THE SECOND DERIVATIVE OF A POLYGENIC FUNCTION*

BY

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In previous papers† I have studied the first derivatives of a polygenic function, especially from the geometric aspect. A polygenic function w (in opposition to an ordinary or monogenic function) of the complex variable $z=x+iy$, is any function of the form

$$w = \phi(x,y) + i\psi(x,y), \tag{1}$$

where the components ϕ and ψ are arbitrary functions (except for suitable continuity assumptions) of the real variables x, y, the Cauchy-Riemann conditions not being assumed.

The first derivative, which we denote by $\gamma=\alpha+i\beta$,

$$\gamma = dw/dz = \lim_{\Delta z\to 0}\Delta w/\Delta z$$

of course then depends not only on the point z but also on the direction of approach θ, that is, on the slope $m=y'=dy/dx$. The main theorem is that, plotted in the α, β plane, the locus of points γ, corresponding to a given point z, is a circle (*the derivative circle*‡), or more accurately a suitably parametrized circle, which I have termed a clock (*the derivative clock*).

Various expressions for γ are convenient, and are here given for reference:

$$\gamma = \frac{dw}{dz} = \frac{(\phi_x + i\psi_x) + m(\phi_y + i\psi_y)}{1 + im} = \frac{w_x + mw_y}{1 + im}; \tag{2'}$$

$$\gamma = \mathfrak{D}(w) + e^{-2i\theta}\mathfrak{P}(w), \tag{2''}$$

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† *A new theory of polygenic functions*, Science, vol. 66 (1927), pp. 581–582; *General theory of polygenic or non-monogenic functions. The derivative congruence of circles*, Proceedings of the National Academy of Sciences, vol. 13 (1928), pp. 75–82; also L. Hofmann and E. Kasner, *Homographic circles or clocks* and E. Kasner, *Appendix on polygenic functions*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 495–503; E. Kasner, *Note on the derivative circular congruence of a polygenic function*, the same Bulletin, vol. 34, pp. 561–565, and two papers to appear in the Proceedings of the Bologna Congress.

‡ See a forthcoming paper by E. R. Hedrick, *On derivatives of non-analytic functions*, Proceedings of the National Academy of Sciences, where the derivative circle is termed the Kasner circle and new properties of it are studied.

where the linear operators

$$\mathfrak{D} = \tfrac{1}{2}(D_x - iD_y), \quad \mathfrak{P} = \tfrac{1}{2}(D_x + iD_y)$$

are introduced, the former defining mean differentiation, the latter the phase operator, so that $\mathfrak{D}(w)$ gives the center, and $\mathfrak{P}(w)$ the phase, of the derivative clock.

So far the real functions ϕ and ψ are assumed to be continuous and to have continuous partial derivatives. If we also assume ϕ and ψ to be analytic (that is, real power series in two variables x, y) we may consider the polygenic function w to be a power series (with complex coefficients) in the two variables (minimal coördinates)

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy;$$

and then it is easily seen that

$$\mathfrak{D}(w) = \partial w/\partial z = w_z, \quad \mathfrak{P}(w) = \partial w/\partial \bar{z} = w_{\bar{z}},$$

so that we have the simple formula*

$$(2''') \qquad \gamma = w_z + e^{-2i\theta} w_{\bar{z}}.$$

The present paper is devoted chiefly to the study of the second derivative d^2w/dz^2 of a polygenic function, and the appropriate geometry; higher derivatives and polygenic functions of more than one complex variable are considered only briefly in the final section. The formulas and the results are considerably more complicated. *The general second derivative depends on the curvature as well as the direction of approach.* Differentiating (2) with regard to z, that is, forming the ratio of the total differentials $d(dw/dz)$ and dz, we obtain

$$\frac{d^2w}{dz^2} = \frac{(1 + iy')(w_{xx} + 2w_{xy}y' + w_{yy}y'^2 + w_y y'') - (w_x + w_y y')iy''}{(1 + iy')^3}.$$

After simplifying this expression, and introducing for d^2w/dz^2 the complex notation $\sigma = \xi + i\eta$, we have

$$(3) \quad \sigma = \xi + i\eta = \frac{w_{xx} + 2w_{xy}y' + w_{yy}y'^2}{(1 + iy')^2} + \frac{w_y - iw_x}{(1 + iy')^3} y'' \equiv \Omega(x, y, y', y'').$$

The second derivative of a polygenic function w of z is thus a function of the point z at which it is formed and of the differential element of the second order (y', y'') along which this point is approached. Corresponding

* See G. Calugaréono, *Sur les fonctions polygènes d'une variable complexe*, Comptes Rendus, vol. 186 (1928), pp. 930–932. N. Nicolesco, *Fonctions complexes dans le plan et l'espace*, Paris thesis, 1928. No geometry is given in these papers, and derivatives of higher order are considered only for rectilinear approach, while in the present paper the path of approach is allowed to be an arbitrary curve.

to the ∞^2 real elements of the second order existing at every point, d^2w/dz_2 assumes ∞^2 values for every value of z.* If we map these values of d^2w/dz^2 in the complex plane $\sigma=\xi+i\eta$, the mapping points will therefore fill out a region of this plane. It is not obvious, but will be proved as a special theorem later, that this region covers the entire plane, so that every point in the σ-plane corresponds to at least one real curvature element (y', y'') at the given point z.

