

EXISTENCE AND OSCILLATION THEOREMS FOR NON-LINEAR DIFFERENTIAL SYSTEMS OF THE SECOND ORDER*

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1. INTRODUCTION

Boundary value problems for second order linear differential systems have been investigated extensively by many authors of the past century. It is the object of the present paper to consider non-linear equations of the second order and to establish existence and oscillation theorems for these equations together with boundary conditions of the two-point type. The coefficients of the differential equations are taken as summable functions of the independent variable and all integrals in the paper are taken in the sense of Lebesgue. The quantities used are confined to the real domain.

2. EXISTENCE THEOREMS

The following two theorems are special cases of general existence theorems that have recently been established by the author.†

THEOREM I. *Given the differential system*

$$(1) \quad dy_i/dx = \sum_{j=1}^{j=n} A_{ij}(x, y_1, \dots, y_n)y_j$$

$$(2) \quad y_i(c) = a_i \quad (i = 1, 2, \dots, n),$$

where the functions A_{ij} are single-valued and continuous in (y_1, \dots, y_n) for each fixed x on $X: a \leq x \leq b$, summable (Lebesgue) in x on X for fixed (y_1, \dots, y_n) , and bounded numerically for all x on X and for all (y_1, \dots, y_n) by a function $M(x)$ that is summable on X , and where $x=c$ is any fixed point of X and the quantities a_1, \dots, a_n are constants, then there exists a unique solution‡ of system (1), (2), provided that there exists a summable function $N(x)$ on X such that

* Presented to the Society, September 7, 1927; received by the editors in August, 1927.

† W. M. Whyburn, *On the fundamental existence theorems for differential systems*, to appear in *Annals of Mathematics* for December, 1928.

‡ By a solution of this system we mean a set of absolutely continuous functions on X that satisfies (2) and satisfies (1) almost everywhere on X .

$$|A_{ij}(x, y_1, \dots, y_n) - A_{ij}(x, z_1, \dots, z_n)| \leq N(x) \sum_{r=1}^{r=n} |y_r - z_r|$$

(i, j = 1, \dots, n)

for all (y_1, \dots, y_n) and (z_1, \dots, z_n) and for every x on X .

THEOREM II. For the system

$$(3) \quad dy_i/dx = \sum_{j=1}^{j=n} A_{ij}(x, y_1, \dots, y_n; \lambda) y_j \quad (i = 1, \dots, n),$$

$$(4) \quad y_i(c, \lambda) = a_i(\lambda)$$

where a_1, \dots, a_n are continuous functions of λ on \bar{L} : $\bar{L}_1 \leq \lambda \leq \bar{L}_2$; the functions A_{ij} are summable in x on X for each fixed (y_1, \dots, y_n) and each fixed λ on \bar{L} , continuous in $(y_1, \dots, y_n, \lambda)$ for all (y_1, \dots, y_n) and all λ on \bar{L} when x has a fixed value on X , bounded numerically by a function $M(x)$ that is summable on X , and finally there is a summable function $N(x)$ on X such that for all λ on \bar{L} and for every (y_1, \dots, y_n) and (z_1, \dots, z_n)

$$|A_{ij}(x, y_1, \dots, y_n; \lambda) - A_{ij}(x, z_1, \dots, z_n; \lambda)| \leq N(x) \sum_{r=1}^{r=n} |y_r - z_r|$$

(i, j = 1, \dots, n),

there exists on $X\bar{L}$ a unique set $y_1(x, \lambda), \dots, y_n(x, \lambda)$ of continuous* functions such that for each fixed λ on \bar{L} these functions are absolutely continuous in x on X and satisfy (3) almost everywhere on X , and such that the set of functions satisfies (4) everywhere on \bar{L} .

COROLLARY. If the hypotheses of Theorem II are met for every closed sub-interval of the open λ -interval L : $L_1 < \lambda < L_2$, then the conclusions of that theorem are valid for the domain XL .

