ON A GENERALIZATION OF THE ASSOCIATIVE LAW*

BY

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1. In my investigations in group theory, I have observed that Lagrange's theorem (that the order of a group is divisible by the order of any subgroup) does not use for its proof the Associative Law in its whole extent; this law can be replaced by a more general postulate, "Postulate A", as I shall call it.

We shall represent our elements by capital italic letters; the operation upon them may be represented by a star $\star$, so that $A\star B$ signifies the result of this operation performed upon A and B. A set of elements closed under any operation $\star$ may be called "a group"; this word is thus used in a more general sense than is usual, since the operation $\star$ is arbitrary. The ordinary groups with a special well known operation may be called "classic" to distinguish them from our generalised groups. Sets and groups will be denoted by capital German letters.

POSTULATE A. In the equation

$$(X\star A)\star B = X\star C, \tag{1}$$

the element C depends upon the elements A and B only and not upon X. (We suppose here that X can be an arbitrary element of a finite group to which A, B and C belong also.)

The Associative Law is obviously a special case of this Postulate A, viz. if $C=A\star B$.

I have investigated the finite groups that are obtained by replacing the Associative Law in the system of postulates of Frobenius† by Postulate A. I have found the following properties of these groups.

I. Besides our operation $\star$ every group $\mathfrak{G}$ of our type has another operation that will be denoted by a little circle $\circ$ and defined as follows: the equation (1) being given, we write

$$C = A\circ B. \tag{2}$$

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† Frobenius (*Über endliche Gruppen*, Berliner Sitzungsberichte, 1895) defines the classic finite groups by the four following postulates: 1. The operation that will be considered is uniform (eindeutig) and applicable to any two elements. 2. This operation is uniformly reversible (eindeutig umkehrbar), i.e. from $AB=AC$ or $BA=CA$ it follows that $B=C$. 3. The Associative Law is true for it. 4. The operation is "limited in its effect" (begrenzt in ihrer Wirkung); that signifies the possibility of forming finite groups of our elements.

It is easy to see, that $\mathfrak{G}$ is also a group relative to the operation $\circ$; we express this fact by writing $\mathfrak{G}(\circ)$; (analogously, $\mathfrak{G}(\star)$). I shall prove that $\mathfrak{G}(\circ)$ is classic.

II. The group $\mathfrak{G}(\star)$ has always a right unit (the same for all its elements).

III. If the group $\mathfrak{G}(\star)$ has also a single left unit for all its elements (that must necessarily coincide with the right unit), then the Associative Law is true for $\mathfrak{G}(\star)$; in this case $\mathfrak{G}(\star)$ is classic and the operations $\star$ and $\circ$ are identical.

It follows that in the systems of postulates of Moore* and Dickson† for the definition of classic groups the Associative Law can be replaced by Postulate A (or its left analogue).

IV. We associate with every element A of our group $\mathfrak{G}$ a substitution

$$\bar{A} \simeq \begin{pmatrix} X \\ X\star A \end{pmatrix}‡$$

whereby X runs over all elements of $\mathfrak{G}$. I prove that all those substitutions $\bar{A}$ (corresponding to each element A of $\mathfrak{G}$) form a substitution group $\bar{\mathfrak{G}}$ which is obviously classic and simply isomorphic with $\mathfrak{G}(\circ)$. Conversely, all such substitutions $\bar{A}$ form a group only if the Postulate A is true for $\mathfrak{G}(\star)$.

V. All groups of our type will be obtained from classic groups by making any substitution in the head-line of Cayley's table of a classic group. Moreover, it is sufficient to make only such substitutions as do not alter the unit of the classic group. Such a substitution may be denoted by α.

VI. $\mathfrak{H}(\star)$ being any subgroup of $\mathfrak{G}(\star)$, $\mathfrak{H}(\circ)$ is also a subgroup of $\mathfrak{G}(\circ)$, i.e. relative to the operation $\circ$. The converse is not true. Every subgroup $\mathfrak{H}$ of $\mathfrak{G}$ relative to $\circ$ is also a group relative to $\star$, if and only if the substitution α, which corresponds to $\mathfrak{G}(\star)$, has the following form:

$$\alpha = \begin{pmatrix} X \\ X^l \end{pmatrix},$$

the numbers l being relatively prime to the orders of corresponding elements X.

