A NOTE ON CLOSEST APPROXIMATION*

BY

DUNHAM JACKSON

Let $f(x)$ be a given continuous function of period 2π, and $T_n(x)$ a trigonometric sum of the nth order, ultimately to be chosen so as to give an approximate representation of $f(x)$, but for the present arbitrary. A numerical measure of the discrepancy between $f(x)$ and $T_n(x)$ can be defined in an infinite variety of ways. One of the most important of such measures is the maximum of the absolute value of $f(x)-T_n(x)$; another is the integral of the square of this difference, extended over an interval of length 2π. If $T_n(x)$ is chosen so as to minimize the latter quantity, it is the partial sum of the Fourier series for $f(x)$. If it minimizes the former, it is the approximating sum in the sense of Tchebychef.

This note is concerned with a measure of discrepancy which is in a certain sense qualitatively intermediate between the two preceding ones, representing a transition from one to the other. It is defined by forming the integral of the square of the error over an interval of length h, where h is a given positive number, in general different from 2π, and then taking the maximum of this integral, regarded as a function of the position of the initial point of the interval.

Let the maximum of the integral, for given $T_n(x)$ and given h, be denoted by I_{nh}. The particular approximating function to be studied is the sum $T_{nh}(x)$ characterized among all trigonometric sums of the nth order as the one which makes I_{nh} a minimum. It reduces to the Fourier sum for $h=2\pi$, the value of the integral being then independent of the situation of the interval, and it approaches the Tchebychef sum (as will be shown presently) when h approaches zero. In view of the fact that (for $0<h<2\pi$) it occupies this intermediate position with respect to well known approximating sums, it is not surprising that $T_{nh}(x)$ converges uniformly toward $f(x)$, under fairly general hypotheses, if h is held fast and n is allowed to become infinite. The main purpose of the paper is to prove that this is the case, and incidentally to point out that the method is susceptible of much more general application. (As far as the validity of the reasoning is concerned, values of h greater than 2π, as well as those less than 2π, may be admitted from the beginning.)

---

* Presented to the Society at Chicago, April 7, 1928; received by the editors in August, 1928.

The existence of a sum $T_{nh}(x)$ with the required minimizing property follows almost immediately from well known existence proofs for the case in which the integral is extended over a period. As the whole problem is trivial when $f(x)$ is identically a trigonometric sum of the nth order, it will be understood that this case is ruled out. It is readily seen that I_{nh} is a continuous function of the coefficients in $T_n(x)$; and then the essential thing is to show that the assignment of an upper bound for I_{nh} prescribes at the same time an upper bound for the magnitude of the coefficients, so that the range within which the minimizing coefficients are to be sought may be regarded as closed. To say that an upper bound is fixed for the size of the coefficients is the same thing as saying that an upper bound is prescribed for the maximum of the absolute value of $T_n(x)$ itself.* The existence of the desired upper bound for $|T_n(x)|$ is known† when $h=2\pi$. More definitely, if

$$\int_0^{2\pi}[f(x) - T_n(x)]^2dx \leqq G,$$

then

$$\max|T_n(x)| \leqq QG^{1/2}, \tag{1}$$

where Q is independent of the coefficients (being fixed when $f(x)$ and n are given).‡ If $h>2\pi$, and if

$$\int_a^{a+h}[f(x) - T_n(x)]^2dx \leqq G,$$

the integral over a period satisfies the same inequality, and (1) follows at once. When $h<2\pi$, let p be an integer such that $ph\geqq 2\pi$. If $I_{nh}\leqq G$, the integral of the square of the error over any interval of length less than or equal to h can not exceed G; a period can be divided into p such intervals, and consequently

$$\int_0^{2\pi}[f(x) - T_n(x)]^2dx \leqq pG,$$

whence max $|T_n(x)|\leqq Q(pG)^{1/2}$, a relation which serves as well as (1) for

* The equivalence of the two statements is well known; for a generalization due to Sibirani, see for example the passage cited in the next footnote.

