BOUNDARY VALUE PROBLEMS FOR POTENTIALS OF A SINGLE LAYER*

BY

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1. Introduction.   The principal object of the following paper is the discussion of a Neumann problem, with reference to a potential of a single layer which is based on a general distribution of matter on a simple closed plane boundary.   Such potentials were introduced by Plemelj.[†]   The result obtained here is of the same order of generality for these boundaries as that obtained by G. C. Evans with the aid of conformal transformations[‡], but the present method is entirely different, and simpler.   The problem is equivalent to a Stieltjes integral equation, which is solved by reduction to the classical Fredholm type.

Consider the function

$$(1) \qquad v(M) = \int_C \log (1/r) d\mu(P),$$

where $\mu(P)$ is a function of limited variation on the simple closed curve C. The notation $\mu(P)$ is used as synonymous with $\mu(s_P)$.   For the measurement of angles the length $r = MP$ is regarded as directed from M to P, where M is a point which remains fixed in the integration.   It is assumed that the direction of C changes continuously as we go along the curve, and that the curvature exists at each point.

Let n, or, more particularly, n_Q, be the normal at a point Q of the curve, directed toward the interior, and M a point on n_Q.   The derivative

$$(2) \qquad \frac{dv(M)}{dn} = \int_C \frac{\cos (n_Q, r)}{r} d\mu(P)$$

generally fails to be continuous as M passes over the point Q.   We shall investigate this discontinuity.

The integral (2) converges when M is coincident with Q.   In order to prove this statement, it suffices to examine the integrand in the neighborhood

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of Q. Set up a system of rectangular coördinates, with origin at Q, and axes along the positive tangent and inner normal, respectively. Measure s from Q. In the neighborhood of Q, the equation of C may be written in the form

$$y = x^2 f(x),$$

where, as may be verified from the hypotheses for C, $f(x)$ and $xf'(x)$ are continuous except at $x=0$, and bounded in a suitable interval containing Q.

Denote by θ the angle measured in the positive direction from the normal at Q to QP. We have then

$$\frac{\cos \theta}{r} = \frac{y}{r^2} = \frac{f(x)}{1 + x^2[f(x)]^2},$$

which is therefore continuous except at Q as a function of x, and bounded in the neighborhood of Q. Hence it has the same properties as a function of s_P when $P \neq Q$. The statement is therefore proved.

From what has been said of the function $(\cos \theta)/r$, we obtain immediately the following corollary, which will be useful later. We state the result as a lemma.

LEMMA. *If $\mu(P)$ is continuous at Q, and C' is an arc of C, containing Q, and of length $< \delta$, then*

$$\int_C \frac{\cos \theta}{r} d\mu(P) = \lim_{\delta \to 0} \int_{C-C'} \frac{\cos \theta}{r} d\mu(P).$$

2. **Boundary values of the normal derivative.** For convenience we denote by $dv(Q+)/dn$ the value, where it exists, of dv/dn as M approaches Q along n from the inside; we define similarly $dv(Q-)/dn$ with reference to the exterior of C. With this notation, we have the following theorem:

THEOREM 1. *If $v(M)$ is given by (1), then*

$$\frac{dv(Q+)}{dn} + \frac{dv(Q-)}{dn} = 2 \int_C \frac{\cos (n_Q, QP)}{QP} d\mu(P),$$

$$\frac{dv(Q+)}{dn} - \frac{dv(Q-)}{dn} = -2\pi\mu'(Q),$$

provided $\mu'(Q)$ exists.

