A GENERAL PROBLEM OF MINIMIZING AN INTEGRAL WITH DISCONTINUOUS INTEGRAND*

BY

C. F. ROOS†

1. **Introduction.** There are several well known minimum problems which do not readily yield to the classical analysis of the calculus of variations. In some of these, such as the brachistochrone problem of determining the curve of quickest descent from a fixed curve to a fixed point, the integrand of the integral to be minimized is a function of an end value which is variable.‡ In others, such as that of finding a point π and a curve through this point and two fixed points such that the area of the surface of revolution formed by rotating this curve and the ordinate of π about the OX axis shall have as small a value as possible, the expression to be minimized is the sum of a point function and a definite integral.§ A more general theory which includes this latter problem as a special case has been given by E. H. Clarke.‖ In order to develop a general dynamical theory of depreciation it is desirable to consider a very general problem which somewhat resembles each of the above types, but which does not yield to the analysis of either.¶

The depreciation problem as well as the others referred to above can all be considered as special cases of a very general problem with discontinuous integrand, which also includes the most general problems of both the Lagrange and Mayer type with general boundary conditions.** In this paper it is proposed to develop a theory for minimizing an integral whose integrand is a discontinuous function sufficiently general in nature to include all of the the above mentioned types of problems and many others.††

The expression to be minimized is the sum of two integrals taken from x_1 to a variable limit ω, and from ω to x_2, respectively. It may also be regarded as a single integral with an integrand which is discontinuous at the point ω, $y(\omega)$ of the minimizing curve. The integrand in addition to containing the usual arguments x, y and dy/dx contains also the end values x_1, $y(x_1)$, x_2, $y(x_2)$ and the corner values ω, $y(\omega)$.

In the work which follows we will find it convenient to use the notation of the tensor calculus to indicate summations. It will be understood, therefore, that when a subscript occurs twice in a term that term is to be summed for all values of the subscript.

2. **Statement of the problem.** It is desired to find among the values ω and arcs $y_i = y_i(x)$ $(i=1, \cdots, n; \; x_1 \leqq x \leqq x_2)$, satisfying m differential equations,

(1) $$\phi_\alpha(x, y, y') = 0 \qquad (\alpha = 1, \cdots, m)$$

in x and y, and having end points satisfying end equations

(2) $$\theta_\mu(\rho_1, \gamma_1, \rho_2, \gamma_2) = 0 \qquad (\mu = 1, \cdots, \rho \leqq 2n + 2),$$

one which minimizes an expression of the form

(3) $$I = \int_{x_1}^{\omega} g(x, y, y', \rho_1, \gamma_1, \rho_2, \gamma_2, \omega, y(\omega))dx$$
$$+ \int_{\omega}^{x_2} h(x, y, y', \rho_1, \gamma_1, \rho_2, \gamma_2, \omega, y(\omega))dx,$$

where for convenience in notation the set $(y_1, \cdots, y_n, y_1', \cdots, y_n')$ has been represented by (y, y'); γ_1 and γ_2 are n-partite numbers which for a particular curve $y_i(x)$ stand respectively for the sets $y_1(x_1), \cdots, y_n(x_1)$, and $y_1(x_2), \cdots, y_n(x_2)$, and, finally, primes denote derivatives with respect to x.

3. **Admissible arcs and variations.** In the analysis which follows we will need the following further hypotheses:*

(a) the functions $y_i(x)$ defining the minimizing arc E_{12} are continuous on the interval x_1, x_2, and this interval can be subdivided into a finite number of parts on each of which the functions have continuous derivatives;

(b) in a neighborhood R of the values x, y, y', ω, $y(\omega)$, x_1, $y(x_1)$, x_2, $y(x_2)$ on the minimizing arc the functions g, h and ϕ_α have continuous derivatives up to and including those of the second order;

---

* Bliss, *Lectures on the Calculus of Variations*, University of Chicago, summer quarter, 1925, mimeographed by O. E. Brown, Northwestern University, Evanston, Ill.; this will be cited as Bliss, *Lectures*.

(c) at every element (x, y, y') on E_{12} the $(m \times n)$-dimensional matrix $\partial \phi_\alpha / \partial y_i'$ has rank m, and

(d) the functions θ_μ have continuous derivatives up to and including those of the second order near the end values $(x_1, y(x_1), x_2, y(x_2))$ of the minimizing arc, and at these end values the $p \times (2n+2)$-dimensional matrix

$$\| \partial \theta_\mu / \partial x_1 \quad \partial \theta_\mu / \partial y(x_1) \quad \partial \theta_\mu / \partial x_2 \quad \partial \theta_\mu / \partial y(x_2) \|$$

has rank p.

An arc $y_i = y_i(x)$, $i = 1, \cdots, n$; $x_1 \leq x \leq x_2$, will be called an *admissible arc* if it is given by functions $y_i(x)$, defined on an interval (x_1, x_2), and by a number ω_0 such that $x_1 < \omega_0 < x_2$ and such that the $y_i'(x)$ are continuous on the intervals (x_1, ω_0) and (ω_0, x_2).

