GENERALIZED VANDERMONDE DETERMINANTS*

BY

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Introduction

The Vandermonde determinant, usually written in this way:

$$\begin{vmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_n \\ \cdot & \cdots & \cdot \\ a_1^{n-1} & \cdots & a_n^{n-1} \end{vmatrix},$$

is an alternant and is equal to the difference-product of its variables. Instead of keeping the indices of the Vandermonde determinant fixed at $0, 1, 2, \cdots, n-1$, let us take any positive integers. This yields another alternant which we shall call a generalized Vandermonde determinant. This determinant may be expressed thus:

$$\begin{vmatrix} a_1^{t_1} & \cdots & a_n^{t_1} \\ a_1^{t_2} & \cdots & a_n^{t_2} \\ \cdot & \cdots & \cdot \\ a_1^{t_n} & \cdots & a_n^{t_n} \end{vmatrix}$$

in which $t_1 < t_2 < \cdots < t_n$. Since this alternant vanishes on equating any two of its variables, we know that every generalized Vandermonde determinant is divisible by the difference-product of its variables and hence by the Vandermonde determinant of these variables. Let us now consider the quotient of any generalized Vandermonde determinant by its Vandermonde determinant. Since both change sign under a transposition, their quotient will remain unchanged, putting it into the class of symmetric functions. A general formula for finding this symmetric function has been the goal of much research in the last half century.

Thus far two ways of treating this subject have been introduced. One of these involves the determination of a general method for finding the quotient of a generalized Vandermonde determinant by the Vandermonde determinant of its variables in terms of symmetric functions. This method

* Presented to the Society, September 9, 1927; received by the editors in February, 1929.

has been used by W. Woolsey Johnson* and, in an entirely different manner, by Thomas Muir.† The second method of approach, used by both Muir‡ and E. D. Roe, Jr.,§ consists of the multiplication of the Vandermonde determinant by a symmetric function and the expression of this product as an integral rational function of generalized Vandermonde determinants in the same variables.

The object of this paper is to develop a new method of treating this problem, consisting in first expressing every generalized Vandermonde determinant as an integral rational function of certain special Vandermonde determinants. This leads to a solution of the problem mentioned above; and it gives, in conjunction with a theorem of Muir's, a direct method for expressing an integral symmetric function of n variables in terms of the elementary symmetric functions. In this treatment of the subject, the determinant

$$\begin{vmatrix} 1 & \cdots & 1 \\ a_1 & \cdots\cdots & a_n \\ \cdots & \cdots & \cdots \\ a_1^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{\substack{i=2 \\ i>j}}^{n} (a_i - a_j) \quad (j=1,2,\cdots,n-1)$$

will be called the principal Vandermondian and will be represented by the notation $|n-1, n-2, \cdots, 1, 0|$. The matrix formed from this determinant by adding the nth power of the variables as an extra row will be called the Vandermonde matrix and will be represented by the symbol $\|n, n-1, \cdots, 1, 0\|$. And the determinant formed from this matrix by omitting the $(n-k+1)$th row, or the $(k+1)$th term in the symbol, will be represented by $|n, n-1, \cdots, n-k+1, n-k-1, \cdots, 1, 0|$ or briefly by V_{nk}, in which the first subscript, n, denotes the number of variables, and the second subscript, k, indicates that the $(k+1)$th term has been omitted from the symbol for the Vandermonde matrix, $\|n, n-1, \cdots, 1, 0\|$.¶ The special case $k=0$ gives the principal Vandermondian V_{n0}. The determinants V_{nk}, where $k \neq 0$, will be called secondary Vandermondians. The first section of this paper is concerned with expressing any generalized Vandermonde determinant in terms of the principal and secondary Vandermondians.

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* American Journal of Mathematics, vol. 7, pp. 345–352, 380–388; Quarterly Journal of Mathematics, vol. 21, pp. 217–224.

† Proceedings of the Royal Society of Edinburgh, vol. 14, pp. 433–445.

‡ Proceedings of the Royal Society of Edinburgh, vol. 22, pp. 539–542.

§ American Journal of Mathematics, vol. 25, pp. 97–106.

¶ Where there is no danger of confusion, the subscript n will be omitted from the symbol V_{nk}.