2. We shall prove now all the assertions of §1.

I. The group $\mathfrak{G}(\circ)$ is obviously uniformly reversible. Again:

$$[(X\star A)\star B]\star C = [X\star(A\circ B)]\star C = X\star[(A\circ B)\circ C]\ ;$$

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* Moore, *A definition of abstract groups*, these Transactions, vol. 3 (1902).

† Dickson, *Definition of a group and a field by independent postulates*, these Transactions, vol. 6 (1905).

‡ The sign $\simeq$ signifies that we denote a complicated expression more simply with a single letter.

and on the other hand

$$[(X\star A)\star B]\star C = (X\star A)\star(B\circ C) = X\star[A\circ(B\circ C)] ;$$

and hence the Associative Law is true for $\mathfrak{G}$ ($\circ$).

II. The classic group $\mathfrak{G}$ ($\circ$) has always a unit E; it is such that

$$(X\star E)\star A = X\star(E\circ A) = X\star A ;$$

and therefore

$$X\star E = X \text{ for every } X ;$$

E is thus the right unit for $\mathfrak{G}$ ($\star$).

III. Let E be a left unit of $\mathfrak{G}$ ($\star$); we have, then,

$$(E\star A)\star B = E\star(A\star B) = A\star B ;$$

and hence by virtue of Postulate A for every element X

$$(X\star A)\star B = X\star(A\star B),$$

i.e. the Associative Law; hence $\mathfrak{G}$ ($\star$) is classic, and $A\circ B = A\star B$.

IV. It follows from (1), by virtue of Postulate A, that $\overline{A}\overline{B} = \overline{C}$; hence $\overline{\mathfrak{G}}$ is a substitution group simply isomorphic with $\mathfrak{G}$ ($\circ$) (see (2)).

Conversely, let $\mathfrak{G}$ ($\star$) be any finite uniformly reversible group and let $\overline{\mathfrak{G}}$ be the set of corresponding substitutions, which form also a (classic) group. Let

$$\overline{A}\overline{B} = \overline{C}, \quad \text{or} \quad \begin{pmatrix} X \\ X\star A \end{pmatrix}\begin{pmatrix} X \\ X\star B \end{pmatrix} = \begin{pmatrix} X \\ X\star C \end{pmatrix};$$

since

$$\begin{pmatrix} X \\ X\star B \end{pmatrix} = \begin{pmatrix} X\star A \\ (X\star A)\star B \end{pmatrix},$$

it follows that

$$(X\star A)\star B = X\star C$$

for each element X of $\mathfrak{G}$ ($\star$); hence Postulate A holds.

V. In the head-line of Cayley's table of $\mathfrak{G}$ ($\star$) we make the following substitution:

$$\alpha \simeq \begin{pmatrix} X \\ E\star X \end{pmatrix}$$

(E being the right unit of $\mathfrak{G}$ ($\star$)). Let $E\star X = X'$. We define the third operation $\times$ as follows:

$$A\star B = A \times B'. \tag{3}$$

The operation $\times$ is uniformly reversible and also associative; in fact we have from (1) and (3):

$$(X \times A') \times B' = X \times C', \tag{4}$$

C' depending on A' and B' only but not on X; let $X = E$; then $(E \times A') \times B' = E \times C'$; but we have $E \times X' = E_\star X = X'$; hence $C' = A' \times B'$, and (4) gives us the Associative Law for $\times$; thus $\mathfrak{G}$ $(\times)$ is classic. Again it follows from (2) that α gives an isomorphism between $\mathfrak{G}$ (o) and $\mathfrak{G}$ $(\times)$.