Let us suppose that we have a one-parameter family of admissible arcs given by functions

(4) $$y_i(x, b), \quad x_1(b), \quad x_2(b) \quad \text{and} \quad \omega(b)$$

defined on an interval (b_1, b_2) such that $y_i(x, b_0) = y_i(x)$, $x_1(b_0) = x_1$, $x_2(b_0) = x_2$ and $\omega(b_0) = \omega_0$, and such that the functions $y_i(x, b)$ have continuous first-order derivatives with respect to x on the intervals $[x_1(b), \omega(b)]$ and $[\omega(b), x_2(b)]$. We define the functions $\eta_i(x) = \partial y_i(x, b_0)/\partial b$, $\xi_1 = x_1'(b_0)$, $\xi_2 = x_2'(b_0)$, $\xi_3 = \omega'(b_0)$ as the *variations of the family along* E_{12}.

Let us denote by $\Omega_i(x)$ the variation $\eta_i(x)$ of $y_i(x, b)$ for the interval $x_1 \leq x \leq \omega$ and by $\pi_i(x)$ the corresponding variation of $y_i(x, b)$ for the interval $\omega \leq x \leq x_2$, and by (u, u') the set (y, y') for $x_1 \leq x \leq \omega$ and by (v, v') the corresponding set (y, y') for $\omega \leq x \leq x_2$. We do not assume that $\Omega_i(\omega) = \pi_i(\omega)$.

The *equations of variation for the functions* ϕ_α are defined by

(5) $$\psi_\alpha = (\partial \phi_\alpha / \partial y_i)\eta_i + (\partial \phi_\alpha / \partial y_i')\eta_i' = 0 \qquad (\alpha = 1, \cdots, m)$$

where it is understood that $\eta_i = \Omega_i$ for $x_1 \leq x \leq \omega$ and $\eta_i = \pi_i$ for $\omega \leq x \leq x_2$, and the coefficients $\partial \phi_\alpha / \partial y_i$, $\partial \phi_\alpha / \partial y_i'$ have as arguments the functions $y_i(x)$ defining the minimizing arc E_{12}.

The equations of variation for the functions θ_μ *are*

(6) $$\begin{aligned} [\partial \theta_\mu / \partial x_1 + (\partial \theta_\mu / \partial y_i(x_1))y_i'(x_1)]\xi_1 &+ [\partial \theta_\mu / \partial y_i(x_1)]\eta_i(x_1) \\ + [\partial \theta_\mu / \partial x_2 + (\partial \theta_\mu / \partial y_i(x_2))y_i'(x_2)]\xi_2 &+ [\partial \theta_\mu / \partial y_i(x_2)]\eta_i(x_2) = 0. \end{aligned}$$

A set of functions $\eta_i(x)$ with the continuity properties described in (a) except at $x = \omega_0$, and satisfying the above equations of variation, (5) and (6), we define as an *admissible set of variations*.

THEOREM 1. *For every set of admissible variations $\eta_i(x)$ along the arc E_{12} there exists a one-parameter family, (4), of admissible arcs containing E_{12} for the value $b=0$ and having the functions $\eta_i(x)$ as its variations along E_{12}. For this family the functions $y_i(x, b)$ are continuous and have continuous derivatives with respect to b for all values, (x, b), near those defining E_{12}, and the derivatives $y_{ix}(x, b)$ have the same property except at the values of x on the curve defined by $x=\omega(b)$, $y_i=y_i(\omega(b), b)$ and, possibly, at the values of x defining other corners of E_{12}.*

This theorem can be proved by slightly modifying a proof given by Bliss.[*] Following his example we enlarge the system $\phi_\alpha=0$ to have the form

$$(7) \qquad \phi_1 = 0, \cdots, \phi_m = 0, \quad \phi_{m+1} = z_{m+1}, \cdots, \phi_n = z_n,$$

where $z_{m+1}, \cdots, z_n$ are new variables, and $\phi_{m+1}, \cdots, \phi_n$ are new functions of (x, y, y') such that the functional determinant $|\partial\phi_i/\partial y_k'|$ is different from zero along E_{12}.[†] By means of the last $n-m$ of these equations the functions $y_i(x)$ belonging to E_{12} define a set of functions $z_r(x)$ $(r=m+1, \cdots, n)$.

We have a corresponding system of equations of variation,

$$(8) \qquad \psi_1 = 0, \cdots, \psi_m = 0, \quad \psi_{m+1} = \zeta_{m+1}, \cdots, \psi_n = \zeta_n,$$

along E_{12}. The last $n-m$ of these define a set $\zeta_r(x)$ corresponding to every set of admissible variations $\eta_i(x)$.