1. Secondary Vandermondians and elementary symmetric functions. Notations

Since the quotient of any generalized Vandermonde determinant by the principal Vandermondian is always a symmetric function, we can express it rationally in terms of elementary symmetric functions. We will now show that every elementary symmetric function of $a_1, \cdots, a_n$ is expressible rationally in terms of the principal and secondary Vandermondians in these variables. Consider the equation

$$p_0x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0, \tag{1}$$

whose roots are $a_1, a_2, \cdots, a_n$. We then have

$$\frac{p_i}{p_0} = (-1)^i \sum a_1a_2 \cdots a_i \qquad (i = 1, 2, \cdots, n). \tag{2}$$

By substituting each of the roots in equation (1), we obtain n identities:

$$\sum_{j=0}^{n} p_j a_i^{n-j} = 0 \qquad (i = 1, 2, \cdots, n). \tag{3}$$

We look upon these identities as stating that $p_0, p_1, \cdots, p_n$ satisfy a system of n equations in the p's. The matrix of this system of n equations in $n+1$ unknowns is the Vandermonde matrix $\|n, n-1, \cdots, 1, 0\|$. Hence we have

$$\frac{p_i}{p_0} = (-1)^i \frac{V_{ni}}{V_{n0}} \qquad (i = 1, 2, \cdots, n). \tag{4}$$

Equations (2) and (4) yield

$$\frac{V_{ni}}{V_{n0}} = \sum a_1a_2 \cdots a_i \qquad (i = 1, 2, \cdots, n). \tag{5}$$

We may therefore state

Theorem I. *The elementary symmetric function $E_i \equiv \sum a_1a_2 \cdots a_i$ of the n variables $a_1, a_2, \cdots, a_n$ is equal to the quotient of the secondary Vandermondian V_{ni} by the principal Vandermondian V_{n0}.*

From this theorem follows

Corollary I. *$p_j = (-1)^j V_{nj}\rho$, where ρ is a proportionality factor.*

Before proceeding farther, we will lay down the following definitions:

Definition I. D_l^n will represent the following lth order determinant, formed from principal and secondary Vandermondians in n variables:

$$D_l^n \equiv \begin{vmatrix} V_1 & V_2 & V_3 & \cdots & V_n & 0 & \cdot & \cdots & 0 \\ V_0 & V_1 & V_2 & \cdots & 0 & V_n & \cdot & \cdots & 0 \\ 0 & V_0 & V_1 & \cdots & \cdot & V_n & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdots & \cdot & 0 & V_0 & V_1 & & V_2 \\ 0 & \cdot & \cdots & \cdot & \cdot & 0 & V_0 & & V_1 \end{vmatrix}_l,$$

where l may be less than, equal to, or greater than n.

Definition II. $D_l^n\{t_1, t_2, \cdots, t_{s-1}, t_s\}$, where $l \geqq t_1 \geqq t_2 \geqq \cdots \geqq t_{s-1} \geqq t_s$, is defined to be the determinant obtainable from D_l^n by increasing the subscripts in the first t_s rows by s, those in the next $t_{s-1}-t_s$ rows by $s-1, \cdots$, those in the next t_1-t_2 rows by 1. In this connection V_m is defined to be 0 if m is greater than n; and a zero which precedes a V_0 should be replaced by V_0 whenever the subscripts in the row in which it stands are increased by 1. For example

$$D_4^5\{3,2,1,1\} \equiv \begin{vmatrix} V_5 & 0 & 0 & 0 \\ V_2 & V_3 & V_4 & V_5 \\ V_0 & V_1 & V_2 & V_3 \\ 0 & 0 & V_0 & V_1 \end{vmatrix}.$$

2. The determinants $|k, n-2, n-3, \cdots, 1, 0|$

We shall now attempt to develop a general formula for the generalized Vandermonde determinant $|k,\ n-2,\ n-3, \cdots, 1,\ 0|$, where $k>n-2$. To establish this general formula we will use mathematical induction. Setting $k=n-1$ yields the principal Vandermondian V_{n0}. Also, if $k=n$, we, have the secondary Vandermondian V_{n1}. In comparing $|k+1, n-2, \cdots, 1, 0|$, where $k>n$, with $|k, n-2, \cdots, 1, 0|$, we shall consider the n systems of $k-n+2$ equations each:

$$\begin{aligned} p_0a_i^{k+1} + p_1a_i^k + \cdots + p_na_i^{k-n+1} &= 0, \\ p_0a_i^k + \cdots + p_{n-1}a_i^{k-n+1} + p_na_i^{k-n} &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ p_0a_i^n + p_1a_i^{n-1} + \cdots + p_{n-1}a_i + p_n &= 0, \end{aligned}$$

where i takes the values 1 to n inclusive. Eliminating the $k-n+1$ variables $a_i^{k-1}, a_i^{k-2}, \cdots, a_i^{n-1}$, from the ith system, we have