Denote by C' a small arc HK containing Q as an interior point. We may write (2) in the form

$$\frac{dv(M)}{dn} = \int_{C-C'} + \int_{C'} \frac{\cos \theta}{MP} d\mu(P) = I_1 + I_2.$$

Consider I_2. Construct the osculating circle $\bar{C}$ at Q. Let ρ be the distance from its center to M, $\bar{P}$ that point of $\bar{C}$ for which $x = x_P$. Let $\bar{r} = M\bar{P}$, and $\bar{\theta}$ the angle from n_Q to $M\bar{P}$, measured in the positive direction. We shall show that

$$\left| \int_{C'} \left[\frac{\cos\theta}{r} - \frac{\cos\bar{\theta}}{\bar{r}} \right] d\mu(P) \right| < \epsilon,$$

uniformly with respect to M, provided that the length of C' is small enough. For this purpose it suffices to show that $(\cos\theta)/r - (\cos\bar{\theta})/\bar{r}$ is bounded uniformly with respect to M. In fact, $\mu(P)$ is of limited variation in the neighborhood of Q, and continuous at Q. The conclusion is apparent if we write

$$\frac{\cos\theta}{r} - \frac{\cos\bar{\theta}}{\bar{r}} = \frac{(\bar{y} - y)\cos(\theta + \bar{\theta})}{r\bar{r}}$$

and note (a), that $\bar{y} - y$ is of order higher than the second with respect to x, and (b), that $|x| < r, \bar{r}$. The details are as follows:

$$\bar{y} = R - (R^2 - x^2)^{1/2},$$

where R is the radius of curvature of C at Q. But the preceding equation may also be written

$$\bar{y} = R\left(1 - \left(1 - \frac{x^2}{R^2} \right)^{1/2} \right)$$

$$= \frac{x^2}{2R} + x^4\Phi(x).$$

Then

$$y - \bar{y} = x^2\left[f(x) - \frac{1}{2R} \right] + x^4\Phi(x),$$

where the expression in brackets is bounded. Moreover $\cos(\theta + \bar{\theta})$ is obviously continuous as a function of x, and uniformly bounded with respect to M. The statement is therefore proved.

Consider therefore the integral

$$I_2 = \int_{\bar{C}'} \frac{\cos\bar{\theta}}{\bar{r}} d\mu(P) = \frac{d}{dn} \int_{\bar{C}'} \log\frac{1}{\bar{r}} d\mu(P)$$

$$= \frac{1}{2} \frac{\partial}{\partial\rho} \int_{\bar{C}'} \log(R^2 + \rho^2 - 2R\rho\cos\psi) d\mu(P)$$

$$= -\frac{1}{2\rho} \int_{\bar{C}'} \frac{-2\rho^2 + 2R\rho\cos\psi}{R^2 + \rho^2 - 2R\rho\cos\psi} d\mu(P),$$

where ψ is the angle subtended by $Q\overline{P}$ at the center O of the circle, and where ρ is the segment OM, directed from O to M. Hence

$$- I_2 + \int_{\overline{C}'} \frac{1}{2\rho}\, d\mu(P) = \frac{1}{2\rho} \int_{\overline{C}'} \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho \cos \psi}\, d\mu(P).$$

If the length of $\overline{C}'$ is sufficiently small, we have, therefore,

$$\left| I_2 + \frac{1}{2\rho} \int_{\overline{C}'} \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho \cos \psi}\, d\mu(P) \right| < \epsilon.$$

But*

$$\lim_{\rho \to R} \frac{1}{2\pi R} \int_{\overline{C}'} \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho \cos \psi}\, d\mu(P) = \mu'(Q).$$

Hence, taking account of the lemma of §1,

$$\frac{dv(Q+)}{dn} = -\pi\mu'(Q) + \int_c \frac{\cos(n_Q, QP)}{QP}\, d\mu(P),$$

and similarly

$$\frac{dv(Q-)}{dn} = \pi\mu'(Q) + \int_c \frac{\cos(n_Q, QP)}{QP}\, d\mu(P).$$

From these equations we have the result stated in the theorem.

