GENERALIZED FACTORIAL SERIES*

BY

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In this paper I consider series of the type

$$(1) \qquad c_0 + \sum_{n=1}^{\infty} c_n[\lambda_1\lambda_2 \cdots \lambda_n][(z + \lambda_1)(z + \lambda_2) \cdots (z + \lambda_n)]^{-1}$$

where $z=x+yi$ is a complex variable and the c_n's a sequence of complex constants. Unless otherwise specified $\lambda_n = \rho_n + \sigma_n i$, where $\rho_n \to \infty$, $\sum_{n=1}^{\infty} (\rho_n)^{-1}$ diverges and $\sigma_n/\rho_n \to 0$.

Probably the most interesting case is when λ_n is real. This was treated by Landau in what is now a classical memoir.† Theorem V of the present paper, even for the case that λ_n is real, is a wide extension of Landau's most fundamental result. He proves uniform convergence over a finite region, whereas it is proved here for a sectorial region‡ bounded by polynomial curves of arbitrary degree. For complex values of λ_n Theorem IV does not require that $\sum_{n=1}^{\infty} (\sigma_n/\rho_n)^2$ converge and in this respect is more general than V. Theorems VI and VII on absolute convergence seem also to be new. Regions of absolute convergence have been established§ but not so general as those found here. In §3 so-called equiconvergence theorems are proved for (1) and related exponential series.|| It is believed that the results given there have not previously been obtained.

For an excellent introduction to the literature see the article in the Encyclopaedia¶ by Hilb and Szász.

1. An important rôle in several sections of this paper is played by the following lemma from the general theory of infinite series. Proof of the lemma is easy and is not given here.

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† Münchener Berichte, vol. 36 (1906), p. 151–218.

‡ Compare Pincherle, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 379–390.

§ Pincherle, loc. cit., p. 386, also F. Nevanlinna, Annales Academiae Scientiarum Fennicae, (A), vol. 18, No. 3, p. 8.

|| Compare Landau, loc. cit., p. 201; Pincherle, loc. cit., p. 386; Hille, Annals of Mathematics, (2), vol. 25, pp. 276–278.

¶ Encyklopädie der Mathematischen Wissenschaften, Band II_3 H_8, pp. 1268–72.

LEMMA 1. *Hypotheses*: (a) $a_n(z)$ *and* $b_n(z)$, $n=1, 2, \cdots$, *are defined at all points of a region*, R; (b) $\sum_{n=1}^{\infty} a_n(z)$ *converges uniformly over* R; (c) *there exist positive constants* n' *and* M *such that when* $m \geqq n'$, $\sum_{n=m}^{n} |\Delta b_n(z)| < M$; (d) *there exists a positive constant* M' *such that* $|b_{n'}(z)| < M'$.

Conclusion: $\sum_{n=1}^{\infty} a_n(z)\, b_n(z)$ *converges uniformly over* R.

The part of $b_n(z)$ will be played in the fundamental theorems of this section by

$$b_n^{(\lambda)}(z_0,z) = (z_0+\lambda_1)(z_0+\lambda_2)\cdots(z_0+\lambda_n)[(z_0+\lambda_1)(z+\lambda_2)\cdots(z+\lambda_n)]^{-1}.$$

We proceed to prove preliminary theorems (I, II, and III) relative to $b_n^{(\lambda)}(z_0, z)$.

PRELIMINARY THEOREM I. *If* $x_0 \geqq 0$ *and* $\rho_i > 0$ *for all values of* i *and in case* $\sum_{n=1}^{\infty} (y_0+\sigma_n)^2 (x_0+\rho_n)^{-2}$ *converges, then* (1) $b_n^{(\lambda)}(z_0, z)$ *remains uniformly finite in* z *over the half-plane, defined by* $x \geqq x_0$, (2) *given any positive constants* a *and* b *and positive integer* m *there exists a corresponding constant* N, *such that*

$$|b_m^{(\lambda)}(z_0,z)| < N/\{a(x-x_0)^m + b\}$$

over the half-plane, defined by $x \geqq x_0$.