The general character of the correspondence between the elements (y', y'') and the points (ξ, η) will be the same for all general points z. But of course as z varies, different points (ξ, η) will correspond to the same values of (y', y'').

We have seen that the second derivative of a polygenic function depends in general on y'' as well as on y'. If it is to be independent of y'', that is, if in (3) Ω is not to contain y'', the condition

$$w_y - iw_x = 0$$

has to be fulfilled. But this one complex equation splits up into the two Cauchy-Riemann equations, so that (the converse being evident) *the second derivative of a polygenic function is independent of y'' when and only when the function is monogenic*, which of course makes d^2w/dz^2 independent of y' also.†

Summary of results

In the following we shall first discuss the ∞^1 values assumed by d^2w/dz^2 at a definite point z as this point is successively approached along the various elements of the same constant curvature κ. The point mapping these values in the σ-plane describes an irrational curve of the eighth order as the slope of the considered element at z is varied continuously. For the special curvature $\kappa=0$, however, that is, for the approach of z along the straight line elements, this curve of the eighth order becomes a *limaçon* described twice.

After this, the corresponding problem for the elements at z tangent to a common fixed slope is discussed. The result is very simple. The locus of points of the σ-plane mapping the various values assumed by d^2w/dz^2 as z is approached along all the elements with the same slope, is a *straight line*; and this line is described (from a suitable initial point) at a rate proportional to the rate at which the curvature of the considered elements varies.

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* We wish to stress the point that we consider only *real* elements of the second order just as in discussing the first derivative we considered only *real* elements of the first order.

† Similarly if we continue to higher derivatives of w, d^nw/dz^n will be a function of $x, y, y', y'', \cdots, y^{(n)}$, and will be independent of $y^{(n)}$ when and only when w is monogenic. So that when d^nw/dz^n is independent of $y^{(n)}$ it is also independent of $y', y'', \cdots, y^{(n-1)}$.

We then proceed to study the envelope of the ∞^1 lines corresponding in the just described manner to the various slopes at z. This envelope proves to be a *cardioid.*

Finally we investigate in two different ways the correspondence established by d^2w/dz^2 between the elements at z and the points of the σ-plane. It is found to be one-to-one in that region of the σ-plane formed by the points inside of the just defined cardioid, while to a point outside the cardioid correspond three curvature elements (y', y'').

THE DIFFERENT REPRESENTATIONS OF d^2w/dz^2

In the following we shall discuss exclusively the properties of d^2w/dz^2 at a fixed point (x, y) of the z-plane so that w_x, w_y, w_{xx}, w_{xy}, w_{yy} are to be considered as certain complex constants and d^2w/dz^2 reduces to a function of y' and y'' only.

Since $d^2w/dz^2=\sigma$ is a function of the differential element of the second order along which the point z is approached, we can represent it in different ways by choosing different quantities to determine the element of approach. Formula (3) gives σ as a function of y' and y'',

$$\sigma = \Omega_1(y', y'').$$

By substituting into Ω_1,

$$y'' = (1 + y'^2)^{3/2}\cdot\kappa,\ y' = \tan\theta = -\, i\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}},$$

we obtain a representation of σ as a function of the curvature κ and the direction angle θ of the element of approach,

$$\sigma = \tfrac{1}{4}[w_{xx} - 2iw_{xy} - w_{yy} + 2(w_{xx} + w_{yy})e^{-2i\theta} + (w_{xx} + 2iw_{xy} - w_{yy})e^{-4i\theta}] - i(w_x + iw_y)e^{-3i\theta}\cdot\kappa.$$

The coefficients in this expression simplify, if instead of x and y we use $z=x+iy$ and $\bar{z}=x-iy$ as independent variables. Then

$$w_x - iw_y = 2w_z,\quad w_x + iw_y = 2w_{\bar{z}},\quad w_{xx} - 2iw_{xy} - w_{yy} = 4w_{zz}$$

and so forth, so that finally

$$\sigma = w_{zz} + 2w_{z\bar{z}}e^{-2i\theta} + w_{\bar{z}\bar{z}}e^{-4i\theta} - 2iw_{\bar{z}}e^{-3i\theta}\cdot\kappa \equiv \Omega_2(\theta,\kappa).$$

A differential element of the second order at the point (x, y) is also determined when the center of its circle of curvature is given. If X and Y denote the coördinates of this center relative to the point (x, y), the well known transformation formulas are

$$y' = -\frac{X}{Y},\quad y'' = \frac{X^2+Y^2}{Y^3}.$$

The substitution of these expressions for y' and y'' into Ω_1 furnishes

$$\sigma = \frac{w_{xx}Y^2 - 2w_{xy}XY + w_{yy}X^2 + (w_y - iw_x)(Y + iX)}{(Y - iX)^2} \equiv \Omega_3(X,Y).$$

By using as before z and $\bar{z}$ as independent variables and correspondingly $Z = X + iY$ and $\bar{Z} = X - iY$, we obtain

$$\sigma = \frac{w_{\bar{z}\bar{z}}Z^2 - 2w_{z\bar{z}}Z\bar{Z} + w_{zz}\bar{Z}^2 + 2w_{\bar{z}}\bar{Z}}{Z^2} \equiv \Omega_4(Z,\bar{Z}).$$

The main importance of the two last representations of σ lies in the fact that they convert the element-point correspondence between the elements at the point z and the points of the σ-plane expressed by Ω_1 and Ω_2 into a point correspondence between the Z-plane and the σ-plane.

