TETRADS OF RULED SURFACES*

BY

A. F. CARPENTER

The projective differential geometry of a configuration composed of three ruled surfaces whose generators are in correspondence in sets of three has been discussed in two recent papers by the author.† It is the purpose of the present paper to present briefly an analytic basis for the projective differential geometry of a configuration composed of four ruled surfaces whose generators correspond in sets of four, each set containing one line element from each surface. This correspondence is brought about by choice of a parameter common to all four surfaces.

For defining system we choose a set of eight ordinary first-order linear and homogeneous differential equations in eight dependent variables, together with four linear and homogeneous equations of order zero:

$$(1)\qquad \alpha_i' = \sum_{j=i}^{i+3} a_{ij}\alpha_j, \qquad \alpha_{i+1}' = \sum_{j=i}^{i+3} a_{i+1,j}\alpha_j \qquad (i = 1,3,5,7),$$

$$(2)\qquad \sum_{l=1}^{8} b_{kl}\alpha_l = 0 \qquad (k = 1,2,3,4),$$

where a_{ij}, b_{kl} are functions of the independent variable x, and where all values of i, j, k or l greater than 8 are understood to be replaced with their residues, mod 8.‡

We take, further,

$$(3)\qquad a_{i,i+2}a_{i+1,i+3} - a_{i,i+3}a_{i+1,i+2} \neq 0 \qquad (i = 1,3,5,7),$$

and

$$(4)\qquad \begin{aligned} D_{l,l+1,l+2,l+3} &\equiv \left| b_{1l}b_{2,l+1}b_{3,l+2}b_{4,l+3} \right| \neq 0, \\ D_{l,l+1,l+4,l+5} &\equiv \left| b_{1l}b_{2,l+1}b_{3,l+4}b_{4,l+5} \right| \neq 0 \qquad (l = 1,3,5,7). \end{aligned}$$

A fundamental system of simultaneous solutions,

$$\alpha_i^{(h)}, \quad \alpha_{i+1}^{(h)} \qquad (h = 1,2,3,4\ ;\ i = 1,3,5,7),$$

* Presented to the Society, San Francisco Section, June 2, 1928; received by the editors February 8, 1929.

† *Triads of ruled surfaces*, these Transactions, vol. 29 (1927). *Complete systems of invariants and covariants for triads of ruled surfaces*, Tôhoku Mathematical Journal, vol. 30 (1929).

‡ Throughout the discussion all single subscripts greater than 8 and all digits greater than 8 in multiple subscripts are understood to be replaced with their residues, mod 8.

of equations (1) can be interpreted as determining four lines $l_{i,i+1}$ given by the four pairs of points

$$(\alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)}, \alpha_i^{(4)}), \quad (\alpha_{i+1}^{(1)}, \alpha_{i+1}^{(2)}, \alpha_{i+1}^{(3)}, \alpha_{i+1}^{(4)}).$$

By virtue of conditions (4) no two of these lines intersect. As x varies, these lines trace four ruled surfaces which, due to conditions (3), are non-developables.

It is to be noted that we can recover from equations (1) and (2), for any one of the four surfaces, the pair of second-order differential equations in two dependent variables which is fundamental in the projective differential geometry of ruled surfaces as developed by Wilczynski. For the surface R_{12} generated by l_{12} we annex to the original twelve equations those obtained by differentiating the pair

$$\alpha_1' = \sum_{j=1}^{4} a_{1j}\alpha_j, \qquad \alpha_2' = \sum_{j=1}^{4} a_{2j}\alpha_j.$$

From the set of fourteen equations thus derived, the twelve variables α_i', α_i, a_{i+1}', α_{i+1} $(i=3, 5, 7)$ can be eliminated, leaving the required pair of second-order equations in α_1, α_2. Finally from (1) and (2) by simple algebraic processes we can obtain the defining system for any of the four triads of surfaces, or any of the six pairs of surfaces, of the tetrad.