† For a formulation covering the case now in question and others to be referred to later, see for example D. Jackson, *A generalized problem in weighted approximation*, these Transactions, vol. 26 (1924), pp. 133–154; pp. 133–139.

‡ This form of statement is not taken directly from the passage cited, but is readily deduced from it.

the purpose in hand. So the existence of the minimizing $T_{nh}(x)$ is assured.

The uniqueness of the minimizing sum can be proved by an argument already used in connection with numerous other problems. Let γ_{nh} denote the minimum of I_{nh}. Suppose there were two distinct sums $T_{nh1}(x)$ and $T_{nh2}(x)$, for each of which the corresponding I_{nh} is equal to γ_{nh}. Let

$$T_{nh3}(x) = \tfrac{1}{2}(T_{nh1} + T_{nh2}).$$

Then

$$f(x) - T_{nh3}(x) = \tfrac{1}{2}\{[f(x) - T_{nh1}(x)] + [f(x) - T_{nh2}(x)]\},$$

and

$$[f(x) - T_{nh3}(x)]^2 \leqq \tfrac{1}{2}\{[f(x) - T_{nh1}(x)]^2 + [f(x) - T_{nh2}(x)]^2\},$$

the equality being ruled out wherever $T_{nh1} \neq T_{nh2}$, and being possible therefore only for a finite number of values of x at most in any finite interval. For any specified interval of length h, the integral of either of the squares on the right can not exceed the corresponding I_{nh}, and so can not exceed γ_{nh}; for any interval $(a, a+h)$, therefore, by virtue of the fact that the inequality holds almost everywhere,

$$\int_a^{a+h} [f(x) - T_{nh3}(x)]^2 dx < \gamma_{nh},$$

which contradicts the definition of γ_{nh}.

It may be pointed out next that for given n the sums $T_{nh}(x)$ are uniformly bounded for all values of h. Let M be the maximum of $|f(x)|$, and let M_{nh} be the maximum of $|T_{nh}(x)|$. From its definition, the quantity γ_{nh}, the maximum of the integral of $[f(x) - T_{nh}(x)]^2$ over an interval of length h, can not exceed the corresponding maximum I_{nh} formed for the particular trigonometric sum $T_n(x) \equiv 0$. But this I_{nh} is merely the integral of $[f(x)]^2$ over some interval of length h, and can not exceed M^2h. So

$$\gamma_{nh} \leqq M^2 h \tag{2}$$

for all values of h.

In the relatively trivial case that $n=0$, $T_{nh}(x) = \text{constant}$, it is seen at once that $|T_{nh}| \leqq 2M$. For if $|T_{nh}|$ were greater than $2M$, $|f(x) - T_{nh}|$ would be everywhere greater than M, and the integral of its square over any interval of length h would be greater than M^2h, in violation of (2).

To proceed to the calculation of an upper bound for M_{nh} when $n>0$, suppose first that $0<h\leqq 1/n$. In this case it is to be shown that $M_{nh} \leqq 4M$. Let x_0 be a value such that $|T_{nh}(x_0)| = M_{nh}$. By Bernstein's theorem, $|T'_{nh}(x)| \leqq nM_{nh}$, and for $|x - x_0| \leqq 1/(2n)$,

(3) $\quad |T_{nh}(x) - T_{nh}(x_0)| \leqq nM_{nh}|x - x_0| \leqq \frac{1}{2}M_{nh}, \quad |T_{nh}(x)| \geqq \frac{1}{2}M_{nh}.$

Let it be assumed, as a supposition to be disproved, that $M_{nh} > 4M$. Then (3) means that $|T_{nh}(x)| > 2M$ throughout an interval of length $1/n$, and so throughout an interval of length h, and in this interval $|f(x) - T_{nh}(x)| > M$. But this implies that $\gamma_{nh} > M^2 h$, in contradiction with (2). So M_{nh} has the upper bound $4M$, independent of h, for the indicated range of values of h.