We have

$$|b_n^{(\lambda)}(z_0,z)| = \frac{|z_0+\lambda_1|\cdots|z_0+\lambda_n|}{(x_0+\rho_1)\cdots(x_0+\rho_n)}\cdot\frac{(x+\rho_1)\cdots(x+\rho_n)}{|z+\lambda_1|\cdots|z+\lambda_n|}\cdot b_n^{(\rho)}(x_0,x)$$

$$\leqq \{[1+([y_0+\sigma_1]/[x_0+\rho_1])^2]\cdots[1+([y_0+\sigma_n]/[x_0+\rho_n])^2]\}^{1/2} \cdot b_n^{(\rho)}(x_0,x) \leqq \overline{N} b_n^{(\rho)}(x_0,x) \leqq \overline{N}.$$

Also

$$|b_m^{(\lambda)}(z_0,z)| \leqq \overline{N} b_m^{(\rho)}(x_0,x) \leqq \overline{N}(x_0+\rho_1)\cdots(x_0+\rho_m)/\{x^m + \rho_1\cdots\rho_m\}$$
$$\leqq c\overline{N}(x_0+\rho_1)\cdots(x_0+\rho_m)/\{cx^m + c\rho_1\cdots\rho_m\} \leqq N/\{a(x-x_0)^m + b\}$$

if

$$c > a, \quad c\rho_1\cdots\rho_m > b \text{ and } N > c\overline{N}(x_0+\rho_1)\cdots(x_0+\rho_m).$$

PRELIMINARY THEOREM II. *If* $x_0 \geqq 0$ *and* $\rho_i > 0$ *for all values of* i, *and in case* $\sum_{n=1}^{\infty} (y_0+\sigma_n)^2 \cdot (x_0+\rho_n)^{-2}$ *converges; then* $\sum_{n=m}^{n} |\Delta b_n^{(\lambda)}(z_0, z)|$ *remains uniformly finite over the sectorial region defined by* $x \geqq x_0$ *and* $|y-y_0| \cdot (x-x_0)^{-1} \leqq a(x-x_0)^m + b$, *where* a *and* b *are any positive constants.*

We have

$$
\begin{aligned}
|\Delta b_n^{(\lambda)}(z_0,z)| &= \frac{|z_0 - z|}{|z+\lambda_{n+1}|}\cdot |b_n^{(\lambda)}(z_0,z)| \\
&\leqq -\overline{N}\frac{|z-z_0|}{x-x_0}\cdot\frac{x+\rho_{n+1}}{|z+\lambda_{n+1}|}\cdot(x_0-x)b_n^{(\rho)}(x_0,x)\cdot\frac{1}{x+\rho_{n+1}} \qquad (2)\\
&\leqq -\overline{N}\frac{|z-z_0|}{x-x_0}\Delta b_n^{(\rho)}(x_0,x).
\end{aligned}
$$

From this

$$\sum_{n=m}^{n}|\Delta b_n^{(\lambda)}(z_0,z)| \leqq \overline{N}\frac{|z-z_0|}{x-x_0}(b_m^{(\rho)}(x_0,x) - b_n^{(\rho)}(x_0,x))$$

$$\leqq \overline{N}\frac{|z-z_0|}{x-x_0}b_m^{(\rho)}(x_0,x) \leqq \frac{|z-z_0|}{x-x_0}\cdot\frac{N}{a_1(x-x_0)^m+b_1} \leqq N$$

if $|z-z_0|\cdot(x-x_0)^{-1}<a_1(x-x_0)^m+b_1$. Choose a_1, and b_1, so that $a_1(x-x_0)^m+b_1>\{[a(x-x_0)^m+b]^2+1\}^{1/2}$, and the theorem follows.

Preliminary Theorem III. *If* $x_0>0$ *and* $\rho_i>0$ *for all values of* i, $\sum_{n=1}^{n}|\Delta b_n^{(\lambda)}(z_0,z)|$ *remain uniformly finite over the sector,* S, *defined by* $x\geqq x_0$, *and* $|y-y_0|\cdot(x-x_0)^{-1}\leqq\tan\omega$, *where* ω *is a fixed positive angle less than* $\pi/2$.