Except for conditions (3) no restrictions need be placed on the functions a_{ij} although it will be convenient to insist that they be analytic. For the functions b_{kl}, however, conditions (4) are not the only restrictions. If the four equations (2) be once differentiated and the derivatives of the dependent variables be replaced with their values from equations (1), there result four new equations of type (2). These, together with (2), constitute a linear homogeneous system of eight equations in the eight variables α_j and consistency demands the vanishing of the determinant D of the coefficients. This determinant involves the a_{ij} together with the b_{kl} and their first derivatives. Any choice of the b_{kl} consistent with the vanishing of this determinant and with conditions (4) will be suitable for our purpose.

The fact that equations (2) must be satisfied by any eight independent particular solutions of equations (1) suggests that we choose the b_{kl} as solutions of the system of differential equations adjoint to system (1) in such a way that

$$(5)\quad b_{kl}' = -\sum_{j=l+6}^{l+9} a_{jl}b_{kj}, \qquad b_{k,l+1}' = \sum_{j=l+6}^{l+9} a_{j,l+1}b_{kj} \quad (l = 1,3,5,7\,;\ k = 1,2,3,4).$$

Since conditions (5) are consistent with (4) and cause the determinant D to vanish, they will be assumed to hold throughout the remainder of the discussion.

Permissible transformations on the variables are of the types standard in the projective differential geometry of ruled surfaces, that on the independent variable being

$$\bar{x} = \xi(x), \tag{6}$$

and those on the dependent variables,

$$\bar{\alpha}_i = c_i\alpha_i + c_{i+1}\alpha_{i+1}, \quad \bar{\alpha}_{i+1} = d_i\alpha_i + d_{i+1}\alpha_{i+1} \qquad (i = 1,3,5,7), \tag{7}$$

the coefficients being functions of x. For all systems obtained from (1) and (2) by transformations (6) and (7), conditions (3), (4), (5) will hold.

From (2), (6) and (7) we find that the expressions

$$b_{ki}\alpha_i + b_{k,i+1}\alpha_{i+1} \qquad (i = 1,3,5,7\ ;\ k = 1,2,3,4) \tag{8}$$

are absolute covariants and that

$$D_{jk,i} \equiv b_{ji}b_{k,i+1} - b_{ki}b_{j,i+1} \quad (i = 1,3,5,7\ ; j = 1,2,3\ ; k = 2,3,4\ ; j < k) \tag{9}$$

are relative invariants, since

$$D_{jk,i} \equiv \Delta_{(i+1)/2}\bar{D}_{jk,i}, \tag{10}$$

where

$$\Delta_{(i+1)/2} = c_i d_{i+1} - c_{i+1}d_i,$$

and since the b_{kl} are invariant under transformation of the independent variable.

We are now in a position to transform system (1), (2) into an equivalent system whose coefficients will be absolute invariants. We choose for the transformation of the dependent variables

$$\bar{\alpha}_i = b_{1i}\alpha_i + b_{1,i+1}\alpha_{i+1}, \ \bar{\alpha}_{i+1} = b_{2i}\alpha_i + b_{2,i+1}\alpha_{i+1} \quad (i = 1,3,5,7). \tag{11}$$

By making use of equations (5) there results the system

$$\begin{aligned} \bar{\alpha}_i' &= -\theta_{12,i}\bar{\alpha}_i + \theta_{11,i}\bar{\alpha}_{i+1} + \theta_{12,i+2}\bar{\alpha}_{i+2} - \theta_{11,i+2}\bar{\alpha}_{i+3}, \\ \bar{\alpha}_{i+1}' &= -\theta_{22,i}\bar{\alpha}_i + \theta_{21,i}\bar{\alpha}_{i+1} + \theta_{22,i+2}\bar{\alpha}_{i+2} - \theta_{21,i+2}\bar{\alpha}_{i+3}, \end{aligned} \tag{12}$$

$$\begin{gathered} \sum_i \bar{\alpha}_i = 0, \qquad \sum_i \bar{\alpha}_{i+1} = 0, \\ \sum_i (D_{32,i}\bar{\alpha}_i - D_{31,i}\bar{\alpha}_{i+1})/D_{12,i} = 0, \quad \sum_i (D_{42,i}\bar{\alpha}_i - D_{41,i}\bar{\alpha}_{i+1})/D_{12,i} = 0 \\ (i = 1,3,5,7), \end{gathered} \tag{13}$$

where $D_{jk,i}$ are given by (9), and where

$$(14)\quad D_{12,i}\theta_{kl,i} = a_{i+6,i}b_{k,i+6}b_{l,i+1} - a_{i+6,i+1}b_{k,i+6}b_{li} + a_{i+7,i}b_{k,i+7}b_{l,i+1} - a_{i+7,i+1}b_{k,i+7}b_{li} \quad (k = 1,2;\ l = 1,2).$$