$$\begin{vmatrix}
p_2 & p_3 \cdots p_n & 0 & \cdots & \cdots & 0 & p_0 a_i^{k+1} + p_1 a_i^k \\
p_1 & p_2 \cdots & p_n & 0 & \cdots & 0 & p_0 a_i^k \\
p_0 & p_1 \cdots & \cdot & p_n & 0 \cdots & 0 & 0 \\
0 & p_0 \cdots & \cdots & \cdot & p_n \; 0 \cdots & 0 & 0 \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & p_0 \cdots & p_n & 0 \\
0 & \cdots & \cdots & \cdots & 0 \; p_0 \cdots & p_{n-1} & p_n a_i^{n-2} \\
0 & \cdots & \cdots & \cdots & 0 \; p_0 \cdots & p_{n-2} & p_{n-1} a_i^{n-2} + p_n a_i^{n-3} \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & 0 \; p_0 \; p_1 & p_2 & p_3 a_i^{n-2} + \cdots + p_n a_i \\
0 & \cdots & \cdots & \cdots & \cdots \; 0 & p_0 \; p_1 & p_2 a_i^{n-2} + \cdots + p_{n-1} a_i + p_n
\end{vmatrix}_{k-n+2} = 0.$$

Multiplying the first row by $(-p_0)$ and adding to it p_1 times the second row and developing according to the elements of the last column, we get

$$a_i^{k+1} p_0^2 \begin{vmatrix}
p_1 & p_2 \cdots p_n & 0 & \cdots 0 \\
p_0 & p_1 \cdots \cdot & p_n & 0 \cdots 0 \\
\cdot & \cdots & \cdots & \cdots \\
0 & \cdots & 0 \; p_0 & p_1
\end{vmatrix}_{k-n+1}$$

$$+ a_i^k p_0 \begin{vmatrix}
p_1^2 - p_0 p_2 & p_1 p_2 - p_0 p_3 & \cdots & p_1 p_{n-1} - p_0 p_n & p_1 p_n & 0 & & \cdots & 0 \\
p_0 & p_1 & \cdots & p_{n-2} & p_{n-1} & p_n & 0 & & \cdots 0 \\
0 & p_0 & \cdots & p_{n-3} & p_{n-2} & p_{n-1} & p_n & 0 & \cdots 0 \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \; 0 & p_0 & p_1
\end{vmatrix}_{k-n+1}$$

$$+ M_2 a_i^{n-2} + M_3 a_i^{n-3} + \cdots + M_{n-1} a_i + M_n = 0,$$

where the M's represent polynomials in the p's. The second determinant, which, multiplied by p_0, is the coefficient of a_i^k in the above-mentioned equation, may be looked upon as the sum of two determinants or as a single determinant of order one higher, giving us

$$a_i^{k+1} p_0^2 \begin{vmatrix}
p_1 & p_2 \cdots p_n & 0 & \cdot & \cdots 0 \\
p_0 & p_1 \cdots \cdot & p_n & 0 & \cdots 0 \\
0 & p_0 \cdots & \cdot & p_n & 0 \cdots 0 \\
\cdot & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 \; p_0 & p_1
\end{vmatrix}_{k-n+1}$$

$$
+a_i^k p_0 \begin{vmatrix} p_1 & p_2 & \cdots & p_n & 0 & \cdot & \cdots & 0 \\ p_0 & p_1 & \cdots & \cdot & p_n & 0 & \cdots & 0 \\ 0 & p_0 & \cdots & \cdot & \cdot & p_n & 0 \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdots & \cdot & \cdot & 0 & p_0 & p_1 \end{vmatrix}_{k-n+2} + M_2 a_i^{n-2} + \cdots + M_n = 0.
$$

Using the corollary to Theorem I to change the p's to the V's, multiplying the odd rows and the even columns of the two determinants through by -1, dividing through by V_0 and by the proper power of the proportionality factor ρ, and introducing the symbol D_r^n, we obtain n equations which may be considered as homogeneous equations in D_{k-n+2}^n, D_{k-n+1}^n, and $n-1$ other variables which differ from the M's by the factor V_0, multiplied by some power of ρ. Solving these equations for the ratio of D_{k-n+2}^n and D_{k-n+1}^n, we find

$$
V_0 \left| k+1, n-2, \cdots, 1,0 \right| : \left| k, n-2, \cdots, 1,0 \right| \doteq D_{k-n+2}^n : D_{k-n+1}^n .
$$