We have

$$|\Delta b_n^{(\lambda)}(z_0,z)| = \frac{|z-z_0|}{|z_1+\lambda_{n+1}|-|z_0+\lambda_{n+1}|}\cdot\Delta|b_n^{(\lambda)}(z_0,z)|.$$

We proceed to show that it is possible to choose a fixed N and M so that, when $n>M$,

$$\left|\frac{|z-z_0|}{|z+\lambda_{n+1}|-|z_0+\lambda_{n+1}|}\right|<N \qquad (3)$$

over S. Replace z by $x+yi$, z_0 by x_0+y_0i, and λ_{n+1} by $\rho_{n+1}+\sigma_{n+1}i$. Then by elementary algebra we find the following inequality equivalent to (3):

$$
\begin{aligned}
&4\rho_{n+1}^2\left[N^4-N^2\left(1+\left(\frac{y-y_0}{x-x_0}\right)^2\right)\right]+4\rho_{n+1}\left[N^4-N^2\left(1+\left(\frac{y-y_0}{x-x_0}\right)^2\right)\right](x+x_0)\\
&\quad+N^4(x+x_0)^2+N^4\left[\frac{y-y_0}{x-x_0}(y+y_0+2\sigma_{n+1})\right]^2\\
&\quad+4N^4\rho_{n+1}\left[\frac{y-y_0}{x-x_0}(y+y_0+2\sigma_{n+1})\right] \qquad (4)\\
&\quad+2N^4\left[\frac{y-y_0}{x-x_0}(y+y_0+2\sigma_{n+1})\right](x+x_0)\\
&\quad-2N^2[x^2+y^2+x_0^2+y_0^2+2\sigma_{n+1}(y+y_0)+2\sigma_{n+1}^2]\left[1+\left(\frac{y-y_0}{x-x_0}\right)^2\right]\\
&\quad+[(x-x_0)^2+(y-y_0)^2]\left[1+\left(\frac{y-y_0}{x-x_0}\right)^2\right]>0,
\end{aligned}
$$

which holds if

$$\begin{aligned}&4\rho_{n+1}^2[N^4 - N^2 \sec\omega] + 4\rho_{n+1}[N^4 - N^2\sec\omega](x + x_0)\\ &+ N^4(x + x_0)^2 - 8N^4\rho_{n+1}(\tan\omega)\cdot[\,|y_0| + |\sigma_{n+1}|\,]\\ &- 4N^4(|y_0| + |\sigma_{n+1}|)(\tan\omega)(x + x_0)\\ &- 2N^2[x^2 + y^2 + x_0^2 + y_0^2 + 2\sigma_{n+1}(y + y_0) + 2\sigma_{n+1}^2]\sec\omega > 0.\end{aligned}$$

Replace y by $|y_0|+m(x-x_0)$ and rearrange the terms. We get

$$\begin{aligned}&\{4\rho_{n+1}^2[N^4 - N^2\sec\omega] - 8N^4\rho_{n+1}(\tan\omega)(|y_0| + |\sigma_{n+1}|)\\ &- 4N^2[y_0^2 - mx_0|y_0|\,]\sec\omega - 4N^2\sigma_{n+1}^2\sec\omega\}\\ &+ \{N^4(x + x_0)^2 - 2N^2[x^2(1 + m^2) + x_0^2(1 + m^2) + 2mx|y_0|\,]\sec\omega\}\\ &+ \{4\rho_{n+1}(N^4 - N^2\sec\omega)(x + x_0) + 4N^2m^2x_0x\sec\omega\\ &- 4N^2|\sigma_{n+1}|(m(x - x_0) + 2|y_0|)\sec\omega\\ &- 4N^4(|y_0| + |\sigma_{n+1}|)(\tan\omega)(x + x_0)\} > 0.\end{aligned}$$