Conversely, let $\mathfrak{G}$ $(\times)$ be now a given classic group; we make in the headline of Cayley's table of $\mathfrak{G}$ $(\times)$ any substitution

$$\beta \simeq \begin{pmatrix} X \\ \overline{X} \end{pmatrix}$$

and define a new operation $\star$ as follows:

$$A \times B = A \star \overline{B}.$$

The operation $\star$ is obviously uniform and uniformly reversible; the Postulate A is also true for $\star$; in fact, if

$$(X \star \overline{A}) \star \overline{B} = X \star \overline{C},$$

we have

$$(X \star \overline{A}) \star \overline{B} = (X \times A) \times B = X \times (A \times B) ;$$

and

$$X \star \overline{C} = X \times C ;$$

hence $A \times B = C$ and thus $\overline{C}$ depends upon $\overline{A}$ and $\overline{B}$ only.

E being the unit of $\mathfrak{G}$ $(\times)$, $\overline{E}$ is the right unit for $\mathfrak{G}$ $(\star)$; we have in fact $A \star \overline{E} = A \times E = A$.

I affirm that we can replace β by another substitution α, which does not alter E, and in this manner define a new operation, say $\square$, so that the group $\mathfrak{G}$ $(\square)$ will be simply isomorphic with $\mathfrak{G}$ $(\star)$ and have the right unit E. We take for α

$$\alpha \simeq \begin{pmatrix} X \\ \overline{X} \end{pmatrix} \begin{pmatrix} \overline{E} \star \overline{X} \\ \overline{X} \end{pmatrix} \begin{pmatrix} \overline{X} \\ X \end{pmatrix} = \begin{pmatrix} X \\ \overline{X} \end{pmatrix} \begin{pmatrix} \overline{E} \star \overline{X} \\ X \end{pmatrix} = \begin{pmatrix} X \\ \overline{X} \end{pmatrix} \begin{pmatrix} \overline{E} \times X \\ X \end{pmatrix} ;$$

let $\overline{E} \times X' = \overline{X}$; we can write then

$$\alpha = \begin{pmatrix} X \\ \overline{X} \end{pmatrix} \begin{pmatrix} \overline{X} \\ X' \end{pmatrix} = \begin{pmatrix} X \\ X' \end{pmatrix} ;$$

and so we define

$$A \times B = A \,\square\, B'.$$

We shall prove that the substitution

$$\alpha_1 \simeq \begin{pmatrix} \overline{X} \\ X' \end{pmatrix}$$

gives an isomorphism between the groups $\mathfrak{G}$ $(\star)$ and $\mathfrak{G}$ $(\Box)$. Let

$$\overline{A}\star\overline{B} = \overline{C}\,; \tag{5}$$

we shall prove that we shall have also

$$A' \,\Box\, B' = C'. \tag{6}$$

It follows from (5) that $\overline{A}\times B=\overline{C}$; but $\overline{A}=\overline{E}\times A'$, $\overline{C}=\overline{E}\times C'$; hence $(\overline{E}\times A')\times B=\overline{E}\times C'$; and since $\mathfrak{G}$ $(\times)$ is classic,

$$\overline{E} \times (A' \times B) = \overline{E} \times C'\,;$$

hence $A'\times B=C'$, and so (6) is established.

VI. Let $\mathfrak{H}=P_1+P_2+P_3+\cdots$,* $\mathfrak{H}$ $(\star)$ being a subgroup of $\mathfrak{G}$ $(\star)$. Let $(X\star P_\kappa)\star P_\lambda=X\star P_\mu$; the elements P_κ and P_λ of $\mathfrak{H}$ being given, the element P_μ exists also in $\mathfrak{H}$; by virtue of Postulate A we have $P_\kappa \circ P_\lambda=P_\mu$; hence $\mathfrak{H}$ $(\circ)$ is also a group.

It follows, hence, that Lagrange's theorem is true for the groups $\mathfrak{G}$ $(\star)$ of our type.