As a consequence of (10) the coefficients of equations (13) are absolute invariants. It can be shown without difficulty that the coefficients of equations (12) are relative invariants of weight one. The effect of a transformation of the independent variable $\bar{x}=\xi(x)$ is to divide all coefficients of differential equations of the first order by ξ'. If then we choose ξ so that $\xi'=$ any $\theta_{kl,i}$, *equations* (13) *will be replaced with a set whose coefficients are all absolute invariants.*

Consider now any four space curves whose points correspond in fours, each four consisting of one point from each curve. If each such quadruple has the property that the four points belonging to it are coplanar, then we shall speak of the four curves as a quadrangular set. The geometric significance of transformation (11) is now apparent. *When the system of defining differential equations is of the form* (12), (13), *the tetrad of ruled surfaces* $R_{i,i+1}$ *is referred to eight covariant curves and these curves constitute two quadrangular sets, each set containing one curve from each surface.*

System (12), (13) is not unique since one of the same form could have been obtained by making use of any of the six transformations

$$(15)\quad \bar{\alpha}_i = b_{ki}\alpha_i + b_{k,i+1}\alpha_{i+1}, \quad \bar{\alpha}_{i+1} = b_{li}\alpha_i + b_{l,i+1}\alpha_{i+1} \quad \begin{pmatrix} i = 1,3,5,7\ ; \\ k = 1,2,3\ ;\quad l = 2,3,4\ ;\quad k < l \end{pmatrix}.$$

In equations (13) there appear as coefficients sixteen absolute invariants involving the b_{kl} only and in equations (12), sixteen relative invariants of weight one, giving in all thirty-one absolute invariants involving the a_{ij} and b_{kl}. The six systems of type (12), (13) thus involve one hundred and eighty-six absolute invariants. But of invariants involving the a_{ij} and b_{kl} only, the theory calls for but thirty-one functionally independent, and investigation discloses that those already obtained in (12), (13) are independent. *Hence these constitute a fundamental system in terms of which all others of this type can be expressed.*

A fruitful source of invariants and covariants in this theory is found in the consideration of that property possessed by four skew straight lines of having only two straight line intersectors. The four lines $l_{i,i+1}$ $(i=1, 3, 5, 7)$ determine two such intersectors. The pairs of points determined on each of the lines $l_{i,i+1}$ by these two intersectors are given by quadratic covariants and these covariants involve, in addition to the α_i, α_{i+1}, the coefficients of equations (2) [or (13)] only.

Considering next the ordinary tangents to the surface $R_{i,i+1}$ at points of $l_{i,i+1}$, there will be two which intersect both of the lines $l_{i+2,i+3}$, $l_{i+4,i+5}$, two which intersect both of the lines $l_{i+2,i+3}$, $l_{i+6,i+7}$, and two which intersect both of the lines $l_{i+4,i+5}$, $l_{i+6,i+7}$. There are thus determined nine pairs of covariant points on each line $l_{i,i+1}$, three of the nine corresponding to tangents to $R_{i,i+1}$ and the remaining six determined by tangents to the other three surfaces. Each such pair of points is given by a quadratic covariant which, in addition to involving the coefficients of equations (2) [or (13)] will be linear in the coefficients of equations (1) [or (12)], that is, of weight one.

Again, of the ordinary tangents to $R_{i,i+1}$ at points of $l_{i,i+1}$ there will be two which are tangents to $R_{i+2,i+3}$ at points of $l_{i+2,i+3}$, two which are tangents to $R_{i+4,i+5}$ at points of $l_{i+4,i+5}$ and two which are tangents to $R_{i+6,i+7}$ at points of $l_{i+6,i+7}$.* There are thus determined three pairs of covariant points on each line $l_{i,i+1}$, each pair given by a covariant which in addition to involving the coefficients of equations (2) [or (13)] will be quadratic in the coefficients of equations (1) [or (12)], that is, of weight two.