Now choose $N>1$ so large that the second brace is positive for all values of $x>x_0$. Then choose M so large that when $n\geqq M$, the first and third braces are positive for all values of $x>x_0$. Then, when $n\geqq M$,

$$\sum_{n=1}^{n}|\Delta b_n^{(\lambda)}(z_0,z)| < N\left|\sum_{n=1}^{n}\Delta|b_n^{(\lambda)}(z_0,z)|\right|$$
$$= N\big|\,|b_{n+1}^{(\lambda)}(z_0,z)| - |b_1^{(\lambda)}(z_0,z)|\,\big|.$$

We must now show that $|b_n^{(\lambda)}(z_0,z)|$ remains uniformly finite in z over S as n varies. Consider the last factor

$$\left|\frac{z_0 + \lambda_m}{z + \lambda_n}\right| = \left[\frac{(x_0 + \rho_n)^2 + (y_0 + \sigma_n)^2}{(x + \rho_n)^2 + (y + \sigma_n)^2}\right]^{1/2}.$$

This is less than 1 in case

$$-(x_0 + x) - 2\rho_n < \frac{y - y_0}{x - x_0}(y + y_0) + 2\sigma_n\frac{y - y_0}{x - x_0}\cdot$$

But

$$((y - y_0)/(x - x_0))(y + y_0) + 2\sigma_n((y - y_0)/(x - x_0))$$
$$\geqq -2|y_0|\tan\omega - 2|\sigma_n|\tan\omega,$$

and

$$-(x_0 + x) - 2\rho_n < -2|y_0|\tan\omega - 2|\sigma_n|\tan\omega$$

if

$$\rho_n > (|y_0| + |\sigma_n|)\tan\omega.$$

This is true for sufficiently large n, independently of z. This completes the proof of the theorem.

Definition of deleted region. We shall often have occasion to speak of a region from which neighborhoods of those points $-\lambda_1, -\lambda_2, \cdots$, which lie in it, have been removed. Such regions will be referred to as deleted regions.

THEOREM IV. *If* (1) *converges when* $z=z_0$, *it converges uniformly over the deleted sector*, S, *defined by* $x\geqq x_0$ *and* $|y-y_0|\cdot(x-x_0)^{-1}\leqq\tan\omega$, *where* ω *is any fixed positive angle less than* $\pi/2$.

Assume $x_0>0$. There is no loss of generality in this. If the contrary is the case, suppose $x_0+\rho_i\geqq\delta>0$ when $i\geqq k$. Drop from consideration the first k terms of the series and factor $\lambda_1\lambda_2\cdots\lambda_k/[(z+\lambda_1)(z+\lambda_2)\cdots(z+\lambda_k)]$ from each remaining term. If the resulting series converges uniformly over S, so does the original series. Replace z_0 by $\bar{z}_0=z_0+\lambda_k$ and λ_{k+m} by $\bar{\lambda}_m=\lambda_{k+m}-\lambda_k$. Make necessary changes in the C_n's. We also assume $\rho_i>0$ when $i\geqq 1$. If this is not true proceed as before rewriting the series a second time.

The proof now follows by Lemma 1, using Preliminary Theorem III and the fact that $b_1(z_0, z)$ remains finite over S.

THEOREM V. *If* $\sum_{k=1}^{\infty}(\sigma_n/\rho_n)^2$ *converges and if* (1) *converges at* $z=z_0$, *then it converges uniformly over any deleted region*, R, *defined by* $x\geqq x_0$ *and* $|y-y_0|\cdot(x-x_0)^{-1}\leqq a(x-x_0)^m+b$, *where* a *and* b *are any positive constants and* m *any fixed positive integer.*

Again assume $x_0>0$ and $\rho_i>0$ for all values of i. As in the proof of Theorem IV this involves no loss of generality.

We know that (1) converges uniformly over any sector, S, as defined in Theorem IV and hence converges at $\bar{x}_0=x_0+\delta$, $\delta>0$. Therefore, by Lemma 1, using Preliminary Theorem II, and the fact $b_n(z_0, z)$ remains finite over R, we see that (1) converges uniformly over the sectorial area, $\bar{R}$, defined by $x\geqq\bar{x}_0$ and $|y|\cdot(x-\bar{x}_0)^{-1}\leqq(x-x_0)^{m_2}$. Choose m_2 and ω so large that every point of R is in at least one of the regions S or $\bar{R}$. The theorem follows.