Let $\mathfrak{H}$ be now a subgroup of $\mathfrak{G}$ relative to $\circ$; we shall analyse the conditions by which $\mathfrak{H}$ is also a group relative to $\star$. Let α be the same substitution as in V, and

$$\mathfrak{H}' \simeq P_1' + P_2' + P_3' + \cdots .$$

(P_1', P_2', P_3', $\cdots$ are elements in $\mathfrak{G}$ corresponding to $P_1, P_2, P_3, \cdots$, by virtue of α.) Since α gives an isomorphism between $\mathfrak{G}$ $(\circ)$ and $\mathfrak{G}$ $(\times)$ ($\times$ being the operation defined by (3)), $\mathfrak{H}'$ $(\times)$ is also a group (relative to $\times$).

Let $\mathfrak{H}$ $(\star)$ be also a group; then

$$P_\kappa\star P_\lambda = P_\kappa \times P_\lambda' \simeq P_\mu .$$

If P_λ' runs over all elements of $\mathfrak{H}'$, then P_μ runs over all elements of $\mathfrak{H}$, and conversely. Hence

$$P_\kappa \times \mathfrak{H}' = \mathfrak{H}$$

(for each P_κ of $\mathfrak{H}$). Consequently $\mathfrak{H}$ is one of the partitions of $\mathfrak{G}$ $(\times)$ relative to $\mathfrak{H}'$ $(\times)$†. This condition is obviously also sufficient for $\mathfrak{H}$ $(\star)$ to be a group.

* The sign $+$ signifies that the elements P_1, P_2, $\cdots$ form a set $\mathfrak{H}$.

† Hilton, *An Introduction to the Theory of Groups of Finite Order*, Oxford, 1908, p. 58.

Since the substitution α does not alter the unit E of $\mathfrak{G}$ $(\times)$, $\mathfrak{H}$ and $\mathfrak{H}'$ must be identically equal to each other, because both of them have a common element E.

We shall now analyse the conditions by which *every* subgroup $\mathfrak{H}$ (o) of $\mathfrak{G}$ (o) is also a group relative to $\star$. Then we must have $\mathfrak{H}'=\mathfrak{H}$ (our notation remains as above) for every subgroup $\mathfrak{H}$ (o). We take $\mathfrak{H}'=\mathfrak{H}$ $(\times)=\{P\}$, a cyclic group, P being an arbitrary element of $\mathfrak{G}$. Since $\{P\}$ must be also a group relative to $\star$, we have

$$P^k\star P = P^k \times P^l; \tag{7}$$

consequently for each element X of $\mathfrak{G}$ also,

$$X\star P = X \times P^l. \tag{7'}$$

More generally,

$$X\star P^\kappa = X \times P^\lambda. \tag{8}$$

To every exponent κ in (8) there corresponds one and only one exponent λ and vice versa. This must be true for each element P of $\mathfrak{G}$; if we take P^k instead of P, we obtain, in the same manner as in (8),

$$X\star P^{k\mu} = X \times P^{k\nu}; \tag{9}$$

for every μ there is a definite ν and vice versa. Let m be the order of P, and d the greatest common divisor of k and m; then m/d is the order of P^k and each exponent $k\mu$ and $k\nu$ in (9) is divisible by d. Conversely, if one of the exponents κ, λ in (8) is prime to m, the other is also prime to m. Consequently the exponent l in (7) or (7') must be prime to m. Thus α has in this case the following form:

$$\alpha = \begin{pmatrix} X \\ X^l \end{pmatrix},$$

where the numbers l are prime to the orders of corresponding elements X. This condition is not only necessary but also sufficient: if it holds, then every cyclic subgroup $\{P\}$ of $\mathfrak{G}$ $(\times)$ is also a group relative to $\star$. But hence *every* subgroup $\mathfrak{H}$ $(\times)$ of $\mathfrak{G}$ $(\times)$ is also a group relative to $\star$. Q and P being any two elements of $\mathfrak{H}$, we have in fact $Q\star P=Q\times P^l$; thus $Q\star P$ belongs also to $\mathfrak{H}$.

We can take, in particular, a substitution α of the following form:

$$\alpha = \begin{pmatrix} X \\ X^\tau \end{pmatrix},$$

where τ is the same for each element X and relatively prime to the order of our group $\mathfrak{G}$.