Secondly, suppose $1/n \leqq h \leqq 2\pi$. From (2), $\gamma_{nh} \leqq M^2 h \leqq 2\pi M^2$. Let p be an integer greater than $2\pi n$, for definiteness the smallest such integer. Then the interval $(0, 2\pi)$ can be subdivided into p intervals, each of length less than $1/n$, and so less than h. But the integral of $[f(x) - T_{nh}(x)]^2$ over any such interval can not exceed γ_{nh}, and therefore

$$\int_0^{2\pi} [f(x) - T_{nh}(x)]^2 dx \leqq p\gamma_{nh} \leqq 2p\pi M^2.$$

Consequently

$$M_{nh} \leqq (2p\pi)^{1/2} QM,$$

where Q has the same meaning as in (1).

Let it be supposed finally that $2k\pi \leqq h \leqq 2(k+1)\pi$, where k is a positive integer. Then

$$\int_0^{2k\pi} [f(x) - T_{nh}(x)]^2 dx \leqq \int_0^{h} [f(x) - T_{nh}(x)]^2 dx \leqq \gamma_{nh}$$
$$\leqq M^2 h \leqq 2(k+1)\pi M^2,$$

and

$$\int_0^{2\pi} [f(x) - T_{nh}(x)]^2 dx = (1/k) \int_0^{2k\pi} [f(x) - T_{nh}(x)]^2 dx$$
$$\leqq 2(k+1)\pi M^2/k \leqq 4\pi M^2,$$

so that

$$M_{nh} \leqq (4\pi)^{1/2} QM.$$

If M_0 is defined as the largest of the quantities $4M$, $(2p\pi)^{1/2}QM$, $(4\pi)^{1/2}QM$ (in other words, the larger of the first two of these quantities, since the third is evidently less than the second), this M_0 is independent of h, and $M_{nh} \leqq M_0$ throughout.

It is possible now to justify the assertion made in one of the introductory paragraphs, that $T_{nh}(x)$ approaches the Tchebychef sum when h approaches zero. Let the Tchebychef sum, reducing the maximum of the error to the smallest possible value, be denoted by $Z_n(x)$. Let ϵ be an arbitrary positive quantity. For the moment, let $U_n(x)$ be a general notation for trigonometric sums of the nth order subject to the restriction that at least one coefficient differs from the corresponding coefficient in $Z_n(x)$ by not less than ϵ. Let $Z_{n1}(x)$ be chosen among all sums $U_n(x)$ as one for which the maximum of $|f(x)-Z_{n1}(x)|$ has the smallest possible value. It is clear from the considerations leading to the proof of existence of the Tchebychef sum that there is a $Z_{n1}(x)$ having the required minimizing property; it is immaterial for the present argument whether the minimum is unique or not. Let L be the maximum of $|f(x)-Z_n(x)|$, and $L_1=L+\eta$ the maximum of $|f(x)-Z_{n1}(x)|$. Since the Tchebychef sum at any rate is unique, it is certain that $\eta>0$. Being continuous and periodic, $f(x)$ is uniformly continuous; let δ_1 be a positive number such that $|f(x')-f(x'')|\leqq\frac{1}{4}\eta$ whenever $|x'-x''|\leqq\delta_1$. Let δ be the smaller of the numbers δ_1 and $\eta/(4nM_0)$, where M_0 is defined as above. It will be shown that if $h<\delta$, $T_{nh}(x)$ can not belong to the class of sums $U_n(x)$; that is, to every $\epsilon>0$ it is possible to assign a $\delta>0$ such that if $h<\delta$, no coefficient in $T_{nh}(x)$ can differ from the corresponding coefficient in $Z_n(x)$ by so much as ϵ.

