ON THE ZEROS OF EXPONENTIAL POLYNOMIALS*

BY

J. F. RITT

By an exponential polynomial, we shall mean a function

$$(1) \qquad a_0 e^{\alpha_0 z} + \cdots + a_m e^{\alpha_m z}$$

with constant a's and with constant α's distinct from one another. The distribution of the zeros of such functions, and of more general functions in which the a's are polynomials in z, rather than constants, has been investigated by Tamarkin, Pólya and Schwengler†. The very elegant results secured by them will be described, to some extent, below. The present writer has treated the question of factorizing an exponential polynomial into a product of exponential polynomials‡.

We present here two results. In §1, we prove that if every zero of one exponential polynomial is also a zero of a second exponential polynomial, the quotient of the second function by the first is an exponential polynomial. In §2, we study the function

$$1 + a_1 e^{\alpha_1 z} + \cdots + a_m e^{\alpha_m z},$$

with real α's such that

$$0 < \alpha_1 < \cdots < \alpha_m.$$

We consider any horizontal strip of the complex plane, and derive an expression for the sum of the real parts of those zeros of the exponential polynomial which are situated in the strip. The result obtained is analogous to the theorem that the product of the zeros of

$$1 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

is $(-1)^m / a_m$.

---

* Presented to the Society, October 27, 1928. Received by the editor of the Bulletin in November, 1928, accepted for publication in the Bulletin, and subsequently transferred to these Transactions.

† Tamarkin, Mathematische Zeitschrift, vol. 27 (1927), p. 1, and earlier papers there referred to; Pólya, Münchner Berichte, 1920; Schwengler, *Geometrisches ueber die Verteilung der Nullstellen* etc., Dissertation, Zurich, 1925.

‡ These Transactions, vol. 29 (1927), p. 584.

1. **Division.** THEOREM. *Let*

$$A(z) = a_0 e^{\alpha_0 z} + \cdots + a_m e^{\alpha_m z}, \qquad B(z) = b_0 e^{\beta_0 z} + \cdots + b_n e^{\beta_n z}. \tag{2}$$

Suppose that $B(z) \not\equiv 0$, *and that* $A(z)/B(z)$ *is an integral function. Then there exists a*

$$C(z) = c_0 e^{\gamma_0 z} + \cdots + c_p e^{\gamma_p z}$$

such that $A(z) = B(z)\, C(z)$.

We begin our proof by describing a result of Tamarkin, Pólya and Schwengler. Let the exponents α in $A(z)$ be plotted in the complex plane, and let the smallest convex polygon $\mathfrak{A}$ which contains them be constructed. Let the sides of $\mathfrak{A}$ be designated by

$$\sigma_1, \cdots, \sigma_l.$$

Let d_i, $i = 1, \cdots, l$, represent a ray which is the image, with respect to the real axis, of a perpendicular to σ_i erected exterior to $\mathfrak{A}$. It is proved by the above-named writers that there exist l half-strips,* each parallel to, and extending in the same direction as, one of the rays d_i, which contain all of the zeros of A. If s_i is the length of σ_i, the number of zeros in the half-strip parallel to d_i whose moduli are less than r is asymptotically equivalent to $rs_i/(2\pi)$.

Consider now the convex polygon $\mathfrak{B}$ corresponding to $B(z)$. As every zero of B is also a zero of A, it is clear, from the asymptotic formula for the number of zeros in a half-strip, that to every side τ of $\mathfrak{B}$ there corresponds a side of $\mathfrak{A}$ which is parallel to τ, at least as long as τ, and whose outward perpendicular has the same direction as the outward perpendicular to τ.

We shall suppose that the α's in (1) are so ordered that α_i comes before α_j if the real part of α_i is less than that of α_j, or if the real parts are equal but the coefficient of $(-1)^{1/2}$ in α_i is less than that in α_j. When the α's are real and non-negative, and $a_m \neq 0$, we shall call α_m the degree of the function (1).

LEMMA. *If* $A(z)$, *and* $B(z) \not\equiv 0$, *are two exponential polynomials with real non-negative exponents, we have*

$$A = QB + R, \tag{3}$$

where Q and R are two exponential polynomials with real non-negative exponents, and where R, if not zero, is of lower degree than B.

If, in (2), we have $\alpha_m < \beta_n$, we have (3) with $Q = 0$, $R = A$. Suppose then that $\alpha_m \geqq \beta_n$.

* By a half-strip is meant an infinite region comprised between two half-lines and a line perpendicular to both of them.

Let $\alpha_m, \cdots, \alpha_{m-i}$ be those α's which are at least as great as β_n. Consider the exponential polynomial

$$(4) \qquad C = A - \frac{(a_m e^{\alpha_m z} + \cdots + a_{m-i}e^{\alpha_{m-i}z})}{b_n e^{\beta_n z}} B,$$

whose exponents are non-negative.

Suppose first that B consists of one term. Then C is either zero, or of smaller degree than B, so that we have, in (4), a representation (3) with Q the fraction in the second member of (4), and $R=C$.