Since $|\partial\phi_i/\partial y_k'|\neq 0$ along E_{12}, the existence theorems for differential equations tell us that the system[‡]

$$(9) \qquad \phi_\alpha = 0, \quad \phi_r = z_r(x) + b\zeta_r(x)$$

determines uniquely a one-parameter family of solutions,

$$(10) \qquad\qquad y_i = U_i(x, b) \qquad\qquad (x_1(b) \leqq x \leqq \omega(b)),$$

with initial conditions

$$y_i(x_1) + b\eta_i(x_1) = U_i(x_1, b),$$

[*] Bliss, *Lectures*, pp. 4–6.

[†] For a proof of the possibility of this adjunction see Bliss, *The problem of Mayer with variable end points*, these Transactions, vol. 19 (1918), p. 312.

[‡] Bolza, *Vorlesungen über Variationsrechnung*, pp. 168 ff.

Bliss, Annals of Mathematics, (2), vol. 6 (1904), p. 49.

Bliss, Bulletin of the American Mathematical Society, vol. 25 (1918), p. 15.

where $x = \omega(b)$ is the value of x along a curve of corners $x = \omega(b)$, $u_i = v_i$ $= y_i(\omega(b), b)$ with $\omega(b) > x_1(b)$ and $\omega(b_0) = \omega_0$ but $\omega(b)$ otherwise arbitrary.

For the interval $(\omega(b) \leq x \leq x_2(b))$ there is a unique one-parameter family of solutions $y_i = V_i(x, b)$ having initial conditions at $x = \omega$,

$$(11) \qquad\qquad V_i(\omega(b), b) = U_i(\omega(b), b).$$

This family consisting of $U_i(x, b)$ on $(x_1 \leq x \leq \omega)$ and $V_i(x, b)$ on $(\omega \leq x \leq x_2)$ contains E_{12} for $b = 0$ and has variations which satisfy the equations (8) with the functions $\zeta_r(x)$. The variations of the family are, therefore, identical with the functions $\eta_i(x)$ originally prescribed, since when the ζ_r are given, there is only one set of solutions of equations (8) with given initial values $\eta_i(x_1)$ at $x = x_1$.

COROLLARY. *For a matrix*

$$\left\| \begin{array}{cccc} \xi_1^{(1)} & \cdots & \xi_1^{+1)} \\ \xi_2^{(1)} & \cdots & \xi_2^{(p+1)} \\ \eta_1^{(1)} & \cdots & \eta_1^{(p+1)} \\ \cdot & \cdots & \cdot \\ \eta_n^{(1)} & \cdots & \eta_n^{(p+1)} \end{array} \right\|,$$

whose columns are sets of admissible variations, there exists a $(p+1)$-parameter family, $y_i = y_i(x, b_1, \cdots, b_{p+1})$, $x_1(b_1, \cdots, b_{p+1}) \leq x \leq x_2(b_1, \cdots, b_{p+1})$, containing E_{12} for $(b_1, \cdots, b_{p+1}) = (0, \cdots, 0)$ and having the sets $\xi_1^{(s)}$, $\xi_2^{(s)}$ and $\eta_i^{(s)}(x)$ as its variations along E_{12} with respect to the parameter b_s. The continuity properties are similar to those described in the above theorem.

This corollary is proved with the equations $\phi_\alpha = 0, \cdots, \phi_r = z_r + b_1 \zeta_r^{(1)} + \cdots + b_{p+1} \zeta_r^{(p+1)}$ replacing equations (9).

4. **First variation of the integral I.** If the one-parameter family of admissible arcs (4) be substitued in (3), we obtain for the first variation of the resulting expression for $b = b_0 = 0$

$$(12) \quad I_1(\xi, \eta) = \int_{x_1}^{\omega_0} [(\partial g/\partial u_i)\Omega_i + (\partial g/\partial u_i')\Omega_i'] dx + \int_{\omega_0}^{x_2} [(\partial h/\partial v_i)\pi_i$$

$$+ (\partial h/\partial v_i')\pi_i'] dx + S_{1i}(g, h)\Omega_i(x_1) + S_{2i}(g, h)\pi_i(x_2) + L_1(g, h)\xi_1$$

$$+ L_2(g, h)\xi_2 + M(g, h)\omega'(0) + N_i(g, h)\frac{dy_i(\omega(0), 0)}{db},$$

where as notation

$$S_{\sigma i}(g,h) = \int_{x_1}^{\omega_0}(\partial g/\partial y_i(x_\sigma))dx + \int_{\omega_0}^{x_2}(\partial h/\partial y_i(x_\sigma))dx \qquad (\sigma = 1,2) ;$$

$$L_\sigma(g,h) = (\sigma - 2)g(x_1) + (\sigma - 1)h(x_2) + \int_{x_1}^{\omega_0}[\partial g/\partial x_\sigma + (\partial g/\partial y_i(x_\sigma))y_i'(x_\sigma)]dx$$

$$+ \int_{\omega_0}^{x_2}[\partial h/\partial x_\sigma + (\partial h/\partial y_i(x_\sigma))y_i'(x_\sigma)]dx ;$$

$$M(g,h) = g(\omega_0) - h(\omega_0) + \int_{x_1}^{\omega_0}(\partial g/\partial\omega)dx + \int_{\omega_0}^{x_2}(\partial h/\partial\omega)dx ;$$

$$N_i(g,h) = \int_{x_1}^{\omega_0}[\partial g/\partial u_i(\omega)]dx + \int_{\omega_0}^{x_2}[\partial h/\partial v_i(\omega)]dx ;$$

