ON THE DEGREE OF APPROXIMATION TO AN ANALYTIC FUNCTION BY MEANS OF RATIONAL FUNCTIONS*

BY

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It is the purpose of this note to establish results for the approximation to an analytic function by means of rational functions analogous to the following theorem for such approximation by means of polynomials:

THEOREM I. *Let J be an arbitrary closed Jordan region of the z-plane, and let $w=\Phi(z)$ be a function which maps conformally the exterior of J onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C_R denote the curve $|\Phi(z)|=R$, $R>1$, that is, the transform in the z-plane of the circle $|w|=R$.*

A necessary and sufficient condition that an arbitrary function $F(z)$ defined in J be regular-analytic in (the closed region) J is that there should exist polynomials $P_n(z)$ of degree n, $n=0, 1, 2, \cdots$, and numbers M, $R>1$, such that the inequality

$$|F(z) - P_n(z)| \leqq \frac{M}{R^n} \tag{1}$$

is valid for every z in J.

If the polynomials $P_n(z)$ are given so that (1) is satisfied, the sequence $P_n(z)$ converges everywhere interior to C_R and uniformly on any closed point set interior to C_R, and thus $F(z)$ is regular† throughout the interior of C_R.

If $F(z)$ is given regular in the closed interior of C_ρ, the polynomials $P_n(z)$ can be chosen so that (1) is satisfied for $R=\rho$ and for every z in J.

Theorem I is based on results and methods of S. Bernstein, M. Riesz, Faber, Fejér, and Szegö, but the formulation as just given is due to the present writer.‡

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† Here and below we tacitly assume that if $F(z)$ is not originally supposed to be defined on the entire point set considered, then the definition in the new points is to be made by analytic extension—or what amounts to the same thing, by means of the convergent series of rational functions, polynomials in the present case.

‡ See Sitzungsberichte der Bayerischen Akademie der Wissenschaften, 1926, pp. 223–229, where detailed references to the literature are given.

A rational function of the form

$$r(z) = \frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}$$

is said to be of degree n. We do not assume a_0 or b_0 different from zero, but do assume of course that the denominator does not vanish identically. If the poles of the function are all distinct, we may write $r(z)$ in the form

$$A + A_0 z + \frac{A_1}{z - \beta_1} + \frac{A_2}{z - \beta_2} + \cdots + \frac{A_n}{z - \beta_n},$$

where the A_i are not all different from zero; if the poles of the function are not distinct, we may write $r(z)$ in the form

$$A_0 + A_0' z + \cdots + A_0^{(k_0)} z^{k_0} + \frac{A_1'}{z - \beta_1} + \frac{A_1''}{(z - \beta_1)^2} + \cdots + \frac{A_1^{(k_1)}}{(z - \beta_1)^{k_1}}$$
$$+ \frac{A_2'}{z - \beta_2} + \cdots + \frac{A_m^{(k_m)}}{(z - \beta_m)^{k_m}}.$$

We proceed to the proof of

THEOREM II. *Let C be an arbitrary rectifiable Jordan curve of the z-plane in whose interior the origin lies, and let the functions $w = \Phi(z)$, $\phi(z)$ map respectively the exterior and interior of C onto the exterior and interior of the unit circle in the w-plane so that the point at infinity and origin of the one plane correspond to those of the other. Let C_R and D_R denote the curves $|\Phi(z)| = R$, $R > 1$, and $|\phi(z)| = 1/R$ respectively, that is, the transforms onto the z-plane of the circles $|w| = R$ and $|w| = 1/R$.*

A necessary and sufficient condition that an arbitrary function $F(z)$ defined on C be regular-analytic on C, is that there should exist polynomials $p_n(z)$ in z and polynomials $q_n(z)$ in $1/z$ of degree n, $n = 0, 1, 2, \cdots$, so that we have

$$|F(z) - p_n(z) - q_n(z)| \leqq M/R^n \tag{2}$$

for all z on C, where M and $R > 1$ are constants independent of z.