For the sake of easier reference, we put the four different representations of σ that we have developed together:

$$\text{(A)}\quad \begin{cases} \sigma = \dfrac{w_{xx} + 2w_{xy}y' + w_{yy}y'^2}{(1 + iy')^2} + \dfrac{(w_y - iw_x)y''}{(1 + iy')^3} \equiv \Omega_1(y', y''), \\ \sigma = w_{zz} + 2w_{z\bar{z}}e^{-2i\theta} + w_{\bar{z}\bar{z}}e^{-4i\theta} - 2iw_{\bar{z}}e^{-3i\theta}\cdot\kappa \equiv \Omega_2(\theta,\kappa), \\ \sigma = \dfrac{w_{xx}Y^2 - 2w_{xy}XY + w_{yy}X^2 + (w_y - iw_x)(Y + iX)}{(Y - iX)^2} \equiv \Omega_3(X,Y), \\ \sigma = \dfrac{w_{\bar{z}\bar{z}}Z^2 - 2w_{z\bar{z}}Z\bar{Z} + w_{zz}\bar{Z}^2 + 2w_{\bar{z}}\bar{Z}}{Z^2} \equiv \Omega_4(Z,\bar{Z}). \end{cases}$$

The rectilinear second derivative

As the fixed point z is approached along the different elements belonging to a certain definite slope, d^2w/dz^2 assumes ∞^1 values. Among these we distinguish the value corresponding to the approach of z along the element of curvature $\kappa = 0$, that is, along the straight line element of that particular slope, and term it the *rectilinear second derivative* of w belonging to that slope. We abbreviate it by σ_0, so that

$$\begin{aligned} \sigma_0 &= \frac{w_{xx} + 2w_{xy}y' + w_{yy}y'^2}{(1 + iy')^2} \equiv \Omega_1(y',0) \\ &= w_{zz} + 2w_{z\bar{z}}e^{-2i\theta} + w_{\bar{z}\bar{z}}e^{-4i\theta} \equiv \Omega_2(\theta,0). \end{aligned}$$

The rectilinear second derivative is a function of the slope y' or of the direction angle θ. As y' varies from 0 to ∞, that is, as θ varies from 0 to π, it assumes ∞^1 values, and the point σ_0 mapping these values in the complex plane σ describes a curve. This curve, which we call L, is most easily discussed by means of the second representation of σ_0:

$$\sigma_0 = w_{zz} + w_{\bar{z}\bar{z}}e^{-4i\theta} + 2w_{z\bar{z}}e^{-2i\theta},$$

in which now θ is a variable going from 0 to π.

A point σ_0 of L is determined as the sum of the three vectors w_{zz}, $w_{\bar{z}\bar{z}}e^{-4i\theta}$ and $2w_{z\bar{z}}e^{-2i\theta}$. As θ varies, w_{zz} remains fixed and the end points of $w_{\bar{z}\bar{z}}e^{-4i\theta}$ and $2w_{z\bar{z}}e^{-2i\theta}$ separately describe circles of radii $|w_{\bar{z}\bar{z}}|$ and $2|w_{z\bar{z}}|$ with the angular velocities -4θ and -2θ. Since the vector $2w_{z\bar{z}}e^{-2i\theta}$ is applied at the end point of the vector $w_{zz}+w_{\bar{z}\bar{z}}e^{-4i\theta}$ and rotates with one half the angular velocity of $w_{\bar{z}\bar{z}}e^{-4i\theta}$, the lines S_1 and S_2 carrying any two vectors $2w_{z\bar{z}}e^{-2i\theta_1}$ and $2w_{z\bar{z}}e^{-2i\theta_2}$ applied as described at the points $\sigma_1 = w_{zz}+w_{\bar{z}\bar{z}}e^{-4i\theta_1}$ and $\sigma_2 = w_{zz}+w_{\bar{z}\bar{z}}e^{-4i\theta_2}$ must intersect at a fixed point P of the circle $\sigma = w_{zz}+w_{\bar{z}\bar{z}}e^{-4i\theta}$; for these lines form the angle $-2(\theta_2-\theta_1)$ and this is equal to one half of the central angle $-4(\theta_2-\theta_1)$ over the points σ_1 and σ_2 of the circle which lie on the lines S_1 and S_2 respectively according to the construction. It follows from this discussion that the curve L is the locus of the points lying on the lines through the point P of the circle $\sigma = w_{zz}+w_{\bar{z}\bar{z}}e^{-4i\theta}$ at the distance $2|w_{z\bar{z}}|$ in either direction from the second point of intersection of these lines with the circle (in *either* direction since θ goes from 0 to π and to the angles θ and $\theta+\pi/2$ correspond the points $\sigma_0 = w_{zz}+w_{\bar{z}\bar{z}}e^{-4i\theta} \pm 2w_{z\bar{z}}e^{-2i\theta}$). This however is the ordinary definition of a *limaçon* with the basal circle $\sigma = w_{zz}+w_{\bar{z}\bar{z}}e^{-4i\theta}$, the pole P, and the determining length $2|w_{z\bar{z}}|$.*

Our discussion not only characterises the curve L as a whole but also indicates its parametrization with regard to θ. Let us distinguish on the limaçon the point Q: $\sigma_0 = w_{zz}+w_{\bar{z}\bar{z}}+2w_{z\bar{z}}$ corresponding to $\theta=0$ as an initial point. Then given the limaçon, the pole P and the point Q, the point corresponding to an arbitrary value of θ is found by turning the line PQ about P through the angle -2θ, as that point of the limaçon into which this continuous motion carries the point of intersection of the rotating line with the limaçon.