The foregoing analysis presupposes that the curvature at Q is not zero. In the latter case, we may replace the arc of the osculating circle by the corresponding segment $\overline{C}'$ of the tangent at Q, and observe that the integrals along C' and along $\overline{C}'$ differ by a quantity which is uniformly small with the length of C'. In fact, $(\cos \theta)/r - (\cos \overline{\theta})/\overline{r}$ is uniformly bounded with respect to M, since y is of order higher than the second with respect to x. Therefore consider the integral†

$$\int_{-x_0}^{x_0} \frac{\cos \overline{\theta}}{\overline{r}}\, d\mu(x)$$

$$= \mu'(Q) \int_{-x_0}^{x_0} \frac{\cos \overline{\theta}}{\overline{r}}\, dx - \int_{-x_0}^{0} \frac{\overline{y}}{\overline{r}^2}\, d(x\eta(x)) - \int_{0}^{x_0} \frac{\overline{y}}{\overline{r}^2}\, d(x\eta(x)),$$

where $\mu(x) = x\mu'(Q) + x\eta(x)$, the last term being a function of x of limited variation, such that if η_0 denotes the upper bound of $|\eta(x)|$ in the interval $-x_0 \leq x \leq x_0$, we have $\lim_{x_0 \to 0} \eta_0 = 0$.

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* Evans, loc. cit., pp. 39, 40.

† This integral is essentially the Poisson Stieltjes integral for the straight line considered as an infinite circle.

Now since $\bar{y}/\bar{r}^2$ decreases with x, $x>0$, we have

$$\int_0^{x_0} \frac{\bar{y}}{\bar{r}^2} d(x\eta(x)) = \frac{\bar{y}}{\bar{r}^2} x_0\eta(x_0) - \int_0^{x_0} x\eta(x)d\left(\frac{\bar{y}}{\bar{r}^2}\right),$$

where

$$\left|\frac{\bar{y}}{\bar{r}^2} x_0\eta(x_0)\right| < \eta_0,$$

and

$$\left|\int_0^{x_0} x\eta(x)d\left(\frac{\bar{y}}{\bar{r}^2}\right)\right| \leqq - \eta_0 \int_0^{x_0} xd\left(\frac{\bar{y}}{\bar{r}^2}\right)$$

$$\leqq \eta_0 x_0\left[\frac{\bar{y}}{\bar{r}^2}\right]_{x=x_0} + \eta_0 \int_0^{x_0} \frac{\bar{y}}{\bar{r}^2} dx$$

$$\leqq \eta_0\left(1 + \frac{\pi}{2}\right).$$

Hence the integral considered is made arbitrarily small with x_0, uniformly for M. The same remark applies to

$$\int_{-x_0}^{0} \frac{\bar{y}}{\bar{r}^2} d(x\eta(x)).$$

But $\int_{-x_0}^{x_0} (\cos\bar{\theta})/\bar{r}\, dx = 2(\theta_0 - \pi)$. Hence, by first taking x small enough, and then taking M near enough to Q, we can make

$$\int_{-x_0}^{x_0} \frac{\cos\bar{\theta}}{\bar{r}} d\mu(x)$$

differ by as little as we please from $\pi\mu'(Q)$, which is what we wished to show.

3. **The Stieltjes integral equation.** We consider again the class of functions which can be expressed in the form (1). It is assumed now that the discontinuities of $\mu(P)$ are regular. If we also denote by C the total length of the curve, we may extend the definition of $\mu(s_P)$ beyond $s_P = C$ by means of the relation

$$\mu(s_P + C) = \mu(s_P) + m,$$

where

$$m = \int_C d\mu(s_P) = \mu(C).$$

The problem now to be considered is expressed in terms of the condition

$$(3) \quad \frac{1+\lambda}{2\lambda} \lim_{\delta\to 0+} \int_{A_\delta}^{B_\delta} \frac{dv}{dn} ds_\delta - \frac{1-\lambda}{2\lambda} \lim_{\delta\to 0-} \int_{A_\delta}^{B_\delta} \frac{dv}{dn} ds_\delta = g(B) - g(A),$$

where $g(s)$ is a given function of limited variation, with regular discontinuities, and where A, B are points of C. We shall regard A as fixed and B as variable on C. In the first integral s_i is an arbitrary simple regular curve lying entirely within C, and joining the points A_i, B_i. The latter points are at the distance δ from A, B, measured along the respective inner normals. In the second integral s_i lies entirely outside C; the points A_i, B_i are on the outer normals at A, B, respectively.