Writing this relation for $k=n-1, n, n+1, \cdots, k-1$, we get

$$
\begin{aligned}
&V_0 V_1 : V_0 = D_1^n : D_0^n = V_1 : 1, \\
&V_0 \left| n+1, n-2, \cdots, 1,0 \right| : V_1 = D_2^n : D_1^n, \\
&V_0 \left| n+2, n-2, \cdots, 1,0 \right| : \left| n+1, n-2, \cdots, 1,0 \right| = D_3^n : D_2^n, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&V_0 \left| k, n-2, \cdots, 1,0 \right| : \left| k-1, n-2, \cdots, 1,0 \right| = D_{k-n+1}^n : D_{k-n}^n .
\end{aligned}
$$

Multiplying the terms of the first ratios in each of these equations by 1, V_0, V_0^2, $\cdots$, V_0^{k-n}, we get the continued proportion

$$
\begin{aligned}
&V_0^{k-n+1} \left| k, n-2, \cdots, 1,0 \right| : V_0^{k-n} \left| k-1, n-2, \cdots, 1,0 \right| : \cdots \\
&: V_0^2 \left| n+1, n-2, \cdots, 1,0 \right| : V_0 V_1 : V_0 = D_{k-n+1}^n : D_{k-n}^n : \cdots : D_2^n : D_1^n : 1 .
\end{aligned}
$$

Hence we find that

$$
V_0^{k-n} \left| k, n-2, \cdots, 1,0 \right| = D_{k-n+1}^n .
$$

We have therefore proved the following theorem:

Theorem II. *The generalized Vandermonde determinant* $|k, n-2, \cdots, 1, 0|$ *multiplied by the* $(k-n)$*th power of the principal Vandermondian is equal to the determinant* D_{k-n+1}^n (*see Definition* I).

3. The determinants $|k, j, n-3, \cdots, 1, 0|$

Our next task is to develop a formula for the determinant in which the two highest exponents of the variables are allowed to vary. We shall attempt to compare $|k, j, n-3, \cdots, 1, 0|$, where $j>n-2$, with $|k, n-2, \cdots, 1, 0|$. To do this we consider n systems of $k-n+1$ equations each:

$$\begin{aligned}
p_0a_i^k + p_1a_i^{k-1} + \cdots + p_{k-j}a_i^j + \cdots + p_na_i^{k-n} &= 0,\\
p_0a_i^{k-1} + \cdots + p_{k-j-1}a_i^j + \cdots + p_na_i^{k-n+1} &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
p_0a_i^n + \cdots + p_n &= 0,
\end{aligned}$$

where i takes the values 1 to n inclusive. Eliminating the $k-n$ sets of variables $a_i^{k-1}, \cdots, a_i^{j+1}, a_i^{j-1}, \cdots, a_i^{n-1}$, we get the following determinant of order $k-n+1$:

$$\begin{vmatrix}
p_1 \cdots p_{k-j-1}\ p_{k-j+1} \cdots p_n\ 0 \cdots 0 & p_0a_i^k + p_{k-j}a_i^j\\
p_0 \cdots p_{k-j-2}\ p_{k-j} \cdots p_n\ 0 \cdots 0 & p_{k-j-1}a_i^j\\
0\ p_0 \cdots p_{k-j-3}\ p_{k-j-1} \cdots p_n\ 0 \cdots 0 & p_{k-j-2}a_i^j\\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\\
0 \cdots 0\ p_0 \cdots\cdots p_n & p_{2n-j-1}a_i^j\\
0 \cdots 0\ p_0 \cdots\cdots p_{n-1} & p_{2n-j-2}a_i^j + p_na_i^{n-2}\\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\\
0 \cdots 0\ p_0\ p_2 \cdots p_{n-j+2} & p_1a_i^j + p_{j-n+3}a_i^{n-2} + \cdots + p_na_i^{j-n+1}\\
0 \cdots 0\ p_1 \cdots p_{n-j+1} & p_0a_i^j + p_{j-n+2}a_i^{n-2} + \cdots + p_na_i^{j-n}\\
0 \cdots 0\ p_0 \cdots p_{n-j} & p_{j-n+1}a_i^{n-2} + \cdots + p_na_i^{j-n-1}\\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\\
0 \cdots\cdots 0\ p_0\ p_1 & p_2a_i^{n-2} + \cdots + p_{n-1}a_i + p_n
\end{vmatrix}_{k-n+1} = 0.$$