3. We shall consider now a special case of Postulate A, that is, however, more general than the Associative Law.

POSTULATE B. In the equation

$$(X \star A) \star B = X \star (A \star B_1), \tag{10}$$

the elements B and B_1 depend only upon each other; every B is completely defined by the corresponding B_1, and conversely.

This postulate can be expressed in another form as follows:

POSTULATE B′. If

$$A \star B = C \star D \tag{11}$$

and if K is an arbitrary element, then

$$A \star (B \star K) = C \star (D \star K). \tag{12}$$

We prove first that Postulate B′ follows from Postulate B. Let R be an element such that

$$A \star (B \star K) = (A \star B) \star R ; \tag{13}$$

R depends upon K only (by Postulate B). Again it follows from (11) that

$$(A \star B) \star R = (C \star D) \star R ; \tag{14}$$

and by Postulate B it follows from (13) that

$$C \star (D \star K) = (C \star D) \star R ; \tag{15}$$

hence, from (11), (13), (14), (15) it follows that (12) holds.

Second, we prove that Postulate B follows also from the Postulate B′. For that purpose we shall prove the following lemma:

LEMMA: *If Postulate* B′ *is true for a* (*uniformly reversible*) *group* $\mathfrak{G}$ ($\star$), *there exists in* $\mathfrak{G}$ ($\star$) *a right unit* (*for all elements of* $\mathfrak{G}$ ($\star$)).

If B is a given element, there always exists in $\mathfrak{G}$ ($\star$) an element E such that

$$B \star E = B. \tag{16}$$

Let D be an arbitrary element of $\mathfrak{G}$ ($\star$); if A is also a given element, there always exists an element C, for which

$$A \star B = C \star D ; \tag{11}$$

by virtue of Postulate B′ we have, then,

$$A \star (B \star E) = C \star (D \star E) ; \tag{17}$$

and by virtue of (16) and (11) it follows from (17) that E is the right unit for *every* element D.

Assume now that $A\star B=C=C\star E$; in the hypothesis of Postulate B′ we have, K being an arbitrary element,

$$A\star(B\star K) = C\star(E\star K) = (A\star B)\star(E\star K) ;$$

and thus we have in (13) $E\star K=R$; this shows that Postulate B holds for our group.

Since the groups with Postulate B form a special case of groups with Postulate A, they can be obtained in the same manner as groups with Postulate A (§1, V). We must now examine what must be the substitution α (§1, V), in order that we may obtain a group $\mathfrak{G}$ $(\star)$ with Postulate B. The answer is given by the following theorem:

THEOREM. *If the group* $\mathfrak{G}$ $(\star)$ *is obtained from the classic group* $\mathfrak{G}$ $(\times)$ *by means of the substitution* α, *Postulate* B *is true for* $\mathfrak{G}$ $(\star)$ *if and only if* α *is an automorphism of the group* $\mathfrak{G}$ $(\times)$. *In this case* α *is also an automorphism for* $\mathfrak{G}$ $(\star)$, *and the operations* $\circ$ *and* $\times$ *coincide with each other.*

Let $\mathfrak{G}$ $(\star)$ be a group with Postulate B. The equation (10) gives a dependence of B and B_1 upon each other; this dependence is given by a substitution, that we denote symbolically by $\binom{X}{X_1}$. Let $A=E$ (the right unit of $\mathfrak{G}$ $(\star)$) in (10); then

$$X\star B = X\star(E\star B_1) ;$$

and hence

$$B = E\star B_1. \tag{18}$$

Let

$$\alpha \simeq \begin{pmatrix} X \\ X' \end{pmatrix};$$

we have then

$$B = E\star B_1 = E \times (B_1)' = (B_1)' ;$$

and hence

$$\alpha = \begin{pmatrix} X_1 \\ X \end{pmatrix} = \begin{pmatrix} X \\ X_1 \end{pmatrix}^{-1}$$

Moreover it follows from (10), if we use the notation $A\star B_1 \simeq C$, that

$$C = A\circ B = A\star B_1 = A \times (B_1)' = A \times B ;$$

thus the operations $\circ$ and $\times$ coincide.