If the polynomials $p_n(z)$ and $q_n(z)$ are given so that (2) is satisfied for all z on C, and if we modify (if necessary) these definitions so that $q_n(\infty) = 0$, then the sequence $p_n(z)$ converges for all z interior to C_R and uniformly on any closed point set interior to C_R, and the sequence $q_n(z)$ converges for all z exterior to D_R and uniformly on any closed point set exterior to D_R; hence the function $F(z)$ is regular in the annular region bounded by C_R and D_R.

If $F(z)$ is given analytic in the closed region bounded by C_ρ and D_ρ, the polynomials $p_n(z)$ and $q_n(z)$ can be determined so that (2) is satisfied with $\rho = R$, for all z on C.

The proof is quite easy, by means of Theorem I. Let the polynomials $p_n(z)$ and $q_n(z)$ be given; assume, as we may do with no loss of generality, $q_n(\infty)=0$. We choose z on an arbitrary curve C' interior to C, and denote by d the minimum distance of C' from C. Then we have, for z on C',

$$(3)\quad \frac{1}{2\pi i}\int_C \frac{F(t)-p_n(t)-q_n(t)}{t-z}dt = f(z)-p_n(z), \text{ where } f(z) = \frac{1}{2\pi i}\int_C \frac{F(t)dt}{t-z}.$$

By (2) we now have

$$|f(z)-p_n(z)| \leqq \frac{Ml}{2\pi dR^n}$$

for z on C', where l is the length of C. It follows by Theorem I that the sequence $p_n(z)$ converges not merely on C', but within a curve C_R', the transform of the circle $|w|=R$ when the exterior of C' is mapped on the exterior of the unit circle of the w-plane so that the points at infinity correspond. The convergence of $p_n(z)$ is moreover uniform on any closed point set interior to C_R'. If now the curve C' is allowed monotonically to approach the curve C, the curve C_R' approaches monotonically the curve C_R.* Thus the sequence $p_n(z)$ converges interior to C_R and uniformly on any closed point set interior to C_R. It will be noticed that in particular the sequence $p_n(z)$ converges uniformly on C.

Consideration for z *exterior* to C of the integral used in (3) yields a similar result on the convergence of the polynomials $q_n(z)$, so the sequence $p_n(z)+q_n(z)$ converges between C_R and D_R, uniformly in any closed region between C_R and D_R, and $F(z)$ is regular-analytic between those two curves.

Conversely, let the function $F(z)$ be given analytic in the closed region bounded by C_R and D_R. Then $F(z)$ is also analytic in a larger closed annular region bounded by $C_{R'}$ and $D_{R'}$, $R'>R$. Cauchy's integral

$$f_1(z) = \frac{1}{2\pi i}\int_{C_{R'}} \frac{f(t)dt}{t-z}, \quad z \text{ interior to } C_{R'},$$

$$f_2(z) = \frac{1}{2\pi i}\int_{D_{R'}} \frac{f(t)dt}{t-z}, \quad z \text{ exterior to } D_{R'},$$

defines functions analytic in the closed regions interior to $C_{R'}$ and exterior to $D_{R'}$ respectively, and if the integrals are taken in the positive (counterclockwise) senses on those curves, we have

$$F(z) = f_1(z) - f_2(z).$$

* This follows easily from a comparison of the Green's functions for the two regions. Compare also Carathéodory, Mathematische Annalen, vol. 72 (1912), pp. 107–144, especially pp. 126–127.

Theorem I applied directly to $f_1(z)$, and to $f_2(z)$ after reciprocal transformation, yields now the polynomials $p_n(z)$ and $q_n(z)$ so that we have

$$(4)\qquad \begin{aligned} |f_1(z) - p_n(z)| &\leqq \frac{M_1}{R^n}, \quad z \text{ on or interior to } C, \\ |f_2(z) + q_n(z)| &\leqq \frac{M_2}{R^n}, \quad z \text{ on or interior to } C, \end{aligned}$$

which yields an inequality of form (2), for z on C.