The values of the rectilinear second derivative of a polygenic function w belonging to the different slopes at the point z are represented in the complex plane σ of d^2w/dz^2 by the points of a limaçon L. When the direction angle of the slope at z varies, the corresponding point on the limaçon moves with twice the angular velocity and in the opposite direction, the angular velocity being measured from the pole of the limaçon as center.

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* See Gino Loria, *Spezielle algebraische und transzendente ebene Kurven*, Leipzig, Teubner, 1902, p. 136.

SPECIAL TYPES OF POLYGENIC FUNCTION

The limaçon L specializes in certain noteworthy simple cases.

(a) When $w_{z\bar{z}}=0$, so that

$$w = f_1(z) + f_2(\bar{z}),$$

that is, what I have termed a *harmonic* polygenic function,* then L becomes a clock of rate -4.† As θ varies from 0 to π, the corresponding point σ_0 describes this clock twice, assuming equal values for two angles θ differing by $\pi/2$.

(b) When $w_{\bar{z}\bar{z}}=0$,

$$w = f_1(z) + \bar{z}f_2(z),$$

and L becomes a clock of rate -2.

(c) When $w_{zz}=0$,

$$w = f_1(\bar{z}) + zf_2(\bar{z}).$$

Then L remains a limaçon but is so far specialised in position as to have the center of its basal circle at the origin.

(d) When $w_{z\bar{z}}=0$ and $w_{\bar{z}\bar{z}}=0$,

$$w = f(z) + A\bar{z},$$

and L reduces to a point.

(e) When $w_{\bar{z}\bar{z}}=0$ and $w_{zz}=0$,

$$w = A + Bz + C\bar{z} + Dz\bar{z},$$

and L becomes a clock of rate -2 with its center at the origin.

(f) When $w_{zz}=0$ and $w_{z\bar{z}}=0$,

$$w = f(\bar{z}) + Az$$

and L becomes a clock of rate -4 with center at the origin.

(g) When $w_{zz}=0$, $w_{z\bar{z}}=0$ and $w_{\bar{z}\bar{z}}=0$, $w = A + Bz + D\bar{z}$, and L reduces to the point at the origin.

(h) When $|w_{z\bar{z}}| = |w_{\bar{z}\bar{z}}|$, that is when the determining length $2|w_{z\bar{z}}|$ of the limaçon is equal to the diameter $2|w_{\bar{z}\bar{z}}|$ of its basal circle, the limaçon L specialises into a cardioid.‡

The converses of these eight theorems are also true, so that the geometric properties stated are completely characteristic of these types of polygenic function.

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* A polygenic function $w=F(z)=\phi(z, y)+i\psi(x, y)$ is called harmonic when ϕ and ψ separately fulfill the Laplace equation. See Proceedings of the National Academy of Sciences, loc. cit., p. 81. The mean derivative of such a polygenic function is a monogenic function.

† The circle $\sigma_0=w_{zz}+w_{\bar{z}\bar{z}}e^{-4i\theta}$ is parametrized with regard to θ in such a manner that as θ varies the point σ_0 on the circle moves with four times the angular velocity and in the opposite sense.

‡ A cardioid is defined as a limaçon for which the determining length is equal to the diameter of the basal circle. See Loria, loc. cit., p. 142.

THE CURVILINEAR SECOND DERIVATIVE CORRESPONDING TO THE ELEMENTS OF A FIXED CURVATURE

After having studied the curve described by σ_0,

$$\sigma_0 = \Omega_2(\theta,0)\,,$$

the points of which map the values of the rectilinear second derivative corresponding to the different slopes at the point z, let us now study the curve described by the point

$$\sigma_\kappa = w_{zz} + 2w_{z\bar{z}}e^{-2i\theta} + w_{\bar{z}\bar{z}}e^{-4i\theta} - 2iw_{\bar{z}}e^{-3i\theta}\cdot\kappa \tag{4}$$

as θ varies and κ remains constant. The point σ_κ maps the values of d^2w/dz^2 corresponding to the different elements of the fixed curvature κ at z. If we call the value assumed by d^2w/dz^2 as the point z is approached along an element of a certain slope and a certain curvature $\kappa\neq 0$ the *curvilinear second derivative of* w corresponding to that slope and curvature, then σ_κ maps the values of the curvilinear second derivative corresponding to the various elements of the curvature κ.

Obviously

$$\sigma_\kappa = \sigma_0 - 2iw_{\bar{z}}\kappa e^{-3i\theta}.$$

The term added to σ_0 represents a point describing the circle of radius $2\kappa\,|w_{\bar{z}}|$ about the origin at the rate -3 as θ varies. The points σ_κ can thus be determined as the vectorial sum of a point describing the limaçon L at the rate -2 in the sense explained above, and a point describing a certain circle about the origin at the rate -3.