For the sake of arriving at a contradiction, suppose $T_{nh}(x)$ is of the form $U_n(x)$, while $h<\delta$. There must be a point x_0 such that $|f(x_0)-T_{nh}(x_0)|\geqq L+\eta$. For $|x-x_0|\leqq\delta$, $|f(x)-f(x_0)|\leqq\frac{1}{4}\eta$. On the other hand, $|T_{nh}(x)|\leqq M_0$, and $|T'_{nh}(x)|\leqq nM_0$. For $|x-x_0|\leqq\delta$, therefore,

$$|T_{nh}(x)-T_{nh}(x_0)|\leqq nM_0\delta\leqq\tfrac{1}{4}\eta,$$

whence it follows further that

$$|[f(x)-f(x_0)]-[T_{nh}(x)-T_{nh}(x_0)]|\leqq\tfrac{1}{2}\eta,$$
$$|f(x)-T_{nh}(x)|\geqq L+\tfrac{1}{2}\eta.$$

The last relation holds throughout an interval of length δ, and so throughout an interval of length h. Consequently γ_{nh}, the maximum of the integral of $[f(x)-T_{nh}(x)]^2$ over an interval of length h, is at least $h(L+\frac{1}{2}\eta)^2$. But if the corresponding maximum I_{nh} is formed for the sum $T_n(x)\equiv Z_n(x)$, the error never exceeds L, and the integral I_{nh} can not exceed hL^2. This contradicts the definition of γ_{nh} as the smallest value of I_{nh}. So the facts are as originally stated.

The proof of convergence for $n=\infty$ is analogous to others which the writer has given in recent papers. Let $t_n(x)$ be an arbitrary trigonometric sum of the nth order, and ϵ_n the maximum of $|f(x)-t_n(x)|$. Let

$$f(x)-t_n(x)=r_n(x),\qquad T_{nh}(x)-t_n(x)=\tau_n(x)\ ;$$

it is to be understood that h is held fast throughout the present demonstration. Let μ_n be the maximum of $|\tau_n(x)|$, and x_0 a point such that $|\tau_n(x_0)|$

$=\mu_n$. According to Bernstein's theorem, $|\tau_n'(x)|\leqq n\mu_n$, and, for values of x in the interval $|x-x_0|\leqq 1/(2n)$,

$$|\tau_n(x)-\tau_n(x_0)|\leqq n\mu_n|x-x_0|\leqq \tfrac{1}{2}\mu_n, \qquad |\tau_n(x)|\geqq \tfrac{1}{2}\mu_n.$$

As one of two conceivable alternatives, suppose $\mu_n\geqq 4\epsilon_n$. Then $|r_n(x)|\leqq\epsilon_n\leqq\frac{1}{4}\mu_n$, and

$$|r_n(x)-\tau_n(x)|\geqq \tfrac{1}{4}\mu_n$$

throughout the interval specified. The length of this interval is $1/n$, *which is less than h, as soon as n is sufficiently large.* The integral of $[r_n(x)-\tau_n(x)]^2$ over the interval is at least $(\frac{1}{4}\mu_n)^2/n$. But from the definitions of the quantities involved, $r_n(x)-\tau_n(x)$ is identical with $f(x)-T_{nh}(x)$, and the integral of $[f(x)-T_{nh}(x)]^2$ over any interval of length $\leqq h$ can not exceed γ_{nh}. Consequently

$$(\tfrac{1}{4}\mu_n)^2/n\leqq\gamma_{nh}, \qquad \mu_n\leqq 4(n\gamma_{nh})^{1/2}.$$

The other alternative is to suppose directly that $\mu_n<4\epsilon_n$. In either case,

$$\mu_n\leqq 4(n\gamma_{nh})^{1/2}+4\epsilon_n,$$

for all values of n from a certain point on. Since μ_n is the maximum of $|\tau_n(x)|$, and since $|r_n(x)|\leqq\epsilon_n$,

$$|f(x)-T_{nh}(x)|\equiv|r_n(x)-\tau_n(x)|\leqq 4(n\gamma_{nh})^{1/2}+5\epsilon_n.$$

The function $f(x)$ being continuous, it is certainly possible, by Weierstrass's theorem, to choose sums $t_n(x)$ for the successive values of n so that $\lim_{n=\infty}\epsilon_n=0$. *For uniform convergence of $T_{nh}(x)$ toward $f(x)$, therefore, it is sufficient that* $\lim_{n=\infty} n\gamma_{nh}=0$.