2. We prove the following theorem:

THEOREM VI. *If* (1) *converges absolutely at* $z=z_0$, *it converges uniformly absolutely over any deleted sectorial region*, R, *defined by* $x\geqq x_0$ *and* $|y-y_0|\cdot(x-x_0)^{-1}\leqq\tan\omega$.

We again assume $x_0>0$ and $\rho_i>0$, $i=1, 2, \cdots$.

Consider the ratio

$$b_n^{(\lambda)}(z_0,z)=|z_0+\lambda_1|\cdots|z_0+\lambda_n|/[|z+\lambda_1|\cdots|z+\lambda_n|].$$

We saw in the proof of Theorem III that this remains uniformly finite. Our theorem follows.

THEOREM VII. *If* (1) *converges absolutely at* $z=z_0$ *and if* $\sum_{n=1}^{\infty}(y_0+\sigma_n)^2\cdot(x_0+\rho_n)^{-2}$ *converges, then* (1) *converges uniformly absolutely over the deleted half-plane defined by* $x\geqq x_0$.

We proceed as in the previous proof except that we refer to Theorem I instead of Theorem III.

3. We now prove

THEOREM VIII. *If s is a positive integer and if* $\sum_{j=1}^{\infty}|\lambda_j|^{-s-1}$ *converges,* then series* (1) *converges at the same points as does*

$$\sum_{n=1}^{\infty}c_n e^{g_{1n}}, \quad \textit{where} \quad g_{1n} = \sum_{\mu=1}^{s}(-1)^{\mu}\frac{z^{\mu}}{\mu}\sum_{j=1}^{n}\lambda_j^{-\mu}. \tag{5}$$

Series (1) *also converges absolutely in the same deleted region as* (5).

Consider z fixed and different from $-\lambda_1, -\lambda_2, \cdots$. Assume $|\lambda_j|\geqq\lambda>|z|$ for all values of j. By reasoning similar to that at the beginning of the proof of Theorem IV there is shown to be no loss of generality in this. We have

$$\frac{\lambda_j}{z+\lambda_j} = e^{-\log(1+z/\lambda_j)} = e^{\gamma},$$

$$\gamma = -\frac{z}{\lambda_j}+\frac{z^2}{2\lambda_j^2}-\cdots+(-1)^s\cdot\frac{z^s}{s\lambda_j^s}+(-1)^{s+1}\frac{z^{s+1}}{\lambda_j^{s+1}}\theta(z/\lambda_j),$$

where $|\theta(z/\lambda_j)|<\Theta$, independent of j. Hence

$$\frac{\lambda_1\cdots\lambda_n}{(z+\lambda_1)\cdots(z+\lambda_n)} = e^{g_{1n}+g_{2n}}, \tag{6}$$

where

$$g_{2n} = (-1)^{s+1}\cdot z^{s+1}\sum_{j=1}^{n}\theta(z/\lambda_j)\lambda_j^{-s-1}.$$

We now refer to Lemma 1. Let

$$a_n = c_n e^{g_{1n}}, \quad b_n = \{\lambda_1\cdots\lambda_n/[(z+\lambda_1)\cdots(z+\lambda_n)]\}e^{-g_{1n}}.$$

From (6)

$$b_n = e^{g_{2n}}.$$

* The proof of this theorem holds in case restrictions previously put on λ_j, except $|\lambda_j|\to\infty$, be removed; s also may be zero.