The resulting curve is not rational. This is seen by substituting into $\sigma=\Omega_1(y', y'')$

$$y'' = (1 + y'^2)^{3/2}\kappa.$$

Then in

$$\sigma_\kappa = \xi_\kappa + i\eta_\kappa = \Omega_1(y', [1 + y'^2]^{3/2}\cdot\kappa)\,,$$

ξ_κ and η_κ are algebraic but not rational functions of y', since the irrational expression $(1+y'^2)^{1/2}$ enters essentially and cannot be eliminated.

To determine the order of the curve described by σ_κ it is necessary to note that if this is to be a complete closed curve, we must let θ vary from 0 to 2π on account of the odd exponent -3 of the last term in (4). The reason is obvious since as θ goes from 0 to π only, the element $(0, \kappa)$ at z is carried by a continuous rotation into the element (π, κ) and these two elements are not the same but lie on opposite sides of their tangent. Generally this variation of θ would not furnish all elements of the curvature κ, but for every slope only that one element into which the above described rotation carries $(0, \kappa)$.†

Equation (4) can be written in the form

$$\sigma_\kappa - w_{zz} = w_{\bar{z}\bar{z}} e^{-2i\theta}(e^{-i\theta} + A)(e^{-i\theta} + B),$$

where A and B are determined by $w_{\bar{z}\bar{z}}$, $2w_{z\bar{z}}$ and $-2iw_{\bar{z}}\kappa$. This expression has an algebraic limiting case

$$\sigma^* = e^{-4i\theta}.$$

Here the point σ^* describes the unit circle about the origin four times as θ goes from 0 to 2π, so that the curve as a whole must be counted as a curve of the eighth order. *The curve* (4) *therefore will be at least of order eight.* That it is at most of order eight is shown as follows. Replace, in (4), $e^{-i\theta}$ by $\zeta = \zeta_1 + i\zeta_2$, and assume for a moment that ζ_1 and ζ_2 are independent variables instead of making them fulfill the relation $\zeta_1^2 + \zeta_2^2 = 1$ which makes $e^{-i\theta} = \zeta$. Then $\sigma_\kappa = \xi_\kappa + \eta_\kappa$ assumes ∞^2 values as ζ_1 and ζ_2 vary and we have a point correspondence between the ξ-plane and the σ_κ-plane which is superposed over the σ-plane. Splitting up into real and imaginary parts, we see that ξ_κ and σ_κ are integral functions of the fourth degree of ζ_1 and ζ_2, so that to every line of the σ_κ-plane corresponds a certain curve of the fourth degree in the ζ-plane. Now the curve described by σ_κ according to (4), is equal to that curve of the σ_κ-plane into which the transformation just described carries the circle $\zeta_1^2 + \zeta_2^2 = 1$ of the ζ-plane. This circle is cut by one of the curves of the fourth order corresponding to the lines of the σ_κ-plane in at most eight points that vary with the chosen curve. Therefore, in virtue of the correspondence, the curve described by σ_κ is intersected by a line of its plane in at most eight points. Thus in view of the fact that the order of this curve was above shown to be at least eight, it is now proved to be exactly eight.

The values of the curvilinear second derivative of a polygenic function w corresponding to the elements of a fixed curvature κ at the point z are mapped in the complex plane σ of d^2w/dz^2 by the points of an irrational curve of the eighth order.‡ *The points of this curve are determined as the vectorial sum of a point*

† The variation of θ from 0 to π only is sufficient to furnish all elements of the curvature κ only in the case $\kappa = 0$.

‡ The seeming contradiction that for a general value of the curvature κ, σ_κ describes a curve of the eighth order while for the special value $\kappa = 0$, σ_0 describes a limaçon, i.e. a curve of the fourth order, is solved by the remark that the limaçon was obtained by varying θ from 0 to π only. If also in the case $\kappa = 0$ we let θ go from 0 to 2π, σ_0 will describe the limaçon twice so that the entire curve will here also be a curve of the eighth order.

of the limaçon L and a point of a certain circle about the origin, the point on the limaçon moving at the rate -2 *in the sense explained above as the slope at z varies, and the point on the circle at the rate* -3.

THE CURVILINEAR SECOND DERIVATIVE CORRESPONDING TO THE ELEMENTS WITH A FIXED COMMON TANGENT

So far we have examined the points mapping the values assumed by d^2w/dz^2 for the approach of the point z along the elements of a certain fixed curvature. We shall now investigate the case in which z is approached successively along the elements of a certain fixed slope, that is, along the elements with a common tangent. That means that we must study the curve described by the point

$$\sigma_\theta = (w_{zz} + 2w_{z\bar{z}}e^{-2i\theta} + w_{\bar{z}\bar{z}}i^{-4i\theta}) - 2iw_{\bar{z}}e^{-3i\theta}\cdot\kappa, \tag{5}$$

as κ varies from $-\infty$ to $+\infty$ and θ remains fixed; or, using the representation of σ by means of $\Omega_1(y', y'')$ and abbreviating y' by m, the curve described by the point

$$\sigma_m = \frac{w_{xx} + 2w_{xy}m + w_{yy}m^2}{(1 + im)^2} + \frac{w_y - iw_x}{(1 + im)^3}\cdot y'', \tag{6}$$

as y'' varies from $-\infty$ to $+\infty$ and $m = \tan\theta$ remains fixed.