That the curve C has a unique normal at each point is implied in the preceding statements. Specifically, it is assumed that C is a simple regular closed curve; moreover C shall be a curve of class Γ,* i.e., there shall exist a positive number Γ, independent of M, such that

$$\int_C \frac{\left| \cos (n_P, MP) \right|}{MP} ds_P < \Gamma,$$

uniformly for all M.

THEOREM 2. *There is a unique function of class (1) which satisfies the condition (3) unless λ belongs to a certain set of characteristic values. These values are independent of the given function $g(B)$, and depend merely on the form of the curve C.*

The value $\lambda = -1$ is not a characteristic value. The corresponding problem is a generalized Neumann problem for the exterior region:

$$\lim_{\delta \to 0-} \int_{A_\delta}^{B_\delta} \frac{dv}{dn} ds_\delta = g(B) - g(A).$$

The value $\lambda = +1$ is a characteristic value. A necessary and sufficient condition that there exist a solution $v(M)$ of class (1) is that

$$\int_C dg(B) = g(C) = 0.$$

The corresponding problem is a generalized Neumann problem for the interior region:

$$\lim_{\delta \to 0+} \int_{A_\delta}^{B_\delta} \frac{dv}{dn} ds_\delta = g(B) - g(A).$$

The solution is unique except for an arbitrary additive constant.

Consider the condition (3). We have

$$\lim_{\delta \to 0+} \int_{A_\delta}^{B_\delta} \frac{dv}{dn} ds_\delta = - \lim_{\delta \to 0+} \int_{A_\delta}^{B_\delta} \frac{dw}{ds_\delta} ds_\delta = \lim_{\delta \to 0+} \left[w(A_\delta) - w(B_\delta) \right],$$

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* G. C. Evans, The Rice Institute Pamphlet, vol. 7 (1920), p. 261.

where w is a function conjugate to v:

$$w(M) = -\int_C \tan^{-1} \frac{y_P - y_M}{x_P - x_M}\, d\mu(P).$$

The function under the integral sign being multiple valued, we write

$$w(M) = -\int_C \theta(M,P)\, d\mu(P),$$

where $\theta(M, P)$ is the angle from MA to MP measured in the positive direction. Then, taking A as the initial point of the integration, θ varies continuously as P traces the curve.

Integrate by parts:

(4)
$$\begin{aligned}
w(M) &= -\left[\mu(P)\theta(M,P)\right]_A^A \\
&\quad + \int_C \mu(P)\frac{d\theta(M,P)}{ds_P}\, ds_P \\
&= \phi(M) + u(M).
\end{aligned}$$

We shall take $\mu(A)=0$, since this can be effected by the addition of a constant. Then

$$\phi(M) = \begin{cases} -2\pi m, & M \text{ inside } C, \\ 0, & M \text{ outside } C. \end{cases}$$

The function $u(M)$ is a *double layer* potential with density a function of limited variation. The integral converges when M is a point on C, since $\mu(P)$ is bounded, and C is of class Γ.

Consider $u(Q+)$ as M approaches a point Q of C along the normal:

$$u(M) = \int_C \mu(P)\frac{d\theta(M,P)}{ds_P}\, ds_P = \int_C \mu(P)\frac{\cos(n_P, MP)}{MP}\, ds_P.$$

In order to study the behavior of $u(M)$ in the neighborhood of Q, we need to extend the traditional analysis to cover the case where $\mu(P)$ has a regular discontinuity at Q.

