ON EXTENDING A CONTINUOUS (1-1) CORRESPONDENCE (SECOND PAPER)*

BY

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I. Introduction

In a previous paper† we have given a condition which is necessary and sufficient in order that it be possible to extend a continuous (1-1) correspondence of two plane continuous curves to a correspondence of their planes. In the present paper, we shall give a similar condition for the case of two plane point sets of a more general type. Theorem 2 of this paper includes as special cases the two theorems of our *First paper*.

Definition. By an *E-set*, we shall mean a closed and bounded plane point set, each component‡ of which is a continuous curve, not more than a finite number of them being of diameter greater than any given positive number.

II. Some properties of an E-set and of the domains complementary to an E-set

1. If M is an E-set, we can let $M=m_0+M_1+M_2+\cdots$, where m_0 is totally disconnected and consists of all those components of M which consist of a single point, and for each value of i, M_i is a component of M. Note that m_0 may consist of an uncountable infinity of components of M, but $M-m_0$ consists of at most a countable infinity of components of M.

2. Of the domains complementary to an E-set M lying in a plane S, one is unbounded and the remainder are bounded. Let D denote the unbounded domain of $S-M$, and for each component M_x of M, let D_x denote the unbounded domain of $S-M_x$. The set D is a subset of each of the sets D_x, and hence is a subset of their common part.

Let N be the point set consisting of the sum of the sets $S-D_x$ corresponding to all the components M_x of M. The set N is an E-set, each component

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† H. M. Gehman, *On extending a continuous* (1-1) *correspondence of two plane continuous curves to a correspondence of their planes*, these Transactions, vol. 28 (1926), pp. 252–265. We shall refer to this paper as *First paper*.

‡ A *component* or *maximal connected subset* of a point set M is a connected subset of M which is not a proper subset of any other connected subset of M.

of which has just one complementary domain. The set N therefore has just one complementary domain,* which is the same as the common part of all the sets D_x. Now since M is a subset of N, the set $S-N$ is a subset of $S-M$, and since $S-N$ is unbounded, it must be a subset of D. As we have shown above that D is a subset of $S-N$ (the common part of the sets D_x), it follows that $D=S-N$.

The boundary of D is the sum of the boundaries of the domains D_x corresponding to those components M_x of M which do not lie in a bounded complementary domain of any other component of M.

3. Let d denote a bounded domain of $S-M$. If M_x is any component of M, the domain d is a subset of some domain of $S-M_x$. If d is a subset of a bounded domain of $S-M_x$, then the diameter of M_x is greater than or equal to the diameter of d. Hence it follows from the definition of an E-set, that in the case of a finite number of components of M, d is a subset of a bounded complementary domain; in the case of the remaining components of M, d is a subset of the unbounded complementary domain.

Let $M_1, M_2, \cdots, M_n$ denote the components of M such that for each component M_i, there exists a bounded domain d_i of $S-M_i$ which contains d. Each domain d_i has an outer boundary† consisting of a simple closed curve J_i. The curves $J_1, J_2, \cdots, J_n$ are mutually exclusive since they belong to different components of M, and since each of them encloses d, it follows that of any pair of them, one encloses the other. Hence one of them, say J_k, is interior to each of the other simple closed curves of the set.

It follows by §2, after an inversion of the plane about a circle lying in d, that the outer boundary of d is the simple closed curve J_k, and that the entire boundary of d consists of the boundary of d_k and the boundary of each of the unbounded domains D_x complementary to a component M_x of M which lies in d_k and does not lie in a bounded complementary domain of any other component of M which lies in d_k.

4. We have shown in §3 that corresponding uniquely to each bounded domain d of $S-M$ is a bounded domain d_k complementary to a component M_k of M, such that d_k contains d and has the same outer boundary. Given any positive number ϵ, not more than a finite number of components of M are of diameter greater than ϵ, and each of these has not more than a finite number of bounded complementary domains of diameter greater than ϵ.