Writing this determinant as the sum of $n+1$ determinants by splitting the last column into the sum of n columns each of which has the same power of a in all of its elements, we have

$$U_0a_i^k + \begin{vmatrix}
p_1 \cdots p_{k-j-1}\ p_{k-j+1} \cdots p_n\ 0 \cdots 0 & p_{k-j}\\
p_0 \cdots p_{k-j-2}\ p_{k-j} \cdots p_n\ 0 \cdots 0 & p_{k-j-1}\\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\\
0 \cdots 0\ p_0\ p_2 \cdots\cdots p_{j-n} & p_1\\
0 \cdots 0\ p_1 \cdots\cdots p_{j-n-1} & p_0\\
0 \cdots 0\ p_0 \cdots\cdots p_{j-n-2} & 0\\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\\
0 \cdots\cdots 0\ p_0\ p_1 & 0
\end{vmatrix}_{k-n+1} a_i^j$$

$$
+\begin{vmatrix}
p_1 \cdots p_{k-j-1} & p_{k-j+1} \cdots p_n & 0 & \cdots 0 \\
p_0 \cdots p_{k-j-2} & p_{k-j} \quad \cdots \cdot & p_n & 0 \cdots 0 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\
0 \cdots\cdots\cdots 0 & p_0 \quad \cdots\cdot & p_{n-1} & p_n \\
0 \cdots\cdots\cdots & 0 \quad p_0 \cdots & p_{n-2} & p_{n-1} \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\
0 \cdots\cdots\cdots & \cdots\cdots 0 & p_0\, p_1 & p_2
\end{vmatrix}_{k-n+1}
\begin{array}{l} a_i^{n-2} + U_3 a_i^{n-3} + \cdots \\ \quad + U_{n-1}a_i + U_n = 0, \end{array}
$$

where the U's are polynomials in the p's. If, in the determinant which is the coefficient of $a_i{}^j$, we move the elements of the last column to their natural position, the $(k-j)$th column, we obtain $(-1)^{j-n+1}$ multiplying a determinant the form of D_{k-n+1}^n but having for its elements p's instead of V's. If, in the coefficient of $a_i{}^{n-2}$ we interchange rows and columns and invert the order of the rows and columns, we get a determinant differing from $D_{k-n+1}^n\ \{j-n+2\}$ only in that we have p's for elements instead of V's. Now apply the corollary to Theorem I to both determinants to change the p's to V's. All of the elements of the coefficient of $a_i{}^j$ will be positive if we multiply the odd rows and even columns by -1. To accomplish the same thing in the coefficient of a_i^{n-2}, we multiply the first $j-n+2$ rows by -1 and then multiply the odd rows and the even columns by -1. This gives us

$$
\begin{aligned}
W_0 a_i^k &+ (-1)^{j-n+1}(-1)^{k-n+1} D_{k-n+1}^n a_i{}^j \\
&\qquad + (-1)^{j-n+2}(-1)^{k-n+1} D_{k-n+1}^n \{j-n+2\} a_i^{n-2} \\
&\qquad + W_3 a_i^{n-3} + \cdots + W_{n-1} a_i + W_n = 0,
\end{aligned}
$$

where the W's are polynomials in the V's. Solving these equations for the ratio of the two D's, we have

$$
\begin{aligned}
&|\, k,j,n-3, \cdots, 1,0 \,| : |\, k,n-2, \cdots, 1,0 \,| \\
&\qquad = (-1)^{j+k-2n+3} D_{k-n+1}^n \{j-n+2\} : -(-1)^{j+k-2n+2} D_{k-n+1}^n .
\end{aligned}
$$

Multiplying the left members of the proportion by V_0^{k-n}, we get

$$
\begin{aligned}
&V_0{}^{k-n} |\, k,j,n-3, \cdots, 1,0 \,| : V_0^{k-n} |\, k,n-2, \cdots, 1,0 \,| \\
&\qquad = D_{k-n+1}^n \{j-n+2\} : D_{k-n+1}^n .
\end{aligned}
$$

But $V_0^{k-n}\,|k, n-2, \cdots, 1, 0\,| = D_{k-n+1}^n$, hence

$$
V_0^{k-n} |\, k,j,n-3, \cdots, 1,0 \,| = D_{k-n+1}^n \{j-n+2\}.
$$

We may summarize this result in

THEOREM III. *The generalized Vandermonde determinant* $|k, j, n-3, \cdots, 1, 0|$ *multiplied by the* $(k-n)$*th power of its principal is equal to the determinant* $D^n_{k-n+1}\{j-n+2\}$ (*see Definition* II).