$g(x_1)$ and $g(\omega_0)$ are the values of g at the points (1) and (ω_0) on E_{12}; $h(\omega_0)$ and $h(x_2)$ are the values of h at the points (ω_0) and (2) on E_{12}; i is an umbral index with range $1, \cdots, n$ when it occurs twice as a subscript, and σ is not umbral in L_σ.

Let us multiply each of the equations (8) by a function $\lambda(x)$; form the sum $\lambda_\alpha\psi_\alpha + \lambda_r\psi_r - \lambda_r\zeta_r$ where α and r are umbral indices with ranges $1, \cdots, m$ and $m+1, \cdots, n$, respectively, and then integrate this sum from x_1 to x_2. We obtain

(13)
$$\int_{x_1}^{\omega_0}[(\partial R/\partial u_i)\Omega_i + (\partial R/\partial u_i')\Omega_i']dx$$
$$+ \int_{\omega_0}^{x_2}[(\partial T/\partial v_i)\pi_i + (\partial T/\partial v_i')\pi_i']dx - \int_{x_1}^{x_2}\lambda_r\zeta dx = 0,$$

where, by definition, $R = \lambda_i\phi_i$, $i = 1, \cdots, n$, on $x_1 \leqq x \leqq \omega$, and $T = \lambda_i\phi_i$ on $\omega \leqq x \leqq x_2$. It seems desirable to split the integral into two parts in this manner because this was done with the original integral I.

By (6) the equations of variation on E_{12} for the functions θ_μ are

(14)
$$K_\mu(\xi,\eta) = [(\partial\theta_\mu/\partial x_1) + (\partial\theta_\mu/\partial y_i(x_1))y_i'(x_1)]\xi_1 + [\partial\theta_\mu/\partial y_i(x_1)]\Omega_i(x_1)$$
$$+ [\partial\theta_\mu/\partial x_2 + (\partial\theta_\mu/\partial y_i(x_2))y_i'(x_2)]\xi_2 + [\partial\theta_\mu/\partial y_i(x_2)]\pi_i(x_2) = 0.$$

If a $(p+1)$-parameter family of admissible arcs $y_i = y_i(x, b_1, \cdots, b_{p+1})$, $x_1(b_1, \cdots, b_{p+1})$, $x_2(b_1, \cdots, b_{p+1})$, $\omega(b_1, \cdots, b_{p+1})$, containing E_{12} for $(b_1, \cdots, b_{p+1}) = (0, \cdots, 0)$, be substituted in I and the functions θ_μ, these become functions of $b_1, \cdots, b_{p+1}$. If I_0 is the desired minimum yielded by E_{12}, then the first members of the equations

$$I(b_1, \cdots, b_{p+1}) = I_0 + q,$$
$$\theta_\mu(b_1, \cdots, b_{p+1}) = 0,$$

where q is a positive number, must have their functional determinant equal to zero for $(b_1, \cdots, b_{p+1}) = (0, \cdots, 0)$, since, otherwise, these equations would have solutions near this set of values for both positive and negative values of q. This functional determinant of rank $r < p+1$ is

$$(15) \quad \begin{vmatrix} I_1(\xi^{(1)}, \eta^{(1)}) & \cdots & I_1(\xi^{(p+1)}, \eta^{(p+1)}) \\ K_1(\xi^{(1)}, \eta^{(1)}) & \cdots & K_1(\xi^{(p+1)}, \eta^{(p+1)}) \\ \cdots & \cdots & \cdots \\ K_p(\xi^{(1)}, \eta^{(1)}) & \cdots & K_p(\xi^{(p+1)}, \eta^{(p+1)}) \end{vmatrix},$$

and from its vanishing we argue that there exists a set of constants $d_1, \cdots, d_p, \lambda_0$, not all zero, such that the equation

$$(16) \quad \lambda_0 I_1(\xi, \eta) + d_\mu K_\mu(\xi, \eta) = 0 \qquad (\mu \text{ umbral with range } 1, \cdots, p)$$

holds for every set of admissible variations $\xi_1, \xi_2, \omega'(0), \eta(x)$, since, otherwise, one of the columns of this determinant could be replaced by another making it of rank $r+1$.*