Write
$$u(M) = \int_{C-C'} + \int_H^Q + \int_Q^K \mu(P)\frac{\cos(n_P, MP)}{MP}\, ds_P,$$

where C' denotes a small arc HK, of which Q is an interior point.

The first of the preceding integrals is continuous at Q, since Q is not on C'. Then

$$\lim_{M \to Q} \int_{C-C'} \mu(P)\frac{\cos(n_P, MP)}{MP}\, ds_P = \int_{C-C'} \mu(P)\frac{\cos(n_P, QP)}{QP}\, ds_P.$$

The second integral may be written

$$\int_H^Q [\mu(P) - \mu(Q-)] \frac{\cos(n_P, MP)}{MP} ds_P + \mu(Q-) \int_H^Q \frac{\cos(n_P, MP)}{MP} ds_P = I_1 + I_2,$$

where $\lim_{P \to Q} \mu(P) = \mu(Q-)$, the limit existing since $\mu(P)$ is of limited varia-
tion. Hence H may be taken near enough to Q so that $|\mu(P) - \mu(Q-)| < \epsilon/\Gamma$,
where $\epsilon > 0$ is arbitrarily small. By hypothesis, C is of class Γ. Consequently

$$|I_1| < \epsilon$$

if s_{HQ} is sufficiently small, and uniformly with regard to M.

Write I_2 in the form

$$I_2 = \mu(Q-) \int_H^Q \frac{d}{ds} \tan^{-1} \frac{y_M - y_P}{x_M - x_P} ds_P$$

$$= \mu(Q-) \left[\tan^{-1} \frac{y_M - y_Q}{x_M - x_Q} - \tan^{-1} \frac{y_M - y_H}{x_M - x_H} \right].$$

The quantity within brackets is the angle from MH to MQ, measured in
the positive direction. As M approaches Q along the normal, this angle ap-
proaches as its limit the angle from the chord QH to the (outer) normal at
Q. Hence if H is taken near enough to Q, the limiting angle may be made as
near $\pi/2$ as we wish. That is,

$$\lim_{M \to Q+} I_2(M) = \frac{\pi}{2} \mu(Q-) + \eta(H),$$

where $\lim_{H \to Q} \eta(H) = 0$.

By a similar process we obtain

$$\int_Q^K \mu(P) \frac{\cos(n_P, MP)}{MP} ds_P$$

$$= \int_Q^K [\mu(P) - \mu(Q+)] \frac{\cos(n_P, MP)}{MP} ds_P + \mu(Q+) \int_Q^K \frac{\cos(n_P, MP)}{MP} ds_P$$

$$= I_3 + I_4.$$

By taking K sufficiently near to Q, $|I_3|$ can be made arbitrarily small, uni-
formly with respect to M. Also

$$\lim_{M \to Q+} I_4(M) = \frac{\pi}{2} \mu(Q+) + \eta'(K), \quad \lim_{K \to Q} \eta'(K) = 0.$$

If M approaches Q along the outer normal at Q, the corresponding limits are given by

$$\lim_{M \to Q^-} I_2'(M) = -\frac{\pi}{2}\mu(Q-) + \eta''(H),$$

$$\lim_{M \to Q^-} I_4'(M) = -\frac{\pi}{2}\mu(Q+) + \eta'''(K),$$

$$\lim_{H \to Q} \eta''(H) = \lim_{K \to Q} \eta'''(K) = 0.$$

We have, therefore,

$$u(Q+) = \int_C \mu(P)\frac{\cos(n_P, QP)}{QP}ds_P + \frac{\pi}{2}[\mu(Q+) + \mu(Q-)]$$

$$= \int_C \mu(P)\frac{d}{ds_P}\tan^{-1}\frac{y_Q - y_P}{x_Q - x_P}ds_P + \pi\mu(Q),$$

since the discontinuities of $\mu(P)$ are regular. Similarly

$$u(Q-) = \int_C \mu(P)\frac{d}{ds_P}\tan^{-1}\frac{y_Q - y_P}{x_Q - x_P}ds_P - \pi\mu(Q).$$