4. THE GENERAL VANDERMONDIAN

By the use of mathematical induction we shall now prove the formula for the determinant in which we vary any number of exponents of the variables. We note the special case in Theorem III and then proceed to compare $|t_1, t_2, \cdots, t_s, n-s-1, \cdots, 1, 0|$ with $|t_1, t_2, \cdots, t_{s-1}, n-s, \cdots, 1, 0|$. Consider the t_1-n systems of equations

$$\sum_{j=0}^{n} p_j a_i^{l-j} \qquad (l = t_1, t_1 - 1, \cdots, n;\ i = 1, 2, \cdots, n).$$

We choose to eliminate all of the variables except $a_i^{t_1}, a_i^{t_2}, \cdots, a_i^{t_{s-1}}, a_i^{t_s}, a_i^{n-s}, a_i^{n-s-1}, \cdots, a_i, 1$, giving us the following (t_1-n+1)th order determinant:

$$\begin{vmatrix} p_1 & p_2 \cdots p_{t_1-t_2-1} & p_{t_1-t_2+1} \cdots & \cdots & \cdots p_{t_1-t_s-1} & p_{t_1-t_s+1} \cdots & p_n & 0 & \cdots 0 & p_0a_i^{t_1} + p_{t_1-t_2}a_i^{t_2} + \cdots + p_{t_1-t_s}a_i^{t_s} \\ p_0 & p_1 \cdots p_{t_1-t_2-2} & p_{t_1-t_2} \cdots & \cdots & \cdots p_{t_1-t_s-2} & p_{t_1-t_s} \cdots & \cdot & p_n & 0 \cdots 0 & p_{t_1-t_2-1}a_i^{t_2} + \cdots + p_{t_1-t_s-1}a_i^{t_s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & \cdots & 0 & p_0 & \cdots & \cdot & \cdot & \cdot & p_{n-1} \cdots + p_{2n-s-t_s}a_i^{t_s} + p_na_i^{n-s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & p_0 \cdot & \cdot & \cdot & p_{s-1} \quad p_{n-t_s}a_i^{t_s} + p_sa_i^{n-s} + \cdots \end{vmatrix}_{t_1+n+1} = 0,$$

where it is to be understood that $p_k = 0$, whenever $k > n$. Writing this determinant as the sum of $n+1$ determinants by regarding the last column as the sum of n columns each of which has the same power of a in all of its elements, we obtain

$$M_1a_i^{t_1} + M_2a_i^{t_2} + \cdots + M_{s-1}a_i^{t_{s-1}}$$

$$+ \begin{vmatrix} p_1 & p_2 \cdots & \cdots & \cdots p_{t_1-t_s-1} & p_{t_1-t_s+1} \cdots & p_n & 0 & \cdots 0 & p_{t_1-t_s} \\ p_0 & p_1 \cdots & \cdots & \cdots p_{t_1-t_s-2} & p_{t_1-t_s} & \cdots & p_n & 0 \cdots 0 & p_{t_1-t_s-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots 0 & p_0 & \cdots p_{n-t_s-1} & p_{n-t_s+1} & \cdots & \cdots & p_{s-1} & p_{n-t_s} \end{vmatrix}_{t_1-n+1} a_i^{t_s}$$

$$+\begin{vmatrix} p_1 & p_2 & \cdots & \cdots & \cdots & p_{t_1-t_s-1} & p_{t_1-t_s+1} & \cdots & p_n & 0 & \cdots & 0 & 0 \\ p_0 & p_1 & \cdots & \cdots & \cdots & p_{t_1-t_s-2} & p_{t_1-t_s} & \cdots & & p_n & 0 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & \cdots & \cdots & & 0 & p_0 & \cdots & \cdots & & \cdots & p_{n-1} & p_n \\ 0 & \cdots & \cdots & \cdots & & & 0 & p_0 & \cdots & \cdots & \cdots & p_{n-2} & p_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & \cdots & \cdots & & & & & 0 & p_0 & \cdots & p_{s-2} & p_s \end{vmatrix}_{t_1-n+1} a_i^{n-s}$$