As notation let ν be an umbral index with range 1, 2. Let us define two new functions G and H by the respective equations

$$G(x, y, y', x_1, \cdots, y(x_2), \lambda) = \lambda_0 g + \lambda_i \phi_i,$$
$$H(x, y, y', x_1, \cdots, y(x_2), \lambda) = \lambda_0 h + \lambda_i \phi_i.$$

Since the variations (ξ, η) satisfy (13), the value of (16) is not altered if we add (13) to it. By the help of (12) we can, therefore, write

$$
\begin{aligned}
(17) \quad & \int_{x_1}^{\omega_0} [(\partial G/\partial u_i)\Omega_i + (\partial G/\partial u_i')\Omega_i' - \lambda_r \zeta_r] dx \\
& + \int_{\omega_0}^{x_2} [(\partial H/\partial v_i)\pi_i + (\partial H/\partial v_i')\pi_i' - \lambda_r \zeta_r] dx + [S_{\nu i}(g, h) \\
& + d_\mu(\partial \theta_\mu/\partial y_i(x_\nu))]\eta_i(x_\nu) + [L_\nu(g, h) + d_\mu(\partial \theta_\mu/\partial x_\nu)]\xi_\nu \\
& + [(\partial \theta_\mu/\partial y_i(x_1))y_i'(x_1)]\xi_1 + [(\partial \theta_\mu/\partial y_i(x_2))y_i'(x_2)]\xi_2 + M(g, h)\omega'(0) \\
& + N_i(g, h) dy_i(\omega(0), 0)/db = 0.
\end{aligned}
$$

So far the functions $\lambda_i(x)$ have been entirely arbitrary.

5. **First necessary conditions.** We now proceed to determine the $\lambda_i(x)$ so that the equations

---

* Bliss, *Lectures*, p. 20.

$$(18) \qquad \partial G/\partial u_i' \; = \int_{x_1}^{x} (\partial G/\partial u_i)dx + C_{i1} \qquad\qquad (x_1 \leqq x \leqq \omega),$$

$$(19) \qquad \partial H/\partial v_i' \; = \int_{x}^{x_2} (\partial H/\partial v_i)dx + C_{i2} \qquad\qquad (\omega \leqq x \leqq x_2)$$

are satisfied for an arbitrarily selected set of constants C_{i1}, C_{i2}. This is possible because of the assumption that $|\partial\phi_i/\partial y_k'|$ is different from zero along E_{12}. For the details of the proof of the possibility of this choice of the λ_i we refer the reader to the mimeographed lecture notes of G. A. Bliss,[*] since the proof which Bliss gives there requires only slight alteration. For this choice of the functions $\lambda_i(x)$ equation (17) becomes

$$(20) \quad \begin{aligned} &-\int_{x_1}^{\omega_0}\lambda_r\zeta_r dx - \int_{\omega_0}^{x_2}\lambda_r\zeta_r dx + [P_i(x_\nu) + d_\mu(\partial\theta_\mu/\partial y_i(x_\nu))]\eta_i(x_\nu) + [L(x_\nu) \\ &+ d_\mu(\partial\theta_\mu/\partial x_\nu)]\xi_\nu + [d_\mu(\partial\theta_\mu/\partial y_i(x_1))y_i'(x_1)]\xi_1 + [d_\mu(\partial\theta_\mu/\partial y_i(x_2))y_i'(x_2)]\xi_2 \\ &+ M(g,h)\omega'(0) + [\partial G/\partial u'(\omega_0)]\Omega_i(\omega_0) - [\partial H/\partial v_i'(\omega_0)]\pi_i(\omega_0) \\ &+ [N_i(g,h)]dy_i(\omega_0,0)/db = 0, \end{aligned}$$

where

$$P_i(x_\nu) = (\nu - 2)(\partial G/\partial u_i'(x_1)) + (\nu - 1)(\partial H/\partial v_i'(x_2)) + S_{\nu i}(g,h) \qquad (\nu = 1,2),$$

and $L(x_\nu)=L_\nu(g, h)$; $\partial G/\partial u_i'(x_1)$ and $\partial G/\partial u_i'(\omega_0)$ denote derivatives with respect to u_i' at the point (1) and (ω_0), respectively, on E_{12}, and $\partial H/\partial v_i'(\omega_0)$ and $\partial H/\partial v_i'(x_2)$ denote derivatives of H with respect to v_i' at the points (ω_0) and (2), respectively, on E_{12}, and where we have chosen the constants C_{i1} and C_{i2} so that equations (18) and (19) are satisfied, i.e., $C_{i1}=\partial G/\partial u_i'(x_1)$ and $C_{i2}=\partial H/\partial v_i'(x_2)$.