If now we take M as B_δ, we obtain from (4)

$$w(B+) + w(B-) = -2\pi m + 2\int_C \mu(P)\frac{d}{ds_P}\tan^{-1}\frac{y_B - y_P}{x_B - x_P}ds_P,$$

$$w(B+) - w(B-) = -2\pi m + 2\pi\mu(B).$$

Taking M as A_δ,

$$w(A+) + w(A-) = -2\pi m + 2\int_C \mu(P)\frac{d}{ds_P}\tan^{-1}\frac{y_A - y_P}{x_A - x_P}ds_P,$$

$$w(A+) - w(A-) = -2\pi m.$$

We have, therefore,

$$\lim_{\delta \to 0+}\int_{A_\delta}^{B_\delta}\frac{dv}{dn}ds_\delta = w(A+) - w(B+)$$

$$= -\pi\mu(B) - \int_C \mu(P)\frac{d}{ds_P}H(A, B; P)ds_P,$$

where

$$H(A, B; P) = \tan^{-1}\frac{y_B - y_P}{x_B - x_P} - \tan^{-1}\frac{y_A - y_P}{x_A - x_P}.$$

Also

$$\lim_{s \to 0-} \int_{A_\delta}^{B_\delta} \frac{dv}{dn}\, ds_\delta = w(A-) - w(B-)$$

$$= \pi\mu(B) - \int_C \mu(P)\frac{d}{ds_P}H(A,B;P)ds_P.$$

Equation (3) may now be written in the form

$$\mu(B) = -\frac{\lambda}{\pi}[g(B) - g(A)] - \frac{\lambda}{\pi}\int_C \mu(P)\frac{d}{ds_P}H(A,B;P)ds_P.$$

Integrate by parts:

$$\mu(B) = -\frac{\lambda}{\pi}[g(B) - g(A)] - \frac{\lambda}{\pi}[\mu(P)H(A,B;P)]_C$$

$$+ \frac{\lambda}{\pi}\int_C H(A,B;P)d\mu(P)$$

$$= -\frac{\lambda}{\pi}[g(B) - g(A)] - m\lambda + \frac{\lambda}{\pi}\int_C H(A,B;P)d\mu(P).$$

Hence we have the following equation:

$$(5) \qquad\qquad \mu(B) = G(B) + \frac{\lambda}{\pi}\int_C H(A,B;P)d\mu(P),$$

where $G(B) = -(\lambda/\pi)[g(B) - g(A)] - m\lambda$.

If $v(M)$ is a function of class (1) which satisfies (3), the function $\mu(B)$, of limited variation and with regular discontinuities, satisfies the Stieltjes integral equation (5). Conversely, if $\mu(B)$ is a solution of (5), of limited variation and with regular discontinuities, the function $v(M)$ given by (1) will satisfy (3).

4. The equivalent classical equation. We now proceed to show that this equation may be solved by means of one of classical form. Write

$$(6) \qquad\qquad \mu(B) = G(B) + R(B).$$

We shall prove that if $\mu(B)$ is of limited variation, then $R(B)$ is absolutely continuous.