Since the expression on the right of (5) is an integral linear function of κ, it follows that *as* κ *varies from* $-\infty$ *to* $+\infty$, σ_θ *describes a straight line and describes this line at a rate proportional to the rate at which* κ *varies.* We abbreviate this line by S_θ or $S_m(m = \tan\theta)$.†

As (5) shows, S_θ is parallel to the line carrying the vector $-2iw_{\bar{z}}e^{-3i\theta}$. If therefore we construct two lines S_{θ_1} and S_{θ_2} which map the values of d^2w/dz^2 corresponding once to the elements with the common tangent of slope θ_1, once to the elements with the common tangent of slope θ_2, S_{θ_1} and S_{θ_2} form the angle $-3(\theta_2-\theta_1)$, that is -3 times the angle $(\theta_2-\theta_1)$ formed by the determining slopes at the point z. From the fact that when θ varies the direction angle of S_θ varies by the triple amount and in the opposite sense, it follows that there are four real slopes θ at z which are parallel to the corresponding lines S_θ. The proof of this mainly comes down to solving the equation

$$\tan(\alpha + \theta) = -\tan(\beta + 3\theta),$$

where α and β are constant angles, for $\tan\theta$. Obviously this equation is of

† As κ varies it naturally also assumes the value $\kappa = y'' = 0$. Thus the line L_m goes through that point of the limaçon L which maps the value of the rectilinear derivative belonging to the slope m.

the fourth degree in tan θ; a short geometric consideration shows that the four roots are real.

The values of the curvilinear second derivative of a polygenic function w corresponding to the elements with a fixed common tangent of slope θ at the point z, are mapped in the complex plane σ of d^2w/dz^2 by the points of a straight line S_θ or S_m. As the curvature of the elements at z varies, the mapping point describes the line S_θ at a rate proportional to the rate of variation of the curvature itself. As the direction angle θ of the slope at z varies, the direction angle of the corresponding line S_θ varies by the triple quantity and in the opposite sense. There are four real slopes at z which are parallel to the corresponding lines S_θ in the σ-plane.

The envelope of the lines L_θ

After discussing the individual lines L_θ we proceed to consider them as a totality and to study the curve which they envelop as θ goes continuously from 0 to π. This envelope will be called E. To determine the character of E it is necessary to obtain the real equation of the lines L_θ. For this purpose we use the representation of σ by means of (6) and abbreviate it by

$$\sigma_m = \sigma_m \xi_m + i\eta_m = a(m) + b(m)y''. \tag{6'}$$

Let us split a and b into their real and imaginary parts $a = a_1 + ia_2$ and $b = b_1 + ib_2$. According to (6′), S_m has the slope b_2/b_1 and goes through the point $(a_1, a_2)(a_1 + ia_2 = \Omega_2(\theta, 0),\ \theta = \arctan m)$; it therefore has the equation

$$(\eta_m - a_2)/(\xi_m - a_1) = b_2/b_1.$$

We write this equation in the form

$$L_m \equiv p\xi + q\eta + r = 0\,;$$

a simple calculation furnishes the following values for p, q and r:†

$$\begin{aligned} p &= Am^3 - 3Bm^2 - 3Am + B, \\ q &= Bm^3 + 3Am^2 - 3Bm - A, \\ r &= m[Bg(m) + Af(m)] + Ag(m) - Bf(m), \end{aligned} \tag{7}$$

where

$$A = \phi_y + \psi_x,\quad B = -\phi_x + \psi_y,$$

$$f(m) = \phi_{xx} + 2m\phi_{xy} + m^2\phi_{yy},\quad g(m) = \psi_{xx} + 2m\psi_{xy} + m^2\psi_{yy}.$$

By (7) the three homogeneous line coördinates of S_m are given as functions of the parameter m. As A and B are constants (only depending on the point z) and $f(m)$ and $g(m)$ are polynomials of the second degree in m, the three homogeneous line coördinates p, q, r of L_m are integral rational functions of the

† For the sake of greater convenience we here and in the sequel omit the factor of proportionality by which the homogeneous coördinates p, q, and r should be multiplied in the parametric representation (7).

third degree in the parameter m. The curve E was defined as the envelope of the lines L_m. Therefore (7), as the parametric representation of these lines, is the parametric representation of E as locus of its tangents. It follows from the rationality and the degree three of p, q, and r with regard to m that the envelope *E is a rational curve of class three* (the genus of a curve is the same whether the curve be considered as locus of its tangents or of its points).

To determine the order of E we introduce the homogeneous point coordinates ξ_m^*, η_m^* and t for the points of E,

$$\xi_m^*/t = \xi_m,\ \eta_m^*/t = \eta_m,$$

and derive the parametric representation of E as locus of its points from (7) in the familiar manner. We have

$$(8)\ \xi^* = qr' - q'r,\ \eta^* = rp' - r'p,\ t = pq' - p'q,$$

where the prime denotes differentiation with regard to the parameter m.

A rational curve of the third class is either of the fourth or of the third order.† We easily find

$$t = -3(A^2 + B^2)(1 + m^2)^2. \tag{9}$$

Since t is actually of the fourth degree, the curve E can be of a lower order only if ξ^*, η^* and t have some polynomial in m as a common factor. According to (9) this factor could only be $m-i$ or $m+i$. A short calculation however shows that ξ^* and η^* do not vanish for $m=i$ or $m=-i$. Thus it is proved that *E is of the fourth order.*

The singularities of a rational curve of the fourth order and the third class consist in an ordinary double tangent and three ordinary cusps of which one is real and two conjugate imaginary.† Without going into details about the double tangent or the real cusp, we shall merely determine the position of the imaginary cusps of E. The discussion which we give in detail later on shows these cusps to lie at *the cyclic points* of the σ-plane.