* H. M. Gehman, *Concerning acyclic continuous curves*, these Transactions, vol. 29 (1927), pp. 553–568. See Lemma F, p. 558.

† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260.

Hence there are not more than a finite number of domains of $S-M$ which are of diameter greater the ϵ.

5. By the methods used in §2 and the Lemma referred to there, and by means of an inversion of the plane in the case of a bounded domain, it follows that if P is a point of M which is a boundary point of a domain of $S-M$, then P is *accessible* from that domain. That is, if Q is any point of the domain, there exists an arc QP every point of which except P is a point of the domain.

III. Interior-preserving correspondences

Definition.* Let M and M' be E-sets lying in planes S and S' respectively, and let T be a continuous (1-1) correspondence such that $T(M)=M'$. Then we shall say that T *preserves interiors* if given any two simple closed curves J of M and J' of M' which are such that $T(J)=J'$ and which enclose respectively the subset K of M and the subset K' of M', then $T(K)=K'$.

Theorem 1. *If M and M' are E-sets lying in planes S and S' respectively, and if T is a continuous* (1-1) *correspondence such that $T(M)=M'$ and such that T preserves interiors, then if B is the boundary of a domain D of $S-M$, the set $T(B)$ is the boundary of a domain D' of $S'-M'$.*

In the case of the unbounded domain of $S-M$, the proof given in *First paper*, page 262, second paragraph, holds without change.

In the case of a bounded domain of $S-M$, the proof given in *First paper*, page 262, third paragraph, holds with the following changes: N consists of a countable set of trees and a (possibly uncountable) collection of components of M. Using the notation of §1 of Part II, we can denote this collection by $m_0+M_1+M_2+\cdots$. If D_i is the unbounded domain of $S-M_i$, let K_i denote $S-D_i$; and if D_i' is the unbounded domain of $S'-M_i'$, let K_i' denote $S'-D_i'$. Each component of the set $H=(L_1+L_2+\cdots)+m_0+(K_1+K_2+\cdots)$ is either one of the sets L_i, a point of m_0, or one of the sets K_i; similarly, each component of $H'=(L_1'+L_2'+\cdots)+m_0'+(K_1'+K_2'+\cdots)$ is one of the sets L_i', a point of m_0', or one of the sets K_i'. Furthermore since T preserves interiors, if a component of H is a set L_a, a point P of m_0, or a set K_b, then one component of H' is the set L_a', the point $T(P)=P'$ of m_0', or the set K_b', and conversely. Hence, as in *First paper*, not every point of I' (the interior of J') is a point of H', and it follows that $I'-H'$ consists of a single component, which is the required domain of $S'-M'$.

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* *First paper*, p. 260.

IV. Correspondences which preserve sides of arcs in the same sense

Definitions. Let M and M' be E-sets lying in planes S and S' respectively, and let T be a continuous (1-1) correspondence such that $T(M)=M'$. We shall say that T *preserves sides of arcs*, if given any simple closed curve J in S containing an arc AB of M, then there exists a simple closed curve J' in S' containing $T(AB)=A'B'$ and such that if J encloses the subset K of M, then J' encloses $T(K)=K'$; and also if given any simple closed curve J' in S' containing an arc $A'B'$ of M', then there exists a simple closed curve J in S containing $T^{-1}(A'B')=AB$ and such that if J' encloses the subset K' of M', then J encloses $T^{-1}(K')=K$.* We shall say that T *preserves sides of arcs in the same sense*, if T preserves sides of arcs, and if given any two simple closed curves J_1 and J_2 in S containing arcs A_1B_1 and A_2B_2 of M respectively, then the corresponding simple closed curves J_1' and J_2' in S' can be constructed in such a way that if X_1, Y_1, Z_1 are three points of A_1B_1 and X_2, Y_2, Z_2 are three points of A_2B_2, such that the sense† $X_1Y_1Z_1$ on J_1 is the same as the sense $X_2Y_2Z_2$ on J_2, then the sense $T(X_1)T(Y_1)T(Z_1)$ on J_1' is the same as the sense $T(X_2)T(Y_2)T(Z_2)$ on J_2'; and similarly for any two simple closed curves in S' containing arcs of M'.