Substitute the value of $\mu(B)$ from (6) into (5):

$$(7)\ G(B) + R(B) = G(B) + \frac{\lambda}{\pi}\int_C H(A,B;P)dG(P) + \frac{\lambda}{\pi}\int_C H(A,B;P)dR(P).$$

It suffices to consider one of the above integrals, say the first. We have

$$H(A,B;P) = \int_A^B \frac{d}{ds_Q} \tan^{-1} \frac{y_Q - y_P}{x_Q - x_P} ds_Q$$

$$= \int_A^B \frac{\cos(n_Q,QP)}{QP} ds_Q,$$

$$\int_C H(A,B;P) dG(P) = \int_C dG(P) \int_A^B \frac{\cos(n_Q,QP)}{QP} ds_Q.$$

We shall prove that we may invert the order of integration in the right hand member.*

Let $\{h_k(Q,P)\}$ denote a sequence of continuous functions such that

$$|h_k Q,P)| < \frac{|\cos(n_Q,QP)|}{QP} \qquad (k = 1,2,\cdots),$$

$$\lim_{k\to\infty} h_k(Q,P) = \frac{\cos(n_Q,QP)}{QP}.$$

We have

(8) $$\int_C dG(P) \int_A^B h_k(Q,P) ds_Q = \int_A^B ds_Q \int_C h_k(Q,P) dG(P).$$

We shall show first that as $k\to\infty$ the right hand member of this equation approaches the limit

$$\int_A^B ds_Q \int_C \frac{\cos(n_Q,QP)}{QP} dG(P).$$

For this purpose we define a sequence $\{q_m(Q,P)\}$ as follows:

$$q_m(Q,P) = \frac{|\cos(n_Q,QP)|}{QP}, \quad \text{if} \quad \frac{|\cos(n_Q,QP)|}{QP} < m,$$

$$= m, \quad \text{otherwise}.$$

The sequence

$$\int_C q_m(Q,P) dT(P),$$

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* The proof is essentially that in Evans, The Rice Institute Pamphlet, vol. 7 (1920), p. 263, but is given here in more detail, for the sake of clearness.

where T is the total variation function of $G(P)$, is non-decreasing with m. Also

$$(8') \qquad \int_A^B ds_Q \int_C q_m(Q,P)dT(P) = \int_C dT(P) \int_A^B q_m(Q,P)ds_Q < \Gamma T(C)$$

and is therefore bounded for all m. Consequently

$$\lim_{m\to\infty} \int_C q_m(Q,P)dT(P) = \int_C \frac{|\cos(n_Q,QP)|}{QP} dT(P)$$

exists nearly everywhere on C, and is a summable function. Moreover, from $(8')$,

$$\int_A^B ds_Q \int_C \frac{|\cos(n_Q,QP)|}{QP} dT(P) = \int_C dT(P) \int_A^B \frac{|\cos(n_Q,QP)|}{QP} ds_Q,$$

by the fundamental properties of the Lebesgue and Daniell integrals.

Consequently, from the definition of $h_k(Q,P)$, we have

$$\lim_{k\to\infty} \int_C h_k(Q,P)dG(P) = \int_C \frac{\cos(n_Q,QP)}{QP} dG(P),$$

the integral on the right being a Daniell integral, which by $(8')$ represents a summable function. Moreover

$$\left| \int_C h_k(Q,P)dG(P) \right| \leqq \int_C \frac{|\cos(n_Q,QP)|}{QP} dT(P),$$

a summable function. Hence, by the Lebesgue limit theorem,

$$\lim_{k\to\infty} \int_A^B ds_Q \int_C h_k(Q,P)dG(P) = \int_A^B ds_Q \int_C \frac{\cos(n_Q,QP)}{QP} dG(P).$$

As for the left hand side of (8),[*] we have

$$\lim_{k\to\infty} \int_A^B h_k(Q,P)ds_Q = \int_A^B \frac{\cos(n_Q,QP)}{QP} ds_Q.$$

In fact $\int_C dT(P)\int_A^B h_k(Q,P)ds_Q$ exists for all k. Moreover $\int_C dT(P)$ $\cdot\int_A^B[|\cos(n_Q,QP)|/(QP)]ds_Q$ exists, since, as we have seen, the inside integral is a summable function. Finally

$$\int_A^B |h_k(Q,P)| \, ds_Q < \int_A^B \frac{|\cos(n_Q,QP)|}{QP} ds_Q.$$

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[*] Evans, The Rice Institute Pamphlet, vol. 7 (1920), p. 257.