Under the assumption that this is proved, *E is now characterised as a rational curve of order four and class three with its imaginary cusps at the cyclic points.* This characterisation is sufficient to show that E is a well known curve of very special type, namely a *cardioid.* The proof is as follows.

There are ∞^8 rational curves of order four and class three.‡ The require-

† This follows immediately from the consideration of the well known dual curve, the rational curve of the third order. A rational curve of the third order has either an ordinary double point or a cusp. In the first case it is of class four, in the second of class three. Furthermore a rational curve of the third order and fourth class has one real and two imaginary inflexion tangents, which together with its double point make up all its singularities.

‡ This again follows from the fact that there are ∞^8 dual curves of order three and class four. For proof see Severi-Loeffler, *Vorlesungen ueber Algebraische Geometrie*, Leipzig, 1921, p. 161–162.

ment that these curves shall have their imaginary cusps at the cyclic points (or at any two definite points) represents a fourfold condition. Thus there are ∞^4 rational curves of order four and class three with their imaginary cusps at the cyclic points. We abbreviate this family of curves by (E). On the other hand the properties characterising the curves (E) are, as is well known, properties of the cardioids, so that the family of cardioids must be contained among the curves (E). The cardioids themselves, however, form a four-parameter family of curves, a general cardioid containing one parameter of size and three of position. Then since the dimension of the family (E) is equal to the dimension of the family of cardioids and at the same time contains this family, the curves (E) must be identical with the cardioids.

Hereby it is proved that *the envelope E as a curve of (E) is a cardioid.*

We now give the discussion referring to the imaginary cusps of E which was postponed above.

Let us form the expression

$$i\xi_m^* + \eta_m^* + \lambda t, \tag{10}$$

where λ is an arbitrary constant. Substituting for ξ_m^*, η_m^* and t their parametric expressions in m according to (8) and (7), where, however, t need not be written out in full, we find for (10)

$$(m - i)^2[(A - iB)\{ - (m - i)r' + 3r\} - 3\lambda(A^2 + B^2)(m + i)^2]. \tag{10'}$$

The equation obtained by putting (10) equal to zero represents a condition on the coördinates ξ_m^*, η_m^*, t; it will be fulfilled by the coördinates of only those points of E which lie on the line

$$i\xi + \eta + \lambda = 0,$$

that is, a certain line through the cyclic point $I_1(\xi/\eta = i)$ of the σ-plane. The values of the parameter m determining these points of E are of course obtained as the roots of (10′) put equal to zero. But this equation obviously always has the double root $m = i$, no matter what value λ is given. That means that all the lines through I_1 cut E twice in the same fixed point, namely the point corresponding to the parameter value $m = i$. Since, however, the only point common to all the lines through I_1 is I_1 itself, it follows that E must have this point for a double point.

The discussion for $I_2(\xi/\eta = -i)$ is analogous. Since we know from the preceding arguments that every double point of E is a cusp, it is thus proved that E has cusps at the cyclic points of the σ-plane.

Combining the results of this paragraph with those relating to the ndividual lines L_m previously obtained, we have the following theorem:

As the slope m at the point z varies, the corresponding line L_m in the σ-plane moves so that it envelops a cardioid E in general position. Given the cardioid and, as an initial tangent, the line L_0 corresponding to the slope $m=0$ at z, the tangent corresponding to an arbitrary value $m = \tan\theta$ of the slope m is determined by letting the initial tangent glide along the cardioid in the negative sense till its direction angle has grown by the quantity -3θ. There are four tangents to the cardioid parallel to the determining slopes at z.

The correspondence between the curvature elements at z and the points of the σ-plane

By means of the formula

$$\sigma = \Omega_1(y', y'')$$

a certain correspondence is set up between the ∞^2 real elements (y', y'') at the point z and the points of the σ-plane. Since Ω_1 is a rational function of y' and y'', it follows that to every element (y', y'') corresponds one and only one point σ. The converse question, how many elements (y', y'') correspond to one point σ, is answered by the following consideration.

We have in these last paragraphs grouped the ∞^2 real elements of the second order at z into ∞^1 groups each of which contained all the elements tangent to the same slope. The corresponding grouping of the points σ was that according to the real tangents of the cardioid E.† Thus a point σ represents as many elements (y', y'') as there are real tangents to E going through it. And since a cardioid is a curve of the third class, this number is equal to one for a point in the inside of E and equal to three for every other point.‡

By means of the second derivative d^2w/dz^2 of a polygenic function w of z, a certain rational correspondence is set up between the real elements of the second order at a definite point z and the points σ mapping the values assumed by d^2w/dz^2 as z is approached along these elements. This correspondence is one-to-one in that region of the σ-plane which is formed by the points inside the cardioid E and three-to-one in the rest of the σ-plane. Thus every point σ at the inside of E maps one value and every other point σ maps three values of d^2w/dz^2.

Since every point of the σ plane maps at least one value of d^2w/dz^2, the statement made at the very beginning of this paper is verified, that *as d^2w/dz^2 is calculated at a point z for all possible real elements (y', y'') of approach, the mapping point σ covers the entire σ-plane.*

† While discussing the lines L_θ we did not stress the fact that they are real, as being obvious since we consider only real values of θ and κ.