Here it is not necessary to choose $q_n(\infty)=0$. That is, however, easy to accomplish if one cares to do it. For we have $f_2(\infty)=0$, so the new function $\mu(z)=z\ f_2(z)$ is analytic on and exterior to D_R. There exist polynomials $g_n(z)$ in $1/z$ such that

$$|zf_2(z) + g_n(z)| \leqq \frac{M_2'}{R^n}, \quad z \text{ on or exterior to } C.$$

We may now set $q_{n+1}(z)=g_n(z)/z$, a polynomial in $1/z$, and we have the equivalent of (4) because $|z|$ has on C a lower bound different from zero.

Theorem II is particularly simple when the curve C is itself the unit circle. In this case the expansion when $F(z)$ is given may be taken as a Fourier or Laurent series. This case has already been studied by de la Valleé Poussin* in much greater detail than we consider here. His results are of course more precise than Theorem II; those results can be used, in fact, to secure more precise results in the present situation, if C is analytic.

A less specific theorem than Theorem II, but more general in some respects, can be proved in a similar way. The reasoning we give includes in fact a proof of the sufficiency of the conditions of Theorems I and II.

THEOREM III. *Let C be an arbitrary rectifiable Jordan curve and let the function $f(z)$ be defined on C. If there exist rational functions $r_n(z)$ of degree $2n$, $n=0, 1, 2, \cdots$, so that we have for all sufficiently large n*

$$(5)\qquad |f(z) - r_n(z)| \leqq \frac{M}{R^n}, \quad R > 1, \text{ for all } z \text{ on } C,$$

and if the poles of the functions $r_n(z)-r_{n-1}(z)$ have no limit point on C, then the function $f(z)$ is meromorphic on C. If the functions $r_n(z)$ have no poles on C, then the function $f(z)$ is regular-analytic on C.

In determining whether a point is a limit point of the poles of the functions

* *Approximation des Fonctions*, Paris, 1919, Chap. VIII.

$r_n(z)-r_{n-1}(z)$, we count each pole according to its multiplicity in the totality of these rational functions.

We proceed with integration as in the proof of Theorem II. Under the present hypothesis the function $f(z)-r_n(z)$ is continuous on C for sufficiently large n, say for $n \geqq N$. We set

$$F(z) = f(z) - r_N(z), \quad r_n'(z) = r_n(z) - r_N(z), \quad n \geqq N,$$

and deduce

$$F(z) - r_n'(z) = f(z) - r_n(z), \quad n \geqq N.$$

Then we have for z on a curve C' interior to C,

$$| f_1(z) - p_n(z) | \leqq \frac{M'}{R^n}, \quad \text{where } f_1(z) = \frac{1}{2\pi i}\int_C \frac{F(t)dt}{t-z}, \tag{6}$$

$$p_n(z) = \frac{1}{2\pi i}\int_C \frac{r_n'(t)dt}{t-z}.$$

We have similarly for z on a curve C'' exterior to C,

$$| f_2(z) - q_n(z) | \leqq \frac{M''}{R^n},$$

where

$$f_2(z) = \frac{1}{2\pi i}\int_C \frac{F(t)dt}{t-z}, \quad q_n(z) = \frac{1}{2\pi i}\int_C \frac{r_n'(t)dt}{t-z},$$

the integrals taken in the positive sense on C. It follows of course that $q_n(\infty)=0$, and also that $r_n'(z)=p_n(z)-q_n(z)$ for z on C, if $p_n(z)$ and $q_n(z)$ are defined on C by analytic extension. For instance the integral defining $p_n(z)$ can be taken over C or over a larger Jordan curve in whose interior C lies, without changing the value of the integral.

The following reasoning is an extension of the reasoning used by Marcel Riesz* in connection with Bernstein's special case of Theorem I. Let us establish a lemma for use in studying the convergence of the sequence $p_n(z)$.