Theorem 2. *If M and M' are E-sets lying in planes S and S' respectively, and if T is a continuous* (1-1) *correspondence such that $T(M)=M'$ and such that T preserves interiors and preserves sides of arcs in the same sense, then T can be extended to a correspondence between the planes S and S', i.e., there exists a continuous* (1-1) *correspondence U, such that $U(S)=S'$, and such that for each point P of M, $U(P)=T(P)$.*

1. If every component of M is an arc or a point, and no interior point of an arc XY of M is a limit point of $M-XY$, then an arc in the plane S may be passed through M, and an arc in S' may be passed through M'.‡ If M and M' are subsets of arcs, it has been proved§ that *any* continuous (1-1) correspondence of M and M' can be extended to a correspondence of the

* In *First paper*, p. 253, we said that *sides are preserved under T*, if T has the above property. It seemed better to say here that *sides of arcs are preserved.* See Part V of this paper.

† J. R. Kline, *A definition of sense on closed curves in non-metrical plane analysis situs*, Annals of Mathematics, vol. 19 (1918), pp. 185–200, and J. R. Kline, *Concerning sense on closed curves in non-metrical plane analysis situs*, Annals of Mathematics, vol. 21 (1919), pp. 113–119.

‡ R. L. Moore and J. R. Kline, *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, vol. 20 (1919), pp. 218–223.

§ R. L. Moore, *Conditions under which one of two given closed linear point sets may be thrown into the other one by a continuous transformation of a plane into itself*, American Journal of Mathematics, vol. 48 (1926), pp. 67–72.

planes, which proves Theorem 2 for sets of this type. The theorem just cited is a special case of Theorem 2, because if the E-sets M and M' are subsets of arcs, any continuous (1-1) correspondence of M and M' preserves interiors (vacuously) and preserves sides of arcs in the same sense.

2. In all other cases, we shall define, for convenience, a positive and a negative sense for points on simple closed curves in the planes S and S'.

If M contains any simple closed curves, we shall select an arbitrary simple closed curve J_1 of M and three arbitrary points A, B, C of J_1 and shall say that the sense XYZ on the simple closed curve J in the plane S is *positive*, if and only if it is the same as the sense ABC on J_1; otherwise the sense XYZ on J is *negative*. In the same way, in the plane S' we shall say that the sense $X'Y'Z'$ on the simple closed curve J' is *positive*, if and only if it is the same as the sense $T(A)T(B)T(C)$ on $T(J_1)$.

If M contains no simple closed curve, but contains an arc AB such that an interior point P of AB is a limit point of $M-AB$, then we shall construct an arbitrary simple closed curve J_1 containing the arc AB of M, and enclosing a subset K of $M-AB$ which has P for a limit point. Then we shall say that the sense XYZ on the simple closed curve J in the plane S is *positive*, if and only if it is the same as the sense APB on J_1. Since T preserves sides of arcs, there exists a simple closed curve J_1' in S' containing the arc $T(AB)$ and enclosing $T(K)$, and we shall define the sense $X'Y'Z'$ on the simple closed curve J' in the plane S' as *positive*, if and only if it is the same as the sense $T(A)T(P)T(B)$ on J_1'. With the help of Lemma A of *First paper* and Kline's definition of sense, it is easily seen that the positive sense in S' as defined thus is independent of the particular selection of the simple closed curve J_1'.

If in every application of the condition that T preserves sides of arcs in the same sense, we let J_1 be the fixed curve J_1 defined above, it follows that if three points of an arc of M which is a subset of the simple closed curve J_2, are in a certain sense (either positive or negative) on J_2, then the three points corresponding under T to the given points are in the same sense on the simple closed curve J_2', regardless of the particular selection of J_2'.