Conversely, suppose that the operations $\circ$ and $\times$ coincide. Let

$$\alpha = \binom{X}{X'} = \binom{X_1}{X};$$

we have then

$$X{\star}A = X \times A'; \tag{19}$$

and

$$(X{\star}A){\star}B = X{\star}(A \times B) = X{\star}(A{\star}B_1),$$

and that is Postulate B, because B and B_1 depend only on each other. Again, by (19),

$$(X{\star}A){\star}B = X{\star}C = (X \times A') \times B' = X \times C';$$

and hence

$$A \times B = C, \quad A' \times B' = C';$$

this shows us that α is an automorphism of $\mathfrak{G}$ $(\times)$.

Conversely, let α be an automorphism of $\mathfrak{G}$ $(\times)$; then

$$(X{\star}A){\star}B = (X \times A') \times B' = X \times (A' \times B')$$
$$= X \times (A \times B)' = X{\star}(A \times B) = X{\star}(A{\star}B_1);$$

and thus Postulate B holds.

It remains to prove that α is in this case an automorphism of $\mathfrak{G}$ $(\star)$ also. We have in fact

$$(A{\star}B)' = (A \times B')' = A' \times (B')' = A'{\star}B'.$$

4. In the theory of uniformly reversible groups we can consider the operations inverse to the operation of a given group. Since the operation of our group is performed upon *two* elements (viz. $X{\star}Y$), two inverse operations exist according as the left or the right of these two elements is unknown to us.

If the commutative law is true for our group, such a group has only *one* inverse operation and only one "inverse group" (i.e. the group relative to the inverse operation). But although a general classic group has two "inverse groups," it has only one inverse *operation* (abstractly considered), because the properties of the operation of a classic group are "symmetric," i.e. the same on both sides; two "inverse groups" of a classic group are simply isomorphic to each other (if our notations are conveniently chosen); this follows from the fact that a classic group is always "anti-isomorphic" to itself, i.e. there always exists such a substitution $\binom{X}{\overline{X}}$ of elements of a classic group, that if A, B correspond respectively to $\overline{A}$, $\overline{B}$, then AB corresponds to $\overline{B}\overline{A}$; we can take, for example, $\overline{X} = X^{-1}$.

The operation of a finite classic group $\mathfrak{G}$ may be denoted by $\times$, the two inverse operations by $\triangle$ and ∇; more precisely,

$$\text{if} \quad A \times B = C, \quad \text{then} \quad C \triangle B = A, \quad C \nabla A = B.$$

Both inverse groups $\mathfrak{G}\ (\triangle)$, $\mathfrak{G}\ (\nabla)$ are finite and uniformly reversible but not associative. Let us consider what influence the associative law of the operation $\times$ makes on the operations $\triangle$ and ∇. Let $(A \times B) \times C = A \times (B \times C) \simeq R$; $A \times B \simeq P$; $B \times C \simeq Q$; then $P \times C = A \times Q = R$. Hence $P \triangle B = A$, $Q \triangle C = B$, $R \triangle C = P$, $R \triangle Q = A$; consequently

$$(R \triangle C) \triangle B = R \triangle Q, \quad \text{and} \quad B \times C = Q.$$

[Or $P \nabla A = B$, $Q \nabla B = C$, $R \nabla P = C$, $R \nabla A = Q$, $(R \nabla A) \nabla B = R \nabla P$, and $P = A \times B$.] This is Postulate A, that is true for the operation $\triangle$ (and for ∇). But the operations $\triangle$ and ∇ are subject to still another postulate, viz.:

POSTULATE J. Every element X satisfies the equation

$$X \triangle X = E \quad (\text{or} \quad X \nabla X = E),$$

where E is a determined element (the unit of the direct operation $\times$).