LEMMA. *Let Γ be an arbitrary Jordan curve of the z-plane, and denote by $w=\Phi(z)$, $z=\Psi(w)$, a function which maps the exterior of Γ onto the exterior of the unit circle γ in the w-plane so that the two points at infinity correspond. Let Γ_R denote the curve $|\Phi(z)|=R$, in the z-plane, for $R>1$. If $P(z)$ is a rational function of degree n whose poles lie exterior to ρ, $\rho>1$, and if we have*

$$| P(z) | \leqq L \text{ for all } z \text{ on } \Gamma,$$

then we have likewise

$$| P(z) | \leqq L\left(\frac{\rho R_1 - 1}{\rho - R_1}\right)^n, \text{ for all } z \text{ on } \Gamma_{R_1},\ R_1 < \rho.$$

* Acta Mathematatica, vol. 40 (1916), pp. 337–347.

The function $P[\Psi(w)]$ has at most n poles for $|w|\geqq 1$, and these all lie exterior to $|w|=\rho$. For convenience in exposition we shall suppose that there are precisely n poles $\alpha_1, \alpha_2, \cdots, \alpha_n$, not necessarily all distinct, and that none lies at infinity. If there are less than n poles, or if infinity is also a pole, there are only obvious modifications to be made in the discussion. In the latter case, for instance, we consider in the right-hand member of (7) the function found by taking the limit as one or more of the α_i become infinite. The function

$$\pi(w) = P[\Psi(w)]\frac{w-\alpha_1}{1-\bar{\alpha}_1 w}\cdot\frac{w-\alpha_2}{1-\bar{\alpha}_2 w}\cdots\frac{w-\alpha_n}{1-\bar{\alpha}_n w} \tag{7}$$

is regular for $|w|>1$ and continuous for $|w|\geqq 1$. The function

$$\frac{w-\alpha_i}{1-\bar{\alpha}_i w}$$

has the absolute value unity for $|w|=1$, so we have

$$|\pi(w)| \leqq L, \text{ for all } w \text{ on } \gamma\ ; \tag{8}$$

since (8) holds for w on γ it also holds throughout $|w|\geqq 1$. The transformation $\zeta=(w-\alpha_i)/(1-\bar{\alpha}_i w)$ transforms $|w|=R_1$ into the circle $|(\zeta+\alpha_i)/(1+\bar{\alpha}_i\zeta)|=R_1$, so we have

$$|\zeta| \geqq \frac{|\alpha_i| - R_1}{R_1|\alpha_i| - 1} \geqq \frac{\rho - R_1}{R_1\rho - 1}, \quad \text{for } |w| = R_1 < \rho.$$

Thus we find from (7) and (8),

$$|P[\Psi(w)]| \leqq L\prod_{i=1}^{n}\left|\frac{1-\bar{\alpha}_i w}{w-\alpha_i}\right| \leqq L\left(\frac{R_1\rho - 1}{\rho - R_1}\right)^n$$

for $|w|=R_1<\rho$, and the Lemma is established.

We return to the proof of Theorem III. From (6) we conclude

$$|f_1(z) - p_{n-1}(z)| \leqq \frac{M'}{R^{n-1}}, \quad |p_n(z) - p_{n-1}(z)| \leqq M'\frac{1+R}{R^n},$$

for all z on C'. If the poles of $p_n(z) - p_{n-1}(z)$ lie exterior to C_ρ, $\rho>1$ (and here we assume the fact not necessarily for all n but merely for all n sufficiently large), these poles lie likewise exterior to C'_ρ, the transform of $|w|=\rho$ when the exterior of C' is mapped on the exterior of γ so that the points at infinity correspond to each other. There are at most $4n-2$ of such poles, so we conclude from the Lemma

$$(9) \qquad |p_n(z) - p_{n-1}(z)| \leqq M'\frac{1+R}{R^n}\left(\frac{R_1\rho - 1}{\rho - R_1}\right)^{4n-2}$$

for z on the curve C'_{R_1}, $R_1<\rho$. We use here the fact that $p_n(z)$ is defined by Cauchy's integral, as in (6), and the integral may be taken not merely over C but over any curve interior to C_ρ, so $p_n(z)-p_{n-1}(z)$ has no poles other than the poles of $r_n(z)-r_{n-1}(z)$ exterior to C_ρ. Thus the sequence $p_n(z)$ converges for z within the curve C'_ν, hence, as we see by allowing C' to approach C monotonically, for z within the curve C_ν, where $\nu=(1+\rho R^{1/4})/(\rho+R^{1/4})$, which is greater than unity. The convergence is uniform on any closed point set interior to C_ν.