‡ These tangents are distinct for a general point σ at the outside of E. For those points lying on the double tangent of E or on E itself, two of them coincide; for the cusp of E and for the points of contact of the double tangent, all three coincide.

THE POINT CORRESPONDENCE BETWEEN THE Z- AND THE σ-PLANE

The results of the preceding paragraph can also be obtained without taking the cardioid E into consideration. So far we have made use only of the first two representations of σ in system (A), p. 807. We now turn to the remaining ones:

$$(11) \quad \sigma = \frac{w_{xx}Y^2 - 2w_{xy}XY + w_{yy}X^2 + (w_y - iw_x)(Y + iX)}{(Y - iX)^2} \equiv \Omega_3(X,Y),$$

$$(11') \quad \sigma = \frac{w_{\bar{z}\bar{z}}Z^2 - 2w_{z\bar{z}}Z\bar{Z} + w_{zz}\bar{Z}^2 + 2w_{\bar{z}}\bar{Z}}{Z^2} \equiv \Omega_4(Z,\bar{Z}).$$

These formulas express a point correspondence between the (X, Y)- or $X+iY=Z$-plane and the $\xi+i\eta=\sigma$-plane, where X and Y are the center coördinates relative to the point z of the circle of curvature of the element (y', y'') at z.

Since in (11) σ is a rational function of X and Y there will be one point σ corresponding to every real point (X, Y). We ask the converse question, how many real points (X, Y) correspond to the same point σ. This is identical with the question of how many values of d^2w/dz^2 are represented by the same point σ, since every point (X, Y) determines one element (y', y'') which again determines one value of d^2w/dz^2.

To solve our problem, we write (11) in the form

$$(12) \quad (\xi + i\eta)(Y - iX)^2 = w_{xx}Y^2 - 2w_{xy}XY + w_{yy}X^2 + (w_y - iw_x)(Y + iX),$$

where ξ and η are now assumed to have some definite values and X and Y vary. Equation (12) represents a complex conic in the (X, Y)-plane. Therefore there will be as many real points (X, Y) corresponding to the fixed point (ξ, η), i.e., points (X, Y) whose coördinates fulfill equation (12), as there are real points on the conic represented by (12). From this number, however, we must naturally exclude such fixed real points as are common to all conics (12) independently of ξ and η. Now the number of real points on (12) is equal to the number of real intersection points of the two real conics obtained by equating the real and imaginary parts of (12) separately to zero. Since the total number of intersection points of these two conics is four, and since they have the fixed intersection $X=Y=0$ independently of (ξ, η), the number of their variable real intersection points is equal either to three or to one.

The discussion as to which points (ξ, η) correspond to a complex conic (12) with three real variable points and which to one with one real variable point would of course lead to the cardioid E in σ. We will not carry it through but merely state the complete theorem as follows:

By means of the second derivative d^2w/dz^2 of a polygenic function w of z a certain rational correspondence is established between the points Z indicating the centers of the curvature circles of the elements (y', y'') at a fixed point z and the points σ mapping the values of d^2w/dz^2 as z is approached along these elements. This correspondence is one-to-one in that region of the σ-plane formed by the inside points of the cardioid E and three-to-one in the remainder of the σ-plane, so that to a point σ in the first region corresponds one and to any other point σ correspond three points Z.

DERIVATIVES OF HIGHER ORDER

The nth derivative of a polygenic function is a complex quantity depending on the given point x, y and on the derivatives $y', y'', \cdots, y^{(n)}$ of the curve of approach. If the highest derivative $y^{(n)}$ is to be absent it turns out that all the other derivatives are absent and the function is necessarily monogenic.

The general form of expression for *the third derivative* of any polygenic function is found to be

$$d^3w/dz^3 = \beta_1 + \beta_2 y'' + \beta_3 y''^2 + \beta_4 y''',$$

where the coefficients involve x, y, y'. Many geometric theorems may be deduced from this. We mention only the simplest, namely that the locus corresponding to fixed values y', y'', and variable y''' is a straight line.

The fourth derivative is

$$d^4w/dz^4 = (B_1 + B_2 y'' + B_3 y''^2 + B_4 y''^3) + (B_5 + B_6 y'')y''' + B_8 y'''',$$

which is seen to be linear in the two highest derivatives y''', y''''. Analogous results are found for all higher derivatives but we shall not take up the detailed theory here.

Polygenic functions of two or more independent complex variables will be studied in another paper. If w is a polygenic function of $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the two partial derivatives of first order $P = \partial w/\partial z_1$ and $Q = \partial w/\partial z_2$ give rise to a pair of circles or clocks (for any given pair of points z_1 and z_2). But on account of the *essentially new feature* $\partial^2 w/\partial z_1 \partial z_2 \neq \partial^2 w/\partial z_2 \partial z_1$ (equality holding in fact only when w is an analytic function), there are *four* distinct partial derivatives of second order. $R = \partial^2 w/\partial z_1^2$, $S = \partial^2 w/\partial z_1 \partial z_2$, $S^* = \partial^2 w/\partial z_2 \partial_1$, $T = \partial^2 w/\partial z_2^2$, and the corresponding geometry is of fascinatingly complicated structure.

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