3. THE SECOND-ORDER SYSTEM

Consider the system

$$(5) \quad dy/dx = K(x, y, z; \lambda)z, \quad dz/dx = G(x, y, z; \lambda)y,$$

where K and G satisfy the hypotheses of Theorem II and its corollary. Let $y(x, \lambda), z(x, \lambda)$ denote the unique solution of (5) that satisfies the conditions $y(a, \lambda) = 0, z(a, \lambda) = 1$, for all values of λ on L .

To system (5) apply the transformation †

* Cf. Carathéodory, *Vorlesungen über reelle Funktionen*, Berlin, 1918, p. 678.

† The use of this transformation in the study of boundary value problems was suggested to me by Professor H. J. Ettlenger.

$$\begin{aligned}y(x, \lambda) &= u(x, \lambda) \sin v(x, \lambda), \\z(x, \lambda) &= u(x, \lambda) \cos v(x, \lambda).\end{aligned}$$

This yields

$$(6) \quad \begin{aligned}dv/dx &= K \cos^2 v - G \sin^2 v, \\du/dx &= \frac{u[K + G]}{2} \sin 2v.\end{aligned}$$

Carathéodory's existence theorem* yields at least one solution of (6) that satisfies $u(c, \lambda) = a_1$, $v(c, \lambda) = a_2$, where $x = c$ is a point of X and a_1 and a_2 are continuous functions of λ on L . Suppose that for some fixed λ on L there is more than one solution of (6) and these initial conditions. Let u_1, v_1 and u_2, v_2 be two such solutions and suppose for definiteness that they are not identical on C : $c \leq x \leq b$. Obviously the case where these solutions are identical on C but differ on $a \leq x \leq c$ can be obtained from the present case by modifying the notation slightly and inserting additional absolute value signs in the inequalities. Let H be a constant that is greater than any of the continuous functions 1, $|u_1|$, $|v_1|$, $|u_2|$, $|v_2|$, on X . Let $K_1 = K(x, u_1 \sin v_1, u_1 \cos v_1)$, $G_1 = G(x, u_1 \sin v_1, u_1 \cos v_1)$, $K_2 = K(x, u_2 \sin v_2, u_2 \cos v_2)$, and $G_2 = G(x, u_2 \sin v_2, u_2 \cos v_2)$. From (6) we have

$$\begin{aligned}v_1 - v_2 &= \int_c^x [K_1 \cos^2 v_1 - K_2 \cos^2 v_2 - (G_1 \sin^2 v_1 - G_2 \sin^2 v_2)] dt, \\|v_1 - v_2| &\leq \int_c^x [|K_1 - K_2| + |G_1 - G_2| \\&\quad + (|K_2| + |G_2|)(|\sin^2 v_1 - \sin^2 v_2|)] dt \\&\leq 2 \int_c^x (N[|u_1 \sin v_1 - u_2 \sin v_2| + |u_1 \cos v_1 - u_2 \cos v_2|] \\&\quad + 2M|\sin v_1 - \sin v_2|) dt \\&< 4 \int_c^x \left[N|u_1 - u_2| + 2(NH + M) \left| \sin \frac{(v_1 - v_2)}{2} \right| \right] dt \\&< 4 \int_c^x H(NH + M)(|u_1 - u_2| + |v_1 - v_2|) dt,\end{aligned}$$

since $|\sin [(v_1 - v_2)/2]| \leq |(v_1 - v_2)/2|$ and H is greater than unity. Similarly

$$|u_1 - u_2| < 4 \int_c^x H(NH + M)(|u_1 - u_2| + |v_1 - v_2|) dt.$$

The foregoing inequalities are valid for each x on C .

* Loc. cit., p. 672 ff. This theorem does not apply to give uniqueness.