$$+ N_{s+1}a_i^{n-s-1} + N_{s+2}a_i^{n-s-2} + \cdots + N_{n-1}a_i + N_n = 0,$$

where the M's and the N's are polynomials in the p's. Let us consider the determinants which are the coefficients of $a_i^{t_s}$ and a_i^{n-s}. Place the elements of the last column of the coefficient of $a_i^{t_s}$ in their natural position, the (t_1-t_s-s+2)th column; then interchange rows and columns; then invert the order of the rows and columns, giving us $(-1)^{t_s+s-n-1}$ multiplying a determinant of the form of $D_{t_1-n+1}\{t_2-n+2, t_3-n+3, \cdots, t_{s-1}-n+s-1\}$, but having for its elements p's in place of V's. If, in the coefficient of a_i^{n-s}, we interchange rows and columns and then invert the order of rows and columns, we get a determinant differing from $D^n_{t_1-n+1}\{t_2-n+2,\ t_3-n+3, \cdots, t_s-n+s\}$ only in that the elements are p's instead of V's. We now apply the corollary to Theorem I to both determinants to change the p's to V's. All of the elements of the coefficient of $a_i^{t_s}$ will be positive if we multiply the first t_2-n+2 rows by -1, the first t_3-n+3 rows again by $-1, \cdots$ the first $t_{s-1}-n+s-1$ rows again by -1; and then multiply the odd rows and even columns by -1. This requires $(\sum_{j=2}^{s-1} t_j-n+j)+t_1-n+1=\sum_{j=1}^{s-1}t_j-n+j$ changes in sign. When this number is added to the $t_s-n+s-1$ changes mentioned above, we get $-1+\sum_{j=1}^{s}t_j-n+j$ changes. We can make all of the changes of the coefficient of a_i^{n-s} positive by multiplying the first t_2-n+2 rows by -1, the first t_3-n+3 rows again by $-1, \cdots$ the first t_s-n+s rows again by -1; and then multiplying the odd rows and the even columns by -1. This requires $\sigma\equiv\sum_{j=1}^{s}t_j-n+j$ changes in sign. Our equation then takes the form

$$\begin{aligned} X_1a_i^{t_1} &+ X_2a_i^{t_2} + \cdots + X_{s-1}a_i^{t_{s-1}} \\ &+ (-1)^{-1+\sigma}D^n_{t_1-n+1}\{t_2+n+2, t_3+n+3, \cdots, t_{s-1}-n+s-1\}a_i^{t_s} \\ &+ (-1)^{\sigma}D^n_{t_1-n+1}\{t_2-n+2, t_3-n+3, \cdots, t_s-n+s\}a_i^{n-s} \\ &+ Y_{s+1}a_i^{n-s-1} + Y_{s+2}a_i^{n-s-2} + \cdots + Y_{n-1}a_i + Y_n = 0, \end{aligned}$$

where the X's and the Y's are polynomials in the V's. We can then say

$$
\begin{aligned}
|t_1,t_2,\cdots,t_s,n-s-1,\cdots,1,0| &: |t_1,t_2,\cdots,t_{s-1},n-s,\cdots,1,0| \\
&= D^n_{t_1-n+1}\{t_2-n-2,t_3-n+3,\cdots,t_s-n+s\} \\
&\quad : D^n_{t_1-n+1}\{t_2-n+2,t_3-n+3,\cdots,t_{s-1}-n+s-1\}.
\end{aligned}
$$

Multiplying the left members of the proportion by $V_0{}^{t_1-n}$ and noting the special case where $s=2$, and $t_{s-1}=k$, and $t_s=j$ in the preceding theorem, we have

$$
\begin{aligned}
V_0{}^{t_1-n}&|t_1,t_2,\cdots,t_s,n-s-1,\cdots,1,0| \\
&= D^n_{t_1-n+1}\{t_2-n+2,t_3-n+3,\cdots,t_s-n+s\}.
\end{aligned}
$$

We may now state our general theorem:

THEOREM IV. *The generalized Vandermonde determinant* $|t_1, t_2, \cdots, t_s, n-s-1, \cdots, 1, 0|$ *multiplied by the* (t_1-n)*th power of the principal Vandermondian is equal to the determinant* $D^n_{t_1-n+1}$ $\{t_2-n+2, t_3-n+3, \cdots, t_s-n+s\}$.

5. Symmetric functions

We shall now use the preceding results to devise a method for expressing any integral rational symmetric function in terms of elementary symmetric functions. To accomplish this, we make use of the following theorem of Muir's: The product of a simple alternant and a single symmetric function of its variables is expressible by a sum of simple alternants, whose indices are got by arranging the variables in every term of the symmetric function in the same order, and adding the indices of each term to the indices of the original alternant, the first to the first, the second to the second, and so on.* For example

$$
\begin{aligned}
|a^0b^1c^2|\ \textstyle\sum a^3b &= |a^3b^2c^2| + |a^3b^1c^3| + |a^1b^4c^2| + |a^0b^4c^3| \\
&\quad + |a^1b^1c^5| + |a^0b^2c^5| = |a^0b^2c^5| - |a^0b^3c^4| - |a^1b^2c^4|.
\end{aligned}
$$