Consequently $\int_C dG(P) \int_A^B [\cos(n_Q, QP)/(QP)]ds_Q$ exists and

$$\lim_{k \to \infty} \int_C dG(P) \int_A^B h_k(Q,P)ds_Q = \int_C dG(P) \int_A^B \frac{\cos(n_Q, QP)}{QP} ds_Q.$$

We obtain therefore from (8), as $k \to \infty$,

$$\int_C dG(P) \int_A^B \frac{\cos(n_Q, QP)}{QP} ds_Q = \int_A^B ds_Q \int_C \frac{\cos(n_Q, QP)}{QP} dG(P).$$

The right hand side, being the indefinite integral of a summable function, is absolutely continuous. So therefore is the left hand side. Similarly for the remaining integral in (7). Hence $R(B)$ is absolutely continuous. If then we write (7) in the form

$$R(B) = \frac{\lambda}{\pi} \int_C H(A,B;P)[dG(P) + dR(P)],$$

the derivative nearly everywhere of $R(B)$ is

$$r(B) = \frac{\lambda}{\pi} \int_C \frac{\cos(n_B, BP)}{BP} [dG(P) + dR(P)].$$

Let $h(B) = (\lambda/\pi)\int_C [\cos(n_B, BP)/(BP)]dG(P)$. Then, since $R(B)$ is absolutely continuous:

$$R(B) = \int_A^B r(Q)ds_Q,$$

we have

(9) $$r(B) = h(B) + \frac{\lambda}{\pi} \int_C \frac{\cos(n_B, BP)}{BP} r(P)ds_P.$$

Thus (9) is a consequence of (5). Conversely, if (9) has a summable solution $r(B)$, then $\mu(B)$ given by (6) is of limited variation and a solution of (5).

But (9) has a unique solution unless λ is one of a set of characteristic values, since (9) is the classical integral equation for the Neumann problem. Moreover it is known that $\lambda = -1$ is not a characteristic value for this equation.

We know that $\lambda = +1$ is a characteristic value, and that a necessary and sufficient condition that (9) have a solution is that

(10) $$\int_C h(Q)ds_Q = 0.$$

But

$$\int_C h(Q)ds_Q = \frac{1}{\pi} \int_C ds_B \int_C \frac{\cos(n_B, BP)}{BP} dG(P).$$

By changing the order of integration,

$$\frac{1}{\pi} \int_C dG(P) \int_C \frac{\cos(n_B, BP)}{BP} ds_B = \frac{1}{\pi} \int_C \pi dG(P) = G(C) = 0.$$

Hence $g(C) = 0$. Conversely, if $G(C) = 0$, the analysis may be retraced, and we have (10). The condition $g(C) = 0$ is therefore necessary and sufficient in order that (5) have a solution for $\lambda = +1$.

The function $v(M)$ determined by (1) is therefore that belonging to a particular solution of (5) by means of (1) plus the function determined by (1) where the $\mu(P)$ is a solution of the homogeneous equation corresponding to (5). But in this case

$$\mu(Q) = \int_A^Q \psi(P)ds_P,$$

where $\psi(P)$ is a solution of the homogeneous equation corresponding to (9). But we know the functions $v(M) = \int_C \log(1/(MP))\psi(P)ds_P$ to be constant. Theorem 2 is therefore proved.

In §§3, 4 we have assumed only that C is of class Γ, and have not required the existence of a curvature as in §§1, 2.*

The foregoing methods may be applied to obtain analogous theorems for potentials of a double layer, not only for the plane, but also for the three-dimensional case. The latter case, both for the single and the double layer, is to be discussed in a paper which the author is writing in conjunction with Professor Evans, who suggested the present investigation.

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* If the curve has a curvature which is bounded for all points of the curve, then it is automatically of class Γ.

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