Let the modulus of continuity of $f(x)$ be denoted by $\omega(\delta)$. It is known* that for all positive integral values of n trigonometric sums of the nth order can be constructed so as to differ from $f(x)$ by less than a constant multiple of $\omega(2\pi/n)$ for all values of x. If the corresponding quantity I_{nh} is calculated in each case, it does not exceed a constant multiple of $[\omega(2\pi/n)]^2$. Hence

$$\gamma_{nh}\leqq C[\omega(2\pi/n)]^2,$$

where C is independent of n. *The sums $T_{nh}(x)$ will converge uniformly toward $f(x)$, if $f(x)$ has a modulus of continuity $\omega(\delta)$ such that* $\lim_{\delta=0}\omega(\delta)/\delta^{1/2}=0$.

---

* See for example D. Jackson, *On the approximate representation of an indefinite integral* . . . these Transactions, vol. 14 (1913), pp. 343–364; p. 350.

Some of the generalizations that come to mind may be mentioned though without any attempt at exhaustiveness. The generalizations are concerned for the most part with the existence and uniqueness of an approximating function for given n, and its convergence toward $f(x)$ as n becomes infinite.

As a matter of course, the square of the error may be replaced by the mth power of its absolute value, in the integrals leading to the definition of the approximating sums. In the proof of uniqueness it is to be assumed, for simplicity at least, that $m>1$; in the convergence theorem (valid for $m>0$) the hypothesis on $f(x)$ is to be appropriately modified. While the direct connection with the Fourier sum lapses, it is still possible in this case to show that the approximating function approaches the Tchebychef sum as a limit when h approaches zero.

A weight function may be introduced in the integrals, the assignment of weights being regarded as fixed either with respect to the origin from which x is measured, or with respect to the initial point of the interval of integration; that is, I_{nh} may be defined as the maximum of

$$\int_a^{a+h} \rho(x)[f(x) - T_n(x)]^2 dx$$

or of

$$\int_a^{a+h} \rho(x-a)[f(x) - T_n(x)]^2 dx$$

as a varies, ρ being a given bounded and measurable function of its argument with a positive lower bound.

The maximum of the integral over a single interval of length h may be replaced by the maximum of the sum of the integrals over N intervals of given aggregate length $h<2\pi$, where N is a given integer, and the intervals are supposed non-overlapping and contained within a single period. (Intervals having a common end point are admitted, and accordingly a set of fewer than N intervals is to be regarded for the purposes of the definition as a special case of a set of N intervals, since one interval can be subdivided to make up the requisite number.)

As an alternative, the class of all intervals of length h, among which a maximum value for the integral is to be sought in defining I_{nh}, may be replaced by the class of all point sets congruent to a given measurable set, the assumption being made, for simplicity at any rate, that this set contains at least one interval.

Finally, let $h_1, h_2, \cdots, h_N$ and $c_1, c_2, \cdots, c_N$ be two given sets of N

positive numbers each; and let the quantities I_{nh} calculated in accordance with the original definition for intervals of lengths $h_1, \cdots, h_N$ respectively be denoted by $I_{n1}, \cdots, I_{nN}$. Then an approximating sum $T_n(x)$ may be determined so as to minimize the combination $c_1I_{n1}+c_2I_{n2}+\cdots+c_NI_{nN}$. (This is suggested as a generalization of the original definition for $h>2\pi$; for example, if $h_1=2\pi$, $h_2<2\pi$, $c_1=c_2=1$, the new definition is equivalent to the old one as formulated for a single interval of length $2\pi+h_2$.)

In all these cases, proofs of existence, uniqueness, and convergence can be carried through without any considerable modification of the treatment given in the text.

University of Minnesota,
Minneapolis, Minn.