Let $v(x) = |u_1 - u_2| + |v_1 - v_2|$ and add the above inequalities to get

$$(7) \quad v(x) < 8 \int_c^x H[HN(t) + M(t)]v(t)dt,$$

$$v(x) < 8 \int_d^x H[HN(t) + M(t)]v(t)dt,$$

where $x=d$ is the last point of C such that $v(x) \equiv 0$ on $c \leq x \leq d$. Let x_1 be any point of $d < x \leq b$ and let x_2 be a point of $d \leq x \leq x_1$ at which the continuous function $v(x)$ takes on its maximum value for that interval. From (7)

$$v(x_2) < v(x_2) \int_d^{x_2} 8H[HN + M]dt \leq v(x_2) \int_d^{x_1} 8H[HN + M]dt$$

and since $v(x_2) > 0$, we may divide by it to get

$$\int_d^{x_1} 8H[HN(t) + M(t)]dt > 1.$$

This inequality must hold for every x_1 on $d < x_1 \leq b$ which evidently is impossible. Hence we have established the following theorem:

THEOREM III. *There exists a unique pair of functions $u(x, \lambda)$, $v(x, \lambda)$ each of which is continuous in (x, λ) on XL , absolutely continuous in x on X for each fixed λ on L , and such that the pair satisfies the conditions $u(c, \lambda) = a_1(\lambda)$, $v(c, \lambda) = a_2(\lambda)$ for every λ on L and satisfies system (6) almost everywhere on X for each fixed λ on L .*

THEOREM IV. *If $u(x, \lambda)$, $v(x, \lambda)$ is the solution of (6) such that $u(a, \lambda) \equiv 1$, $v(a, \lambda) \equiv 0$, then $u(x, \lambda)$ does not vanish on XL .*

Assume that $u(c, p) = 0$. By Theorem III there exists a unique solution U, V of (6) on $a \leq x \leq c$ satisfying the conditions $U(c, p) = 0$, $V(c, p) = v(c, p)$. The method of successive approximations yields a solution of this system for which $u(x, p) \equiv 0$ on $a \leq x \leq c$ and hence this solution must be identical with the one that we are considering. This, however, contradicts the fact that $u(a, p) = 1$ and yields our theorem.

In the discussion which follows we will assume that $K(x, y, z; \lambda)$ and $-G(x, y, z; \lambda)$ are positive for all values of x and λ on XL and for all values of y and z . It follows immediately from (6) that for each fixed value of λ on L the function v is an increasing function of x on X . This gives a one-to-one continuous correspondence between the points of $v(a) \leq v \leq v(b)$ and the points of X . Furthermore, this correspondence preserves order so that if x_1 and x_2 are any two points of X such that $x_1 < x_2$, then $v(x_1) < v(x_2)$. Since

$u(x, \lambda) > 0$ on XL the zeros of y coincide with the zeros of $\sin v$ and the zeros of z coincide with those of $\cos v$. If for a fixed λ on L , $(2m-1)\pi/2 \leq v(b) \leq m\pi$, where m is a positive integer, then y will have zeros for $v(x) = i\pi$ while z has zeros for $v(x) = (2i+1)\pi/2$, where $i=1, 2, \dots, m-1$. It follows immediately from the correspondence between v and x that the zeros of y and z separate each other on X .

CONDITION A. *There exists a value $\lambda = p$ on L such that if q is any fixed value of λ on $L_1 < \lambda \leq p$ and M and \bar{M} , respectively, denote the upper bounds of K and $-G$ for this value of λ , then $M + \bar{M} < \pi/(b-a)$.*

CONDITION B. *There exists a subinterval $D: c \leq x \leq d$ of X such that if $m(\lambda)$ denotes the smaller of the lower bounds of K and $-G$ on D for each fixed λ , then $\lim_{\lambda \rightarrow L_2} m(\lambda) = +\infty$.*

Condition A insures that for every λ on $L_1 < \lambda \leq p$, $v(b, \lambda) < \pi$ and consequently that $y(x, \lambda)$ does not vanish on $a < x \leq b$ for fixed values of λ on $L_1 < \lambda \leq p$. This follows since $v(b, \lambda) = \int_a^b (K \cos^2 v - G \sin^2 v) dt \leq \int_a^b [M + \bar{M}] dt < (b-a)\pi/(b-a)$.