3. By Theorem 1, there exists a (1-1) correspondence between the domains of $S-M$ and those of $S'-M'$. We shall now show how to define the correspondence U for points of a pair of corresponding domains and their boundaries, this being done in such a way that for each point P of M, $U(P)=T(P)$.

For simplicity, we shall define U for points of the unbounded domains of $S-M$ and $S'-M'$, which we shall denote by D and D' respectively. A similar method suffices for a pair of corresponding bounded domains, after an inversion of the plane.

4. As in §2 of Part II, let $N=S-D$ and let $N'=S'-D'$. If each component of N is considered as an element and each point of $S-N$ is considered as an element, the set of all such elements is topologically equivalent to the set of all points in a plane Σ.* The set of points in Σ corresponding to the components of N is a closed and totally disconnected set Γ. Similarly, if the components of N' and the points of $S'-N'$ are considered as elements, the set of all such elements is equivalent to the set of all points of a plane Σ', and the points of Σ' corresponding to the components of N' form a closed and totally disconnected set Γ'.

Since T preserves interiors, there exists a (1-1) correspondence V between the components of N and the components of N' such that a point P of M lies in a component N_x of N if and only if $T(P)$ lies in the component $V(N_x)$ of N'. The correspondence V considered as a correspondence between the points of Γ and the points of Γ' is continuous and (1-1). We have shown above that a continuous (1-1) correspondence between two closed and totally disconnected sets such as Γ and Γ', can be extended to their planes; that is, there exists a continuous (1-1) correspondence W such that $W(\Sigma)=\Sigma'$, and such that for each point P of Γ, $W(P)=V(P)$.

The correspondence W, considered as a correspondence between S and S', is a continuous (1-1) correspondence between the points of $S-N$ and the points of $S'-N'$. Furthermore W is such that if J is any simple closed curve in $S-N$, then J encloses the component N_x of N if and only if the simple closed curve $W(J)$ lying in $S'-N'$ encloses the component $V(N_x)$ of N'. This is equivalent to the statement that J encloses the point P of M if and only if $W(J)$ encloses the point $T(P)$ of M'.

5. For any given positive number ϵ, we can enclose† each component N_x of N by a simple closed curve J_x which has no points in common with N and which is such that every point of J_x and every point which is interior to J_x is at a distance less than ϵ from a point of N_x. Since N has the Heine-Borel property, some finite subset of this collection of simple closed curves encloses all points of N. Hence‡ there exists a finite set of simple closed curves $J_1, J_2, \cdots, J_k$ enclosing all points of N, each curve being exterior to

* See Theorem 25, p. 427, of the paper by R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428. See also H. M. Gehman, *Concerning acyclic continuous curves*, loc. cit., pp. 558–559.

† R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

‡ R. L. Moore and J. R. Kline, loc. cit., Lemma 1, p. 219.

every other one of the set, and every point lying on or interior to any one of the curves being at a distance less than ϵ from some point of N.

Under the correspondence W, these simple closed curves correspond in S' to a set of simple closed curves $J_1', J_2', \cdots, J_k'$ which enclose all points of N' and are mutually exterior. If it happens that the distance from some point lying on or interior to one of these curves to any point of N' is greater that or equal to ϵ, we can construct in S' a finite set of simple closed curves $C_1', C_2', \cdots, C_h'$ which enclose all points of N', are mutually exterior, are such that every point lying on or interior to one of the curves is at a distance less than ϵ from some point of N', and are such that each of the curves is interior to one of the curves J_i'. On account of this last condition, the set of simple closed curves $C_1, C_2, \cdots, C_h$ in S, where $C_i = W^{-1}(C_i')$, has all the properties of the set $J_1, J_2, \cdots, J_k$.

6. We shall now define the correspondence U for points of S which are not interior to any of the curves $C_1, C_2, \cdots, C_h$ as any continuous (1-1) correspondence such that $U(S) = S'$, $U(C_i) = C_i'$ for $i = 1, 2, \cdots, h$, and if $X_iY_iZ_i$ are three points of one of the curves C_i, then the sense $X_iY_iZ_i$ on C_i is the same as the sense $U(X_i)U(Y_i)U(Z_i)$ on C_i, *sense* being positive or negative as defined in §2. Note that the correspondence W can always be defined so as to have this last property, in which case we can let $U(P) = W(P)$ for all points P of S which are not interior to any of the curves C_i.