Since the extremal E_{12} is continuous at $x=\omega_0$ the equations $y_i(\omega(b), b)=u_i(\omega(b), b)=v_i(\omega(b), b)$ must hold, and, hence, by a differentiation

$$(21) \qquad dy_i(\omega_0,0)/db = u_i'(\omega_0)\omega'(0) + \Omega_i(\omega_0) = v_i'(\omega_0)\omega'(0) + \pi_i(\omega_0).$$

If we substitute the values of $\Omega_i(\omega_0)$ and $\pi_i(\omega_0)$ defined by (21) in (20), we obtain as a necessary condition for a minimum of the integral I

$$(22) \quad \begin{aligned} &-\int_{x_1}^{\omega_0}\lambda_r\zeta_r dx - \int_{\omega_0}^{x_2}\lambda_r\zeta_r dx + [P_i(x_\nu) + d_\mu(\partial\theta_\mu/\partial y_i(x_\nu))]\eta_i(x_\nu) + [L(x_\nu) \\ &+ d_\mu(\partial\theta_\mu/\partial x_\nu)]\xi_\nu + [d_\mu(\partial\theta_\mu/\partial y_i(x_1))y_i'(x_1)]\xi_1 + [d_\mu(\partial\theta_\mu/\partial y_i(x_2))y_i'(x_2)]\xi_2 \\ &+ [-u'(\omega_0)\partial G/\partial u_i'(\omega_0) + v_i'(\omega_0)\partial H/\partial v_i'(\omega_0) + M(g,h)]\omega'(0) \\ &+ [\partial G/\partial u_i'(\omega_0) - \partial H/\partial v_i'(\omega_0) + N_i(g,h)]dy_i(\omega_0,0)/db = 0. \end{aligned}$$

---

[*] Bliss, *Lectures*, p. 7.

Since $\omega'(0)$ and $\Omega_i(\omega_0)$ are arbitrary, equations (21) show us that $dy_i(\omega_0, 0)/db$ is arbitrary. We can, therefore, say that the expression (22) must vanish for every choice of $\eta_i(x_\nu)$, ξ_ν, $\omega'(0)$, $dy_i(\omega_0, 0)/db$ and ζ_r, $r = m+1, \cdots, n$. It follows then that all of the coefficients of these quantities must vanish and, hence, in particular, that $\lambda_{m+1} \equiv \cdots \equiv \lambda_n \equiv 0$. It follows further with the help of (12) that

$$- u_i'(\omega_0)(\partial G/\partial u_i'(\omega_0)) + v_i'(\omega_0)(\partial H/\partial v_i'(\omega_0)) + G(\omega_0) - H(\omega_0)$$

$$+ \int_{x_1}^{\omega_0} (\partial G/\partial \omega)dx + \int_{\omega_0}^{x_2} (\partial H/\partial \omega)dx = 0,$$

(23)

$$\partial G/\partial u_i'(\omega_0) - \partial H/\partial v_i'(\omega_0) + \int_x^{\omega_0} (\partial G/\partial u_i(\omega_0))dx$$

$$+ \int_{\omega_0}^{x_2} (\partial H/\partial v_i(\omega_0))dx = 0 \qquad (i = 1, \cdots, n),$$

where as already stated u_i refers to the extremal arc y_i for the interval $x_1 \leqq x \leqq \omega_0$, and v_i refers to the same arc y_i but for the interval $\omega_0 \leqq x \leqq x_2$; further that in the matrix

(24)
$$\left\| \begin{array}{cccc} L(x_1) & P_i(x_1) & L(x_2) & P_i(x_2) \\ \theta_{\mu x_1} + \theta_{\mu y_{i1}} y_{i1}' & \theta_{\mu y_{i1}} & \theta_{\mu x_2} + \theta_{\mu y_{i2}} y_{i2}' & \theta_{\mu y_{i2}} \end{array} \right\|,$$

where $\theta_{\mu x_1} + \theta_{\mu y_{i\nu}} y_{i\nu}'$ stands for $\partial \theta_\mu/\partial x_1 + (\partial \theta_\mu/\partial y_i(x_\nu))y_i'(x_\nu)$, every determinant of order $p+1$ vanishes, and finally that the equations (18) and (19) are satisfied. The condition (24) must hold since the multipliers $1, d_1, \cdots, d_p$ satisfy all the linear equations whose coefficients are columns of the matrix. The rank of this matrix is unchanged when one column is multiplied by a factor and added to another, so that we can state the following theorem.

THEOREM 2. *For every minimizing arc for this general problem there exist sets of constants C_{i1} and C_{i2}, $i = 1, \cdots, n$, and functions*

$$G(x, y, y', x_1, y(x_1), \cdots, y(x_2), \lambda) = \lambda_0 g + \lambda_\alpha \phi_\alpha \qquad (x_0 \leqq x \leqq \omega_0),$$

$$H(x, y, y', x_1, y(x_1), \cdots, y(x_2), \lambda) = \lambda_0 h + \lambda_\alpha \phi_\alpha \qquad (\omega_0 \leqq x \leqq x_2),$$

such that the equations

$$\partial G/\partial u_i' = \int_x^x (\partial G/\partial u_i)dx + C_{i1} \qquad (x_1 \leqq x \leqq \omega),$$

$$\partial H/\partial v_i' = \int_x^{x_2} (\partial H/\partial v_i)dx + C_{i2} \qquad (\omega \leqq x \leqq x_2)$$

are satisfied at every point of E_{12}, $x_1 \leqq x \leqq x_2$. The constant λ_0 and the functions