Condition B insures that $\lim_{\lambda \rightarrow L_2} v(b, \lambda) = +\infty$, since $v(b, \lambda) \geq \int_c^d (K \cos^2 v - G \sin^2 v) dt > m(\lambda) (d-c)$ and $\lim_{\lambda \rightarrow L_2} m(\lambda) = +\infty$.

THEOREM V. *If K and $-G$ are positive on XL and for every y and z and if Conditions A and B are satisfied, then there exists an infinite collection, k_1, k_2, \dots , of sets of characteristic numbers* for the system (5), (8) such that if p_i is any number in k_i , then $y(x, p_i)$ and $z(x, p_i)$ each have exactly i zeros on $a < x \leq b$, and finally for each index i there exists an index j such that for every n greater than j , every number in k_n is greater than every number in k_i . System (8) is*

$$(8) \quad y(a, \lambda) = 0, \quad y(b, \lambda) = 0.$$

As λ increases from a value between L_1 and p to L_2 , $v(b, \lambda)$ increases from a value less than π to $+\infty$. If i is any positive integer, it follows from the continuity of $v(b, \lambda)$ in λ that for at least one value of λ on L , $v(b, \lambda) = i\pi$. Let k_i be the totality of values of λ on L for which $v(b, \lambda) = i\pi$ and let p_i be any value in k_i ; then since $v(x, p_i)$ is an increasing, continuous function of x on X , it follows that for one and only one value of x on X , $v(x, p_i) = j\pi$, $j=1, 2, \dots, i$. Also, since $\lim_{\lambda \rightarrow L_2} v(b, \lambda) = +\infty$, we can determine q_i such that for every λ on $q_i \leq \lambda < L_2$, $v(b, \lambda) > i\pi$ and hence no values of k_i can lie on $q_i \leq \lambda < L_2$. We can also determine j such that $v(b, \lambda) < j\pi$ on $p \leq \lambda \leq q_i$.

* By a characteristic number we mean a value of λ for which the system has a non-identically vanishing solution.

and hence no values in k_j, k_{j+1}, \dots lie on $p \leq \lambda \leq q_i$. Hence for any $n > j$, every number in k_n is greater than every number in k_i . This completes the proof of Theorem V.

A consideration of the values of λ for which $v(b, \lambda) = (2i-1)\pi/2$ yields the following theorem:

THEOREM VI. *Under the hypotheses of Theorem V there exists an infinite collection of sets, h_2, h_3, \dots , of characteristic numbers for the system (5) $y(a, \lambda) = 0, z(b, \lambda) = 0$ such that if p_i is any value in $h_i, y(x, p_i)$ and $z(x, p_i)$ each have exactly i zeros on X and such that for each i there exists an index j such that for every n greater than j , every number in h_n is greater than every number in h_i .*

By letting $y = u(x, \lambda) \cos v(x, \lambda), z = u(x, \lambda) \sin v(x, \lambda)$ and repeating the foregoing discussion, we get existence and oscillation theorems of the above type for the system (5) together with the boundary conditions $z(a, \lambda) = 0, y(b, \lambda) = 0$ and with the boundary conditions $z(a, \lambda) = 0, z(b, \lambda) = 0$.

We close this section with an example of a non-linear system to which all of the foregoing theory applies. Consider the system

$$\begin{aligned} dy/dx &= \lambda(\sin^2 y + 2)z/x^{2/3}, & X: 1 \leq x \leq b, \\ dz/dx &= -\lambda(2 \cos^2 z + 3)y/x^{4/5}, & L: 0 < \lambda < +\infty, \end{aligned}$$

together with boundary conditions of any of the four types considered in this section. One easily sees that all of the hypotheses of the foregoing theorems are met by this system and hence Theorems II, III, IV, V, VI are valid. Many other examples can be constructed to illustrate the systems covered by this treatment.