If we know that $r_n(z)$ has at most n poles exterior to C, at least for sufficiently large n, we may replace the exponent $4n-2$ in (9) by $2n-1$, and take correspondingly $\nu=(1+\rho R^{1/2})/(\rho+R^{1/2})$. If these n poles of $r_n(z)$ exterior to C are coincident and do not change position with n, or more generally if we know that $p_n(z)-p_{n-1}(z)$ has at most n poles, we may replace the exponent in $4n-2$ in (9) by n, and thus set $\nu=(1+\rho R)/(\rho+R)$.

The convergence of the sequence $q_n(z)$ exterior to some curve $D_{\nu'}$ can be proved similarly. It follows that the original sequence $r_n(z)$ converges uniformly in an annular region in whose interior C lies, so $f(z)$ is meromorphic on C and is regular on C if the $r_n(z)$ have no poles on C.

We have really proved here a much more precise theorem than Theorem III. We do not state the most general theorem possible, but merely note that Theorem V (below) is also valid in the limiting case that the region C (of Theorem V) becomes a rectifiable Jordan curve.

Theorem III remains true if C is no longer a Jordan curve, but has a finite number of double points. For in Theorem III we have shown the original sequence of rational functions to converge uniformly not merely on C but in an annular region in whose interior C is contained. In the modified situation it is still true that the curve C lies interior to a region of uniform convergence of the original sequence $r_n(z)$.

Theorem III is not true if we omit the restriction that the poles of the $r_n(z)$ should have no limit point on C. We illustrate this fact by a simple example. Let C be an arbitrary Jordan curve. If we move a line not intersecting C parallel to itself until it just touches C, we have a point A of C on this line L, yet the entire interior of the curve C is in a half-plane bounded by L. Choose coördinate axes so that A is the origin and L is the axis of imaginaries, the interior of C to the left of L. The series

$$(10) \qquad \sum_{m=0}^{\infty} \frac{1}{4^m(z - 1/2^m)}$$

converges uniformly for all values of z on or to the left of the axis of imaginaries, for we have, for such points,

$$\left| z - \frac{1}{2^m} \right| \geqq \frac{1}{2^m}, \quad \frac{1}{|4^m(z - 1/2^m)|} \leqq \frac{1}{2^m}.$$

Thus (10) converges uniformly on C. Successive partial sums of series (10),

$$r_n(z) = \sum_{m=1}^{n} \frac{1}{4^m(z - 1/2^m)},$$

yield, for z on C, an inequality of type (5). The sum of series (10) is, however, not analytic at the origin and hence not regular everywhere on C. For as is well known, and indeed follows from Theorem III, series (10) represents a function regular over the entire plane, except that there are poles at the points $z = 1/2^m$ and hence there is an essential singularity at the origin.

Theorem I can be extended so that J is not a Jordan region, but a Jordan arc or indeed any closed bounded point set S whose complement with respect to the entire plane is simply connected. We can find a similar result for the approximation on S of functions by means of rational functions. The only necessary modification in the proof already given for Theorem III is to notice that although $\Psi(w)$ is not necessarily continuous for $|w| \geqq 1$, its absolute value is continuous. Let us state

THEOREM IV. *Let S be an arbitrary closed bounded point set which consists of more than a single point and whose complement with respect to the entire plane is simply connected. If there exist rational functions $r_n(z)$ of degree n, $n = 0, 1, \cdots$, such that we have*

$$|f(z) - r_n(z)| \leqq M/R^n, \quad R > 1,$$

for all z on S and for all sufficiently large n, and if the poles of the functions $r_n(z) - r_{n-1}(z)$ have no limit point on S, then $f(z)$ is meromorphic on S. If the functions $r_n(z)$ have no poles on S, then $f(z)$ is regular on S.