This remains finite. Moreover

$$|\Delta b_n| = |e^{g_{2n}}(e^{g_{3n}} - 1)|,$$

where

$$g_{3n} = (-1)^s \cdot z^{s+1}\lambda_{n+1}^{-s-1}\,\theta(z/\lambda_{n+1}).$$

Hence,

$$|\Delta b_n| = |b_n|\,|e^{g_{3n}} - 1| \leqq |b_n|\,(e^{g_{4n}} - 1) \leqq M\,|\lambda_{n+1}|^{-s-1},$$

where $g_{4n} = |z|^{s+1}\lambda_{n+1}^{-s-1}\,\Theta$ and M is independent of n. This, however, is the general term of a convergent series. All hypotheses of Lemma 1 are fulfilled, and we have proved that (1) converges at all points at which (5) converges. The converse is readily proved in the same way.

To prove the portion of the theorem relative to absolute convergence, simply take the ratio of corresponding terms. We get what we have just been calling b_n. This remains finite in both cases for a fixed z.

THEOREM IX. *Series* (1) *has the same abscissa of convergence* and the same abscissa of absolute convergence as the Dirichlet series*

$$(7)\qquad \sum_{n=1}^{\infty} c_n e^{g_{5n}}, \qquad g_{5n} = -z\sum_{j=1}^{n}\lambda_j^{-1}.$$

In this theorem we assume λ_j *to be real.*

Let $z=x$ be real and as in the previous proof assume $\lambda_j>\lambda>|x|$. Then

$$(8)\qquad \lambda_1\cdots\lambda_n/[(x+\lambda_1)\cdots(x+\lambda_n)] = e^{g_{6n}},$$

where

$$g_{6n} = -x\sum_{j=1}^{n}\lambda_j^{-1} + (x^2/2)\sum_{j=1}^{n}\theta(x/\lambda_j)\cdot\lambda_j^{-2}$$

and

$$0 < \theta(x/\lambda_j) < 1.$$

Suppose (7) to converge when $x=\alpha$. Let $\delta>0$. When $x=\alpha+\delta$ we write (1) by means of (8) in the form

$$c_0 + \sum_{n=1}^{\infty} c_n e^{g_{7n}} e^{g_{8n}},$$

$$g_{7n} = -\alpha\sum_{j=1}^{n}\lambda_j^{-1},$$

$$g_{8n} = -\delta\sum_{j=1}^{n}\lambda_j^{-1} + \tfrac{1}{2}(\alpha+\delta)^2\sum_{j=1}^{n}\theta\lambda_j^{-2}.$$

* Schnee, Berlin Dissertation, Göttingen, 1908, p. 74 ff.

When n is sufficiently large, $e^{g_{8n}}$ is a positive decreasing monotonic sequence. To show this, increase n by 1. This multiplies $e^{g_{8n}}$ by a number less than 1. Then, by a well known* theorem, series (1) converges when $x=\alpha+\delta$. In other words, since s although positive is as small as desired, the abscissa of convergence of (1) is not greater than that of (7).

Next, suppose (1) to converge when $x=\alpha$. When $x=(\alpha+\delta)$ we write the general term of (7)

$$c_n e^{g_{9n}} e^{g_{10,n}} = c_n\{(\lambda_1 \cdots \lambda_n)/[(\alpha + \lambda_1) \cdots (\alpha + \lambda_n)]\} e^{g_{10,n}},$$

where

$$g_{9n} = -\alpha \sum_{j=1}^{n}(1/\lambda_j) + (1/2)\alpha^2 \sum_{j=1}^{n}\theta(\alpha/\lambda_j)(1/\lambda_j^2),$$

$$g_{10,n} = -\delta \sum_{j=1}^{n}(1/\lambda_j) - (1/2)\alpha^2 \sum_{j=1}^{n}\theta(\alpha/\lambda_j)(1/\lambda_j^2).$$

For sufficiently great n, $e^{g_{10,n}}$ is a positive decreasing monotonic sequence. Increasing n by unity multiplies it by a factor which is less than 1. Hence, reasoning as before, the abscissa of convergence of (7) is not greater than that of (1). This completes the proof.

Only trivial modifications of this proof are necessary to establish the portion of the theorem relative to absolute convergence.

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* See, for example, Bromwich, *An Introduction to Infinite Series*, 2d edition, p. 58. Also readily proved by means of Lemma 1.

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