4. THE LINEAR SYSTEM

The case where K and G do not depend on y and z is of especial interest in that it gives a new treatment of boundary value problems of the Sturmian type.* The linear case appears as a special case of all of the foregoing treatment and many of the hypotheses are automatically satisfied for this case. It is noted that in the present treatment all of the conditions on K and $-G$ are symmetric, which is to be expected when one notes the manner in which these coefficients enter into the system. Examples can be given of systems where conditions of the above type are highly desirable.†

* Cf. Bôcher, *Leçons sur les Méthodes de Sturm*, Paris, 1917, p. 66ff.

† Such an example arises in the minimum surface of revolution problem of the calculus of variations. Conditions for non-oscillation are desired for the system $y' = (1+\lambda^2)^{1/2} \cosh [x(1+\lambda^2)^{1/2} + \sinh^{-1}\lambda]z, z' = -(1+\lambda^2)^{1/2} \operatorname{sech} [x(1+\lambda^2)^{1/2} + \sinh^{-1}\lambda]y$. The usual conditions of K and G being positive cannot be satisfied for real values of x and λ while Condition A of the above theory can be applied to obtain the desired information.

THEOREM VII. *If for each fixed x_1 on $a < x \leq b$, K and $-G$ are non-decreasing functions of λ on L and there exists a sub-interval $D(x_1)$ of $a < x \leq x_1$ almost everywhere on which either δK or $-\delta G$ is positive, where δf is the increment in f due to a positive increment $\delta\lambda$ in λ , then $v(x_1, \lambda)$ is an increasing function of λ on L .*

Choose $\delta\lambda$ so that $\lambda + \delta\lambda$ lies on L . From (6)

$$d(\delta v)/dx = [\delta K \cos^2(v + \delta v) - \delta G \sin^2(v + \delta v)] - (K + G) \sin(2v + \delta v) \sin \delta v.$$

Using the continuity of v in (x, λ) and restricting the size of $\delta\lambda$, we may solve this equation to get

$$\delta v(x_1, \lambda) = e^{-\omega} \left\{ \int_a^{x_1} [\delta K \cos^2(v + \delta v) - \delta G \sin^2(v + \delta v)] e^{\tau} dt \right\}$$

where

$$\omega = \int_a^{x_1} \{ [K + G](\sin(2v + \delta v))(\sin \delta v)/\delta v \} dt,$$

$$\tau = \int_a^t \{ [K + G](\sin(2v + \delta v))(\sin \delta v)/\delta v \} ds,$$

from which we conclude immediately that $\delta v(x_1, \lambda) > 0$.

Under the hypotheses of Theorem VII there exists a one-to-one, order-preserving correspondence between the points of the v -interval and those of L for each fixed x on $a < x \leq b$. This correspondence enables us to state that k^s and h_i of Theorems V and VI contain one and only one value of λ and that these values are ordered in the following fashion:

$$L_1 < k_1 < h_2 < k_2 < h_3 < \cdots < h_i < k_i < \cdots < L_2.$$

A transformation that was used by the author in another paper* can be used to show that $\phi(x, \lambda) = \alpha(x, \lambda)y - \beta(x, \lambda)z$ and $\psi(x, \lambda) = \gamma(x, \lambda)y - \delta(x, \lambda)z$, where $\alpha, \beta, \delta, \gamma$ have simple properties as functions of x and λ , satisfy equations of the type of (5) (K and G are linear). Hence the foregoing treatment can be used to prove existence and oscillation theorems where the boundary conditions are $\phi(a, \lambda) = \phi(b, \lambda) = 0$.

* These Transactions, vol. 30, pp. 630-640.