$\lambda_\alpha(x)$, $\alpha = 1, \cdots, m$, *are not all identically zero on* E_{12}, *and are continuous except possibly at values of* x *defining corners of* E_{12}. *At the point* ω, $y_i(\omega)$, *at which there exists a corner, the equations* (23), *which are the analogues of the Weierstrass-Erdmann corner conditions, must be satisfied. Furthermore, the end values of* E_{12} *must be such that all determinants of order* $p+1$ *of the matrix*

$$(25) \quad \left\| \begin{array}{cccc} L(x_1) - y'_{i1}P_i(x_1) & P_i(x_1) & L(x_2) - y'_{i2}P_i(x_2) & P_i(x_2) \\ \theta_{\mu x_1} & \theta_{\mu y_{i1}} & \theta_{\mu x_2} & \theta_{\mu y_{i2}} \end{array} \right\|$$

vanish.

These last conditions are the *transversality conditions*. It is easily seen that if the functions g and h are independent of the end and corner variables they reduce to the corresponding conditions obtained by Bliss.*

The corner conditions as obtained in this paper differ from the corner conditions usually obtained because of the occurrence of the integrals involving derivatives with respect to corner variables, so that if g and h are independent of these variables, the integral terms drop out. For such problems as Miss Sinclair's and E. H. Clarke's these derivatives reduce to terms simple enough to be integrated by inspection.†

It will be interesting to examine a few special cases of the very general problem of this paper. In as much as the paper was suggested by the problem of depreciation it will be well to begin with a short examination of this important special case.

6. **Some special cases.** In order to determine the best time at which to replace a machine, which is in operation, by another machine whose operating expense is different we must maximize an expression which can be thrown in the general form (3). This problem of replacement in dynamical economics is that of finding among the arcs $y_i = y_i(x)$ satisfying a differential equation of demand, $D(x, y_1, y'_1, y_2, y'_2) = 0$, where x is the time, y_1 is the rate of production at the time x, and y_2 is the price at the time x, and end conditions

$$\theta_\mu(x_1, y_1(x_1), y_2(x_1), x_2, y_1(x_2), y_2(x_2)) = 0 \qquad (\mu = 1, \cdots, p \leq 6),$$

a set which maximizes an expression of the form

$$V = \int_{x_1}^\omega [y_1 y_2 - Q_1(x, y_1, y_2, y'_1, y'_2)] e^{\beta(x_1 - x)} dx$$

$$+ \int_\omega^{x_2} [y_1 y_2 - Q_2(x, y_1, y_2, y'_1, y'_2)] e^{\beta(x_1 - x)} dx$$

$$+ C_1 e^{\beta(x_1 - \omega)} + C_2 e^{\beta(x_1 - x_2)},$$

---

* Bliss, *The problem of Mayer with variable end points*, loc. cit.

† See fourth and fifth footnotes on page 58.

where x is the time; V is the value at $x=x_1$ of the composite machine, consisting of a machine operating for the period $x_1 \leqq x \leqq \omega$, and a replacement machine operating for the period $\omega \leqq x \leqq x_2$; y_1 is the number of units of output per unit time; y_2 is the price of one unit of output at the time x; Q_1 is the cost of producing y_1 units of output for the machine called the operating machine; Q_2 is the cost of producing y_1 units of output for the replacement machine, and exp. $(\beta(x_1-x))$ etc. are discount factors to allow for interest.

Since ω and x_1 are constant so far as integration with respect to x is concerned, we may write this expression in the form

$$V = \int_{x_1}^{\omega} \left[(y_1 y_2 - Q_1)e^{\beta(x_1-x)} + \frac{C_1 e^{\beta(x_1-\omega)} + C_2 e^{\beta(x_1-x_2)}}{\omega - x_1} \right] dx$$
$$+ \int_{\omega}^{x_2} (y_1 y_2 - Q_2)e^{\beta(x_1-x)} dx.$$

When V is in this form, the replacement problem is easily seen to be a special case of the general problem of this paper. In as much as I have presented a mathematical theory of depreciation and replacement in another paper, there is no need to carry the problem further here.[*]