Let $w = \Phi(z)$ denote a function that maps the complement of S onto the exterior of the unit circle in the w-plane, so that the points at infinity correspond to each other. Let S_R denote the curve $|\Phi(z)| = R$, $R > 1$. If the poles of the functions $r_n(z) - r_{n-1}(z)$ have no limit point interior to S_ρ, then the sequence $r_n(z)$ converges interior to S_ν, where $\nu = (1 + \rho R^{1/2})/(\rho + R^{1/2})$, and the convergence is uniform on any closed point set interior to S_ν. Hence $f(z)$ is meromorphic in S_ν, and if $r_n(z)$ has no poles interior to S_ν, $f(z)$ is regular interior to S_ν. If the function $r_n(z) - r_{n-1}(z)$ has at most n poles, for n sufficiently large, we may set $\nu = (1 + \rho R)/(\rho + R)$, and in particular if the only poles of $r_n(z)$ are at infinity, we may set $\rho = \infty$ and $\nu = R$, so that the sequence $r_n(z)$ converges interior to S_R and $f(z)$ is regular interior to S_R.

Theorem IV yields almost at once

THEOREM V. *Let C be a closed region bounded by Jordan curves $C_0, C_1, \cdots, C_{k-1}$, such that no two of these curves have a common point and so that $C_1, C_2, \cdots, C_{k-1}$ lie interior to C_0. A necessary and sufficient condition that a function $f(z)$ be regular-analytic in the (closed) region C is that there should exist rational functions $r_n(z)$ of degree kn, $n=0, 1, 2, \cdots$, with no poles or limit point of poles on C, such that we have*

$$|f(z) - r_n(z)| \leqq M/R^n, \quad R > 1, \text{ for all } z \text{ on } C. \tag{11}$$

A necessary and sufficient condition that a function $f(z)$ be meromorphic in the region C is that there should exist rational functions $r_n(z)$ of degree kn, $n=0, 1, 2, \cdots$, with no limit point of poles of the functions $r_n(z)-r_{n-1}(z)$ on C, such that (11) *obtains for sufficiently large n.*

Denote by $z_1, z_2, \cdots, z_{k-1}$ arbitrary but fixed points interior to $C_1, C_2, \cdots, C_{k-1}$ respectively, and denote by $w=\phi_i(z)$ a function which maps conformally the interior of C_i on the interior of the unit circle on the w-plane so that $z=z_i$ corresponds to $w=0$. Denote by $w=\phi_0(z)$ a function which maps the exterior of C_0 on the exterior of the unit circle in the w-plane so that the points at infinity correspond. Let $\Gamma_R^{(0)}$ denote the curve $|\phi_0(z)|=R>1$, and $\Gamma_R^{(i)}$ the curve $|\phi_i(z)|=1/R<1$, $i=1, 2, \cdots, k-1$.

