_1(a)|<\epsilon$, $|V_1'(a)|<\epsilon$, $W_1(a)/W_1'(a)<\epsilon$, $|W_1(b)/W_1'(a)|<\epsilon$, $|W_1'(b)/W_1'(a)|<\epsilon$, uniformly with respect to λ in an interval $\lambda_{n-1}+\epsilon \leqq \lambda \leqq \lambda_n - \epsilon$. Then the determinant may be written

$$
\Delta = V_1'(b) W_1'(a)[A_{42} + \epsilon E],
$$

in which E denotes a function that is bounded for λ in the given interval. Since $W_1'(a)$ does not vanish when a is large and negative and $V_1'(b)$ does not vanish for λ in the given interval when b is large, we see at once that Δ does not vanish in this interval. But in the next interval $\lambda_n+\epsilon \leqq \lambda \leqq \lambda_{n+1}-\epsilon$ the sign is changed and therefore Δ vanishes between $\lambda_n-\epsilon$ and $\lambda_n+\epsilon$. Therefore the roots of Δ approach λ_n $(n=0, 1, 2, \cdots)$. Moreover we can easily show that Δ does not vanish more than once in the interval $\lambda_n-\epsilon<\lambda<\lambda_n+\epsilon$, so that the characteristic numbers l_n of §1 are ultimately all simple and $\lim l_n = \lambda_n$.

From (12), (13), (19), and (20) it will be seen that (21) will be satisfied by taking $h \neq 0$, $k=0$ when a and b are infinite. Therefore the characteristic functions $u_n(x)$ of §1 approach the functions $U_n(x)$. We therefore have

THEOREM II. *As the ends of the interval recede to* $-\infty$ *and* $+\infty$ *the characteristic numbers all become simple and*

$$
\lim l_n = \lambda_n \quad \lim u_n(x) = U_n(x) \qquad (n = 0,1,2,\cdots),
$$

the limits being entirely independent of the boundary conditions (4).

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