Using this theorem, we can express as a function of generalized Vandermondians the product of any integral rational symmetric function by the principal Vandermondian in the same number or larger number of variables. Since every generalized Vandermondian is expressible in terms of principal and secondary Vandermondians, and these in turn are expressible in terms of elementary symmetric functions, we can express every integral rational symmetric function in terms of elementary symmetric functions. For

* Muir's *Theory of Determinants*, vol. IV, p. 151.

example, to compute $\sum a^2bc$ in four variables, we multiply the principal Vandermondian in four variables by the given function:

$$\begin{aligned}|a^0b^1c^2d^3| \cdot \sum a^2bc = |a^2b^1c^3d^4| + |a^1b^3c^2d^4| + |a^1b^2c^4d^3| \\ + |a^0b^2c^3d^5| = |a^0b^2c^3d^5| - 3|a^1b^2c^3d^4|.\end{aligned}$$

Using the notation used in the preceding sections,

$$\begin{aligned} V_0 \sum a^2bc &= |5\,3\,0\,2| - 3|4\,3\,2\,1| \\ &= \frac{\begin{vmatrix} V_3 & V_4 \\ V_0 & V_1 \end{vmatrix}}{V_0} - 3V_4 \end{aligned}$$

or

$$\begin{aligned} \sum a^2bc &= \frac{V_1V_3 - V_0V_4}{V_0^2} - \frac{3V_4}{V_0} \\ &= E_1E_3 - 4E_4. \end{aligned}$$

We shall now prove that the result for n variables, where $n>r$, can be obtained if a symmetric function of degree r is multiplied by the principal Vandermondian in r variables. That is to say, when we express $\sum a_0^{m_0}a_1^{m_1} \cdots a_t^{m_t}$ for n variables, where $m_0 \geqq m_1 \geqq \cdots \geqq m_t$ and $\sum_{i=0}^{t} m_i = r$, in terms of elementary symmetric functions, none of the subscripts of the E's can exceed r. To show this, we multiply $|a_0{}^0a_1{}^1 \cdots a_{n-1}^{n-1}|$ by $\sum a_0^{m_0}a_1^{m_1} \cdots a_t^{m_t}$. The product will consist of a number of generalized Vandermondians. The highest exponent which can occur in the symbols for any of these general Vandermondians is m_0+n-1 and the number of exponents in these symbols which are increased above what they are for the principal Vandermondian is at most t, since there are only t exponents in the given symmetric function. Hence the highest subscript in any of the D's is $m_0+n-1-n+1$ and no D followed by a brace can have more than t elements in the brace. Hence the highest subscript in any V can not exceed m_0+t, but since $m_1, m_2, \cdots, m_t$ are all greater than or equal to 1, it follows that $r=m_0+\sum_{i=1}^{t} m_i \geqq m_0+t$, so that no subscript of any V can exceed r.

Before concluding the paper, it may not be amiss to present another example. In order to express $\sum a^3b^2$ in terms of elementary symmetric functions, we multiply it by the principal Vandermondian in 5 variables:

$$\begin{aligned}|a^0b^1c^2d^3e^4| \sum a^3b^2 = |a^3b^1c^2d^5e^4| + |a^0b^4c^2d^3e^6| \\ + |a^2b^1c^5d^3e^4| + |a^0b^1c^5d^3e^6| + |a^0b^3c^2d^6e^4| + |a^0b^1c^4d^3e^7| + |a^0b^1c^2d^5e^7|.\end{aligned}$$

Changing notation, we get

$$V_0 \sum a^3b^2 = |75210| - |74310| - |65310| + 2|64320| - 2|54321|$$

$$= \frac{\begin{vmatrix} V_2 & V_3 & V_4 \\ V_1 & V_2 & V_3 \\ 0 & V_0 & V_1 \end{vmatrix}}{V_0^2} - \frac{\begin{vmatrix} V_3 & V_4 & V_5 \\ V_0 & V_1 & V_2 \\ 0 & V_0 & V_1 \end{vmatrix}}{V_0^2} - \frac{\begin{vmatrix} V_3 & V_4 \\ V_1 & V_2 \end{vmatrix}}{V_0} + 2\frac{\begin{vmatrix} V_4 & V_5 \\ V_0 & V_1 \end{vmatrix}}{V_0} - 2V_5$$

$$= \frac{V_1V_2^2 - 2V_1^2V_3 - V_0V_2V_3 + 5V_0V_1V_4 - 5V_0^2V_5}{V_0^2}.$$

Or

$$\sum a^3b^2 = E_1E_2^2 - 2E_1^2E_3 - E_2E_3 + 5E_1E_4 - 5E_5.$$

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