If the rational functions $r_n(z)$ are given so that (11) *is satisfied for all z on C for n sufficiently large, and if the poles of the $r_n(z)-r_{n-1}(z)$ have no limit point exterior to $\Gamma_\rho^{(i)}$ and interior to $\Gamma_\rho^{(0)}$, then the sequence $r_n(z)$ converges in the region Γ exterior to $\Gamma_\nu^{(i)}$ and interior to $\Gamma_\nu^{(0)}$, where $\nu=(1+\rho R^{1/(2k)})/(\rho+R^{1/(2k)})$, and the convergence is uniform on any closed point set interior to this region. Hence $f(z)$ is meromorphic in Γ, and if $r_n(z)$ has no poles exterior to $\Gamma_\nu^{(i)}$ and interior to $\Gamma_\nu^{(0)}$, $f(z)$ is regular in Γ. If $r_n(z)$ has not more than n poles interior to $\Gamma_\rho^{(i)}$ and not more than n poles exterior to $\Gamma_\rho^{(0)}$, then we may set $\nu=(1+\rho R^{1/2})/(\rho+R^{1/2})$. If $r_n(z)-r_{n-1}(z)$ has not more than n poles interior to $\Gamma_\rho^{(i)}$ and not more than n poles exterior to $\Gamma_\rho^{(0)}$, then we may set $\nu=(1+\rho R)/(\rho+R)$. In particular if the only poles of $r_n(z)$ are at $z_1, z_2, \cdots, z_{k-1}$ and at infinity, not more than n in each point, we may set $\rho=\infty$, and $\nu=R$, so that the sequence $r_n(z)$ converges in the region Γ interior to $\Gamma_R^{(0)}$ and exterior to $\Gamma_R^{(i)}$; the function $f(z)$ is regular in Γ. In any case, for n sufficiently large the function $r_n(z)-r_{n-1}(z)$ can be expressed as the sum of k functions each of the form $p_n^{(i)}(z)-p_{n-1}^{(i)}(z)$, $i=0, 1, \cdots, k-1$, which has poles only interior to C_i (or for $i=0$, only exterior to C_0), and such that the sequence $p_n^{(i)}(z)$ converges everywhere exterior to $\Gamma_\nu^{(i)}$ (or for $i=0$, everywhere interior to $\Gamma_\nu^{(0)}$).*

If $f(z)$ is given regular in the closed region Γ interior to $\Gamma_\nu^{(0)}$ and exterior to

$\Gamma_\nu^{(i)}$, *the functions* $r_n(z)$ *may be found with poles only in the points* z_1, $z_2, \cdots, z_{k-1}$ *and at infinity so that* (11) *is satisfied for all* n *for all* z *in* C, *with* $R=\nu$. *If* $f(z)$ *is given meromorphic in this closed region* Γ, *the function* $r_n(z)$ *may be found with poles only in the poles of* $f(z)$ *in this region, in* $z_1, z_2, \cdots, z_{k-1}$ *and at infinity, so that* (11) *is satisfied for sufficiently large* n *for* z *in* C *and* $R=\nu$.

In Theorem V it is not essential to suppose the bounding curves C_0, $C_1, \cdots, C_{k-1}$ rectifiable, for auxiliary rectifiable curves $C_0', C_1', \cdots, C_{k-1}'$ near to $C_0, C_1, \cdots, C_{k-1}$ respectively can be constructed interior to C. The theorem is true for the region bounded by these auxiliary curves. If these curves are allowed to approach $C_0, C_1, \cdots, C_{k-1}$ monotonically, we obtain Theorem V as stated. It is likewise possible to consider boundaries more general than Jordan curves, by this same limiting process.

Perhaps the necessity of the condition of Theorem V should be briefly discussed. If $f(z)$ is regular in Γ, Cauchy's integral for $f(z)$ for z in Γ automatically expresses $f(z)$ as the sum of k functions each analytic in Γ and which are respectively regular everywhere interior to $\Gamma_\nu^{(0)}$ and exterior to $\Gamma_\nu^{(i)}$. Theorem I yields the result desired, if we use the transformation $z'=1/(z-z_i)$ for the function regular exterior to $\Gamma_\nu^{(i)}$.

If $f(z)$ is meromorphic in the closed region Γ, we can write $f(z)=F(z)+r(z)$, where $F(z)$ is regular in Γ and $r(z)$ is rational, let us say of degree p. There exist rational functions $r_m'(z)$ of degree km such that we have

$$|F(z) - r_m'(z)| \leqq \frac{M'}{\nu^m} \text{ for all } m \text{ and for all } z \text{ on } C.$$

That is, we have

$$|f(z) - r_{m+p}(z)| \leqq \frac{M'}{\nu^m} \text{ where } r_{m+p}(z) = r_m'(z) + r(z).$$

We can consider $r_{m+p}(z)$ of degree $k(m+p)$, and write this last inequality

$$|f(z) - r_{m+p}(z)| \leqq \frac{M'}{\nu^m} = \frac{M}{\nu^{m+p}}, \quad \text{where } M = M'\nu^p.$$

This inequality holds for all z on C for sufficiently large index $n=m+p$ and is of type (11).

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