We shall next define U for points of D and of the boundary of D which are interior to the simple closed curve C, where C denotes any one of the curves $C_1, C_2, \cdots, C_h$.

7. If each component of M interior to C is either an arc or a point, and if no interior point of an arc XY of M is a limit point of $M - XY$, then just as in §1, we can construct an arc interior to C which contains all points of M which are interior to C, and hence there exists a continuous (1-1) correspondence U_1 such that (1) $U_1(S) = S'$, (2) for each point P of C, $U_1(P) = U(P)$, and (3) for each point P of M interior to C, $U_1(P) = T(P)$. We shall then let $U(P) = U_1(P)$ for all points of S interior to C.

8. If the subset of M interior to C is not of the above type, let M_1 be a component of M which is interior to C and which either contains an arc XY such that some interior point of XY is a limit point of $M - XY$, or contains a simple closed curve. If M_1 contains no simple closed curves, let A and B be the end points of the maximal arc of M_1 that contains the arc XY. If M_1 contains a simple closed curve, the component of the boundary of D which is a subset of M_1 contains a simple closed curve J_1. In this case, let A and B

be two points of J_1 which are not feet of trees* with respect to J_1 of the component of the boundary of D which contains J_1.

In either case, let E and F be two points of C. We can construct two non-intersecting arcs AE and BF in $S-C-M+A+B+E+F$.† Let $E'=U(E)$, $F'=U(F)$, $A'=T(A)$, and $B'=T(B)$, and in $S'-C'-M'+A'+B'+E'+F'$ we can construct two non-intersecting arcs $A'E'$ and $B'F'$.

Let G and H be two points which separate E and F on C and let $G'=U(G)$ and $H'=U(H)$. There are two and only two domains of $S-(C+AE+BF+M)$ which are interior to C and have boundary points on C. The boundary of one of these domains R_1 consists of the arc EGF of C, the arcs AE and BF, and a subset of M_1 containing an arc AKB which with the other three arcs forms a simple closed curve which is the outer boundary of the domain R_1. The other domain R_2 is bounded by the arc EHF of C, the arcs AE and BF, and a subset of M_1 containing an arc ALB which with the other three arcs forms a simple closed curve which is the outer boundary of the domain R_2. We have chosen A and B in such a way that either AKB and ALB are identical, or AKB and ALB have no interior points in common. It is also true that the boundary of R_1 (and that of R_2) contains the same points of M_1 regardless of how the arcs AE and BF are constructed.

Similarly in S', there are two and only two domains of $S'-(C'+A'E'+B'F'+M')$ which are interior to C' and have boundary points on C'. If the domain whose boundary contains G' is denoted by R_1', then the outer boundary of R_1' consists of $A'E'+E'G'F'+B'F'$ and one of the arcs $A'K'B'$, $A'L'B'$; the outer boundary of the other domain R_2' consists of $A'E'+E'H'F'+B'F'$ and the other one of the arcs $A'K'B'$, $A'L'B'$. The arcs $A'K'B'$, $A'L'B'$ either are identical or have no interior points in common. In the latter case, since the arcs $A'K'B'$ and $A'L'B'$ form a simple closed curve, it follows that the outer boundary of R_1' contains $A'K'B'$ because the following senses are either all positive or all negative: AKB on $AKBLA$, EGF on C, EGF on $EGFBKAE$, $A'K'B'$ on $A'K'B'L'A'$, and $E'G'F'$ on C'. It is also true that the subset of M_1' that lies in the boundary of R_1' (and that of R_2') is the same, regardless of how the arcs $A'E'$ and $B'F'$ are constructed. Since T preserves interiors and preserves sides of arcs in the same sense, it follows that a point P of M_1 is a point of the boundary of R_i $(i=1, 2)$, if and only if $T(P)$ is a point of the boundary of R_i'.