Considering finally the asymptotic tangents to $R_{i,i+1}$ at points of $l_{i,i+1}$, there are two which intersect $l_{i+2,i+3}$, two which intersect $l_{i+4,i+5}$, and two which intersect $l_{i+6,i+7}$.† There are thus determined on each line $l_{i,i+1}$ six pairs of covariant points, three pairs corresponding to asymptotic tangents to $R_{i,i+1}$ and the other three determined by asymptotic tangents to the other three surfaces. Each such pair of covariant points is given by a covariant of weight three.

To complete the count we may if we wish include the pair of flecnode points on $l_{i,i+1}$ which will correspond to a covariant of weight four. There are thus associated with each of the four lines $l_{i,i+1}$ $(i=1, 3, 5, 7)$ twenty quadratic covariants. The coincidence of the two points in any pair will be conditioned upon the vanishing of the discriminant of the corresponding covariant. All such discriminants are invariants.

Of the sequence of n points given by the expression

$$\phi_{(i+1)/2} = \kappa_{(i+1)/2}\alpha_i + \lambda_{(i+1)/2}\alpha_{i+1} \qquad (i = 1,3,5, \cdots, 2n-1),$$

those for which $n=4m+1, m=0,1,2, \cdots$, lie on the line l_{12} and the remaining lie in sets of three on the lines l_{34}, l_{56}, l_{78}, one on each line in order. If each point is such that the intersector tangent‡ at that point passes through the

* Lane, *Ruled surfaces with generators in one-to-one correspondence*, these Transactions, vol. 25, pp. 290, 291.

† See reference above.

‡ Lane, *Ruled surfaces in correspondence.*

next point in order, the n points constitute an intersector sequence.* We will suppose that the last point of the sequence is given by an n of the form $4m+1$ so that of the n points $m+1$ of them, including the first and the last, lie on l_{12}. If all $m+1$ are distinct, the n points constitute an open sequence of order m; if the last coincides with the first but the remaining $m-1$ are distinct, the n points constitute a closed sequence of order m.

A study of the conditions under which intersector sequences of orders $m=1, 2, 3, \cdots$ are closed gives rise to invariants involving only the a_{ij} and of weights expressible as functions of m. Methods of obtaining invariants of this type are available in a previous paper.†

We have seen above how on each line $l_{i,i+1}$ there are, in addition to the pair of flecnode points, nineteen other pairs which, from the method of their determination, are analogous to the flecnode points. The totality of these points for all line elements of $R_{i,i+1}$ constitutes nineteen surface curves each in general of two branches. Analogues to the complex points and the complex curves of a ruled surface are equally easy to define.

The four lines $l_{i,i+1}$ determine four quadrics $Q_{i,i+1}$ $(i=1, 3, 5, 7)$, each line lying on three of the four quadrics. Any two lines on R_{12} together with the three lines l_{34}, l_{56}, l_{78} determine a linear complex. As the two lines on R_{12} approach coincidence this complex approaches a limit C_{12}. There are thus determined four linear complexes $C_{i,i+1}$ $(i=1, 3, 5, 7)$, and each line $l_{i,i+1}$ belongs to all four complexes.

To each point of l_{12} there corresponds the tangent plane at that point to Q_{34} and this plane contains l_{12}. Similarly, to each point of l_{12} there corresponds its polar plane with respect to C_{12} and this plane also contains l_{12}. The tangent plane and the polar plane for the same point are in general distinct but for two points on l_{12} they coincide. This pair of points is given by a quadratic covariant. Since the three quadrics and the four complexes can be paired as above in twelve different ways, there are thus determined on each line $l_{i,i+1}$ twelve pairs of covariant points giving rise on each surface $R_{i,i+1}$ to twelve curves which are analogues of the complex curve. The possibilities for such analogues are by no means exhausted. But there is no need to pursue this matter further. Additional sources of invariants and covariants readily suggest themselves and the theory as here outlined can be developed in as great detail as desired.

* *Triads of ruled surfaces.*

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UNIVERSITY OF WASHINGTON,
SEATTLE, WASH.