As already mentioned, problems which require that the sum of a point function and a definite integral be minimized are special cases of a general problem discussed by Clarke.[†] His problem is that of finding among the arcs $y_i = y_i(x)$ satisfying end conditions $x_1 = \alpha_1$, $x_2 = \alpha_2$, $y_i(x_1) = \beta_{i1}$, $y_i(x_2) = \beta_{i2}$ $(i=1, \cdots, n)$, a set which minimizes the sum of a definite integral and a function of a point,

$$J = \phi(\omega, y(\omega)) + \int_{x_1}^{x_2} f(x, y, y') dx.$$

Although Clarke does not discuss the Lagrange problem analogous to the case above which he studied, the analysis of my paper is sufficiently general to apply to his problem. We may write J in the form

$$J = \int_{x_1}^{\omega} \left[f(x, y, y') + \frac{\phi(\omega, y(\omega))}{\omega - x_1} \right] dx + \int_{\omega}^{x_2} f(x, y, y') dx,$$

and again obtain a special case. For Clarke's problem the corner conditions (23) become

* Roos, *A mathematical theory of depreciation and replacement*, loc. cit.
† Clarke, *On the minimum of the sum* etc., loc. cit.

$$\phi_\omega + f(\omega, y(\omega), u'(\omega)) - f(\omega, y(\omega), v'(\omega)) - u'(\omega)f_u'(\omega, y(\omega), u'(\omega))$$
$$+ v'(\omega)f_v'(\omega, y(\omega), u'(\omega)) = 0,$$

$$\phi_{y_i(\omega)} + f_{u_i'}(\omega, y(\omega), u'(\omega)) - f_{v_i'}(\omega, y(\omega), v'(\omega)) = 0 \quad (i = 1, \cdots, n),$$

and these are identical with the conditions obtained by Clarke, except for notation.

Let us next consider special cases for which the functions g and h are independent of the corner variables and, furthermore, for which $g \equiv h$, i.e., for which there is no corner at $x = \omega$.

An important example of this type of problem is the following proposed by Bolza.

Minimize the expression

$$U = \int_{x_0}^{x_1} H(y_1, \cdots, y_n, y_1', \cdots, y_n')dx + G(y_{10}, \cdots, y_{n0}, y_{11}, \cdots, y_{n1}),$$

when the admissible curves are defined by functions $y_i = y_i(x)$ $(x_0 \leqq x \leqq x_1)$ which satisfy differential equations $\phi_\alpha(y_i, y_i') = 0$ $(\alpha = 1, \cdots, m)$, while the end points, y_{i0}, y_{i1}, satisfy conditions $\theta_\mu(y_{i0}, y_{i1}) = 0$ $(\mu = 1, \cdots, p)$. Bolza admits so-called finite equations $\psi_k(y_1, \cdots, y_n) = 0$ $(k = 1, \cdots, r < n - m)$, in addition to the equations $\phi_\alpha = 0$, but, as Bliss has shown, this gives no additional generalization.* Bolza's problem is the special case for which ϕ_α and θ_μ of the general problem do not depend upon x explicitly and for which

$$g \equiv h \equiv \frac{G(y_{10}, \cdots, y_{n0}, y_{11}, \cdots, y_{n1})}{x_1 - x_0} + H(y_1, \cdots, y_n, y_1', \cdots, y_n').$$

The integrand is continuous at $x = \omega$ and the corner conditions vanish identically unless $y_i(x)$ has a corner there, in which case they reduce to the Weierstrass-Erdmann corner conditions.†

In as much as Bolza has shown that his problem is sufficiently general to include as special cases the most general Lagrange problem with general boundary conditions for $G \equiv 0$, and the most general Mayer problem with general boundary conditions for the case $H \equiv 0$, $G \equiv y_{10}$, it appears that the problem treated in this memoir is sufficiently general to include a very large class of the problems of the calculus of variations.

7. **Further generalizations.** In the replacement problem discussed in the last paragraph we supposed that there was only one machine producing

---

* Bliss, *The problem of Mayer with variable end points*, loc. cit.
† J. Hadamard, *Leçons sur le Calcul des Variations*, vol. 1, p. 188.

the commodity or service under consideration at any time x. If we suppose several machines to be simultaneously in operation producing the same article, there are two problems of interest. If all owners agree to cooperate in making the sum of their profits a maximum the problem is a special case of the one considered in this paper. On the other hand if the machines are competing, the problem is one for which it is required to obtain partial maxima of as many integrals as there are competing machines. I have already considered the competition case for the special case in which g and h are independent of variable corner and end values.*

In any actual case we do not have all machines in operation for the same period of time, for new competitors are constantly entering and others are dropping out. Such a problem becomes difficult even for the cooperation case because we must maximize an expression which is the sum of say n integrals whose limits may or may not be the same and whose integrands depend upon different numbers of variables x and y_i. Each time a newcomer produces, the demand equation contains an additional variable, e.g., this producer's rate of production.

---

* Roos, *Generalized Lagrange problems in the calculus of variations*, these Transactions, April, 1928. See also Roos, *A mathematical theory of competition*, American Journal of Mathematics, vol. 47 (1925), pp. 163–175, and Roos, *A dynamical theory of economics*, Journal of Political Economy, vol. 35 (1927), pp. 632–656.

UNIVERSITY OF CHICAGO,
    CHICAGO, ILL.