However it may happen that for a particular choice of the arcs AE, BF, $A'E'$, $B'F'$, a component M_2 of M lies in R_1, while $T(M_2)=M_2'$ lies in

* *First paper*, p. 262.

† Part II, §5.

R_2'. We shall now show how, keeping $A'E'$ and $B'F'$ fixed, a new pair of arcs AE and BF may be constructed so that a component M_2 of M lies in R_1 if and only if M_2' lies in R_1'.

9. Let us suppose that the arcs AE, BF, $A'E'$, $B'F'$ have been constructed. Let N_i' $(i=1, 2)$ denote the set consisting of all components of M' that lie in R_i', and let $N_i = T^{-1}(N_i')$. Each of the sets M_1+N_1, M_1+N_2, $M_1'+N_1'$, $M_1'+N_2'$ is closed.

We shall now show that if we denote by N_{1i} $(i=1, 2)$ the subset of N_1 that lies in R_i, then $N_{12}+A+B$ is closed. Since the set $M_1+N_1 = M_1+N_{11}+N_{12}$ is closed, and no point of N_{11} is a limit point of N_{12}, it follows that M_1+N_{12} is closed. Since N_{12} is a subset of R_2 and $T(N_{12})$ is a subset of R_1', it follows that any point P of M_1 which is a limit point of N_{12} must be a boundary point of R_2, such that $T(P)$ is a boundary point of R_1'. If AKB and ALB have no interior points in common, A and B are the only points of M_1 which have this property. If $AKB = ALB$, every point of the arc has this property, but if an interior point of the arc AB of M_1 were a limit point of N_{12}, the correspondence T would not preserve sides of arcs in the same sense. Hence in both cases $N_{12}+A+B$ is closed.

Let us join H to L by an arc in R_2+H+L. This arc divides R_2 into two domains R_3 and R_4 whose boundaries contain the points A and B respectively. If we let N_{1i} $(i=3, 4)$ denote the subset of N_{12} which lies in R_i, then the sets $N_{13}+A$ and $N_{14}+B$ are closed.

Since A is a component of the closed set $N_{13}+A$, we can (by using Zoretti's theorem) let $N_{13} = K_1+K_2+\cdots$, where (1) K_i and K_j have no points in common, if $i \neq j$, (2) K_i and $N_{13}-K_i$ are closed, (3) $K_1+\cdots+K_i$ contains all components of N_{13} which contain points at a distance from A greater than $1/2^i$. Note that since T preserves interiors, N_{13} contains all components of M which lie in a bounded complementary domain of a component of N_{13}. Hence we can assume that the sets K_1, K_2, $\cdots$ also satisfy condition (4): if a component M_x of M lies in a bounded complementary domain of a component of K_i, then M_x is a component of K_i.

For each value of i, the sets K_i and $M-K_i$ are closed, and we can therefore, as in §5, enclose K_i by a finite number of mutually exterior simple closed curves J_{i1}, J_{i2}, $\cdots$, J_{in_i}, lying in R_3, of diameter less than $1/2^{i-2}$, such that the exterior of each one contains $M-K_i+J_{11}+\cdots+J_{1n_1}+J_{21}+\cdots+J_{i-1,n_{i-1}}$. For simplicity, we shall denote each of the curves J_{hk} just defined, by J_i, where $i = n_1+n_2+\cdots+n_{h-1}+k$. The point set consisting of $A+J_1+J_2+\cdots$ is closed; in fact it is an E-set. Also the set consisting of M, the simple closed curve C, the arcs AE, BF, HL, all the curves J_i, and any finite number of arcs, forms an E-set.