THEOREM 1. *A finite uniformly reversible group $\mathfrak{G}\ (\star)$ is an "inverse" to a classic group, if and only if it is subject to the postulates* A *and* J.

Only one part of this theorem remains for us to prove. Let $\mathfrak{G}\ (\star)$ be subject to the postulates A and J. We use the same notation as before; if $A \star B = C$, then $C \triangle B = A$. We must prove that $\mathfrak{G}\ (\triangle)$ is classic. Obviously the operation $\triangle$ is uniform and uniformly reversible. Again, we have $(X \star A) \star B = X \star C \simeq Z$; C depends upon A and B only; let $X \star A \simeq Y$; then $Y \star B = X \star C = Z$; $Z \triangle C = X$, $Z \triangle B = Y$, $Y \triangle A = X$; thus

$$(Z \triangle B) \triangle A = Z \triangle C,$$

which is Postulate A for the operation $\triangle$. It follows from Postulate J, that the group $\mathfrak{G}\ (\triangle)$ has a left unit E; and hence (see §1, III) $\mathfrak{G}\ (\triangle)$ is classic.

We consider a special case, when our classic group is abelian. We obtain then

THEOREM 2. *A finite uniformly reversible group $\mathfrak{G}\ (\star)$ is an "inverse" to an abelian group, if and only if it is subject to the postulates* B *and* J.

Let $\mathfrak{G}\ (\star)$ be subject to the postulates B and J; by the preceding theorem the inverse group $\mathfrak{G}\ (\triangle)$ is classic; it remains for us to show that $\mathfrak{G}\ (\triangle)$ is commutative. We have

$$(X \star A) \star B = X \star (A \star B_1), \tag{10}$$

B and B_1 depending only upon each other. Let $A{\star}B_1 = C$; then (as in the preceding theorem) $B \triangle A = C$, $C \triangle B_1 = A$; hence

$$B \triangle A \triangle B_1 = A. \tag{20}$$

We write this without brackets, because the Associative Law is true for $\triangle$; (20) is true for each element A; we take $A = E$ (unit); then $B \triangle B_1 = E$; $B_1 = B^{-1}$; and thus from (20) it follows that $A \triangle B = B \triangle A$; i. e., the Commutative Law holds for $\triangle$.

Conversely, let $\mathfrak{G}$ $(\triangle)$ be an abelian group; we must prove that B and B_1 in (10) depend only upon each other. But (20) gives $A{\star}B^{-1} = B \triangle A = C$; hence $B_1 = B^{-1}$ in (10), and Postulate B holds for $\star$.

The postulates B and J are characteristic for the operation of division. Thus it is possible, for example, to construct an abstract theory of proportions.

Supplement

Example I. A group with Postulate A but not classic (see Table 1). This group is obtained from the symmetric group of 6th order by making in the head-line of Cayley's table of this group (see Table 2) the following substitution:

$$\begin{pmatrix} E\,A\,B\,C\,D\,F \\ E\,C\,D\,A\,F\,B \end{pmatrix}.$$

Example II. A group with Postulate B but not classic (see Table 3). This group is obtained from the same symmetric group by making in the head-line of Table 2 the following substitution:

$$\begin{pmatrix} E\,A\,B\,C\,D\,F \\ E\,B\,A\,C\,F\,D \end{pmatrix}.$$

This substitution gives an automorphism of the symmetric group of 6th order.

	E	A	B	C	D	F
E	E	C	F	A	B	D
A	A	D	B	E	F	C
B	B	F	C	D	E	A
C	C	E	A	F	D	B
D	D	A	E	B	C	F
F	F	B	D	C	A	E

Table 1

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	F	D	C	B
B	B	D	E	F	A	C
C	C	F	D	E	B	A
D	D	B	C	A	F	E
F	F	C	A	B	E	D

Table 2

	E	A	B	C	D	F
E	E	B	A	C	F	D
A	A	F	E	D	B	C
B	B	E	D	F	C	A
C	C	D	F	E	A	B
D	D	C	B	A	E	F
F	F	A	C	B	D	E

Table 3

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