Let us next select for each value of i, a point Q_i of the curve J_i, and a point P_i whose distance from A is less than $1/2^i$, which is an interior point of the arc EA, and which is (for $i>1$) an interior point of the arc $P_{i-1}A$. We can then construct an arc P_iQ_i in $P_i+Q_i+R_3-(P_1Q_1+\cdots+P_{i-1}Q_{i-1})-(J_1+J_2+\cdots)$ of diameter less than $1/2^{h-2}$, where h is determined by the equation $i=n_1+\cdots+n_{h-1}+k$ given above. The set consisting of A and all the arcs P_iQ_i also forms an E-set.

No point of a given arc P_iQ_i is a limit point of M or of any subset of $\theta_i=(J_1+\cdots+J_{i-1}+J_{i+1}+\cdots)+(P_1Q_1+\cdots+P_{i-1}Q_{i-1}+P_{i+1}Q_{i+1}+\cdots)$. We can therefore enclose P_iQ_i by a simple closed curve C_i in R_1+R_3 whose exterior contains $M+\theta_i+C_1+\cdots+C_{i-1}$ and which is of diameter less than $1/2^{h-2}$, the relation between i and h being given above. The curve C_i contains two arcs $P_{i1}Q_{i1}$, $P_{i2}Q_{i2}$ such that (1) P_{i1} and P_{i2} are interior points of the arc EA lying between P_{i-1} and P_i and between P_i and P_{i+1} respectively, (2) Q_{i1} and Q_{i2} are points of J_i such that the sense $Q_{i1}Q_iQ_{i2}$ on J_i is the same as the sense EGF on $EGFBKAE$, (3) the interior of the simple closed curve formed by these two arcs and the arcs $P_{i1}P_{i2}$ of AE and $Q_{i1}Q_iQ_{i2}$ is a subset of R_3-M and contains no points interior to J_i.

The set consisting of $AE-(P_{11}P_{12}+P_{21}P_{22}+\cdots)+P_{11}Q_{11}+P_{21}Q_{21}+\cdots+P_{12}Q_{12}+P_{22}Q_{22}+\cdots+(J_1-Q_{11}Q_1Q_{12})+(J_2-Q_{21}Q_2Q_{22})+\cdots$ is an arc* from A to E which we may denote by $(AE)_1$. By our method of construction, the domain R_1 determined by using $(AE)_1$ in place of AE, contains $N_{11}+N_{13}$, while the distribution of points of N_2 between the domains R_1 and R_2 has not been changed. In the same way a new arc from B to F, which we may denote by $(BF)_1$, can be constructed so that the domain R_1 determined by $(AE)_1$ and $(BF)_1$ contains $N_{11}+N_{13}+N_{14}=N_1$. If any points of N_2 lie in the domain R_1 determined by $(AE)_1$ and $(BF)_1$, we can in the same way determine new arcs $(AE)_2$ and $(BF)_2$, such that the domain R_1 determined by them contains N_1, and the domain R_2 contains N_2.

10. Then as in *First paper*, pp. 264–265 and pp. 256–258, we can obtain a finite set of points $G_1=E$, G_2, $\cdots$, $G_m=F$, $\cdots$, G_n lying in the order $G_1G_2\cdots G_m\cdots G_nG_1$ on the simple closed curve C, and a set of points $H_1=A$, H_2, $\cdots$, $H_m=B$, $\cdots$, H_n of M_1, and a set of arcs G_1H_1, $\cdots$, G_nH_n, such that (1) the arc G_iH_i has no interior points in common with $C+M$, (2) the simple closed curve C_i formed by the arcs G_iH_i, $G_{i+1}H_{i+1}$, the arc G_iG_{i+1} of C that contains none of the points G_1, $\cdots$, G_{i-1}, G_{i+2}, $\cdots$, G_n, and an arc H_iH_{i+1} of the boundary of D, encloses no tree of $M_1-H_iH_{i+1}$

* This may be established with the aid of Theorem 2 of H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, Annals of Mathematics, vol. 27 (1926), pp. 381–384.

which is of diameter greater than $\epsilon/9$, or whose corresponding set under T is of diameter greater than $\epsilon/9$, and (3) a point P of M is interior to C_i if and only if $T(P)$ is interior to the simple closed curve C_i' formed by the arcs $U(G_iG_{i+1})=G_i'G_{i+1}'$ of C', the arc $T(H_iH_{i+1})=H_i'H_{i+1}'$ of M_1, and arcs $G_i'H_i'$, $G_{i+1}'H_{i+1}'$ lying in $S'-C'-M'+G_i'+H_i'+G_{i+1}'+H_{i+1}'$.

Then by using the method of §5 and the method of *First paper*, p. 265 and pp. 258–259, we can express the points of D and its boundary which are interior to C as the sum of a finite number of domains containing no points of M, a finite number of domains of diameter less than ϵ, and the boundaries of these domains, the same being true for points of D' and its boundary which are interior to C'. By the method of §9, a point P of M lies in a domain in S, if and only if $T(P)$ lies in the corresponding domain in S'. The correspondence U is then defined as in *First paper*, p. 265 and pp. 259–260, so as to satisfy the conclusion of Theorem 2.

V. Logical relations between various kinds of correspondences

Definitions. Let M and M' be E-sets lying in planes S and S' respectively, and let T be a continuous (1-1) correspondence such that $T(M)=M'$. We shall say that T *preserves sides*, if given any simple closed curve J in S containing a connected* subset L of M, then there exists a simple closed curve J' in S' containing $T(L)=L'$ and such that if J encloses the subset K of M, then J' encloses $T(K)=K'$; and similarly for any simple closed curve J' in S' containing a connected subset L' of M'. We shall say that T *preserves sides in the same sense*, if T preserves sides, and if given any two simple closed curves J_1 and J_2 in S containing connected subsets L_1 and L_2 of M respectively, then the corresponding simple closed curves J_1' and J_2' in S' can be constructed in such a way that if X_1, Y_1, Z_1 are three points of L_1 and X_2, Y_2, Z_2 are three points of L_2, such that the sense $X_1Y_1Z_1$ on J_1 is the same as the sense $X_2Y_2Z_2$ on J_2, then the sense $T(X_1)T(Y_1)T(Z_1)$ on J_1' is the same as the sense $T(X_2)T(Y_2)T(Z_2)$ on J_2'; and similarly for any two simple closed curves in S' containing connected subsets of M'.

1. The following seven theorems, which follow from the definitions, give the complete set of logical relations connecting the five kinds of correspondences which have been defined in this paper.

Theorem 3. *If T preserves sides in the same sense, then T preserves sides.*

Theorem 4. *If T preserves sides of arcs in the same sense, then T preserves sides of arcs.*

* Since M is closed, L is necessarily either an arc or the curve J.

THEOREM 5. *If T preserves sides, then T preserves sides of arcs.*

THEOREM 6. *If T preserves sides in the same sense, then T preserves sides of arcs in the same sense.*

THEOREM 7. *If T preserves sides, then T preserves interiors.*

THEOREM 8. *If T preserves interiors and preserves sides of arcs, then T preserves sides.*

THEOREM 9. *If T preserves interiors and preserves sides of arcs in the same sense, then T preserves sides in the same sense.*

2. On account of Theorems 3, 6, 7, and 9, we can make a purely formal change in the statement of Theorem 2 by replacing "T preserves interiors and preserves sides of arcs in the same sense" by "T preserves sides in the same sense." However the conclusion of Theorem 2 no longer remains true if we replace this phrase by either "T preserves sides" or "T preserves sides of arcs in the same sense."

3. If the E-set M is connected, i.e., if M is a continuous curve, then it follows from the theorems of *First paper*, that the conditions (1) T preserves sides in the same sense, (2) T preserves sides of arcs in the same sense, (3) T preserves sides, (4) T preserves sides of arcs, are logically equivalent. Each of them implies that T preserves interiors but not conversely.*

* *First paper*, Lemma B, p. 261. Using the definitions of the present paper, Lemma B is *If M is connected and T preserves sides of arcs, then T preserves interiors.*

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