Suppose now that B has at least two terms. If C is not zero, its degree is either less than β_n or equal to $\alpha_m-(\beta_n-\beta_{n-1})$. If C is zero, or of degree less than β_n, we have, in (4), a representation (3). Otherwise, we repeat the process just described, subjecting C to the treatment received by A. As $\beta_n-\beta_{n-1}$ is a fixed quantity ,we arrive in a finite number of steps at a representation (3).

It is a simple consequence of the asymptotic formula for the distribution of the zeros of exponential polynomials that the representation (3) is unique.

We return now to the A and B of our theorem, whose exponents may, of course, be complex. We assume that $A \not\equiv 0$. Grouping together those terms of A whose exponents have like real parts, we write

$$(5) \qquad A = P_1 e^{u_1 z} + \cdots + P_j e^{u_j z},$$

where the u's are real, increasing with their subscripts, and where the P's are of the type

$$(6) \qquad g_1 e^{v_1 iz} + \cdots + g_p e^{v_p iz},$$

with real v's which increase with their subscripts.

As it does not disturb the zeros of A, or affect the divisibility problem which we are studying, to multiply A by an exponential, we suppose that u_1, and the smallest v in P_j are both zero.

Similarly, we suppose that

$$(7) \qquad B = Q_1 e^{w_1 z} + \cdots + Q_h e^{w_h z},$$

with stipulations identical with those made above for A.

The quantity u_j is the difference between the abscissas of any rightmost point and any leftmost point of $\mathfrak{A}$. Then because to every side of $\mathfrak{B}$ there corresponds a side of $\mathfrak{A}$ at least as long, and having the same direction, it is clear that $u_j \geqq w_h$.

Let us suppose, for the present, that B consists of at least two terms, that is, in (7), $h \geqq 2$, Q_1, $Q_h \not\equiv 0$.

Let $u_j, \cdots, u_{j-r}$ be those u's which exceed $u_j-(w_h-w_{h-1})$. We shall prove that the quotients of $P_j, \cdots, P_{j-r}$ by Q_h are exponential polynomials of the type (6).

If Q_h is a constant, this is certainly so. Suppose that Q_h is not a constant. Then $\mathfrak{B}$ has a right-hand vertical side whose length is the greatest v in Q_h. Then $\mathfrak{A}$ must have a right-hand side of at least the same length. That is, the greatest v in P_j is not less than that of Q_h. By the lemma above,

$$P_j = SQ_h + R$$

where S and R are of type (6), with non-negative v's, and where, if $R \neq 0$, the greatest v in R is less than that of Q_h.

We say that R is zero. Suppose that this is not so. Then

$$A - SBe^{(u_j-w_h)z} = \cdots + Re^{u_jz}. \tag{8}$$

The terms which precede the last in the second member of (8) are products of polynomials (6) by exponentials e^{dz} with every $d \geqq 0$ and less than u_j. Now the first member of (8) has every zero of B. But the right hand vertical side of the polygon for the first member of (8) is shorter than the corresponding side of $\mathfrak{B}$. This shows that $R \equiv 0$.

If $u_{j-1} > u_j-(w_h-w_{h-1})$, we have

$$A - SBe^{(u_j-w_h)z} = \cdots + P_{j-1}e^{u_{j-1}z},$$

and it follows as above the P_{j-1} is the product of Q_h by a polynomial of type (6)*. Similarly, $P_{j-2}, \cdots, P_{j-r}$ are such products.

Consider now

$$D = A - \frac{P_je^{u_jz} + \cdots + P_{j-r}e^{u_{j-r}z}}{Q_he^{w_hz}}B.$$

If it is not identically zero, it is of the form

$$D = S_1e^{t_1z} + \cdots + S_ke^{t_kz},$$

where the S's are of type (6), where the t's are non-negative and increasing, and where $t_k \leqq u_j-(w_h-w_{h-1})$. If D is not zero, we can, since D has every zero of B, repeat the above procedure. As w_h-w_{h-1} is a fixed positive quantity, we can repeat our process only a finite number of times, so that, at some stage, we must reach a function like D above, which is zero. When that happens, we have A expressed as the product of B by an exponential polynomial.

When $h=1$, in (7), we have $B=Q_h$. It follows, as above, that every P

* The presence of negative v's in P_{j-1} would be of no significance.

is the product of B by a function of type (7). Our theorem is thus completely proved.

2. **Real exponents.** We deal with functions of the type

$$f(z) = 1 + a_1 e^{\alpha_1 z} + \cdots + a_m e^{\alpha_m z},$$

where the a's are any constants with $a_m \neq 0$, and the α's are any real numbers such that

$$0 < \alpha_1 < \cdots < \alpha_m.$$

Because $f(z)$ is close to unity when x $(z = x + yi)$ is large and negative, and and close to ∞ when x is large and positive, there exist two vertical lines between which all of the zeros of $f(z)$ are comprised.

Let $R(u, v)$ be the sum of the real parts of those zeros of $f(z)$ for which $u < y < v$, where u and v are any real numbers with $v > u$. We shall prove that

$$R(u,v) = -\frac{(v - u)\log|a_m|}{2\pi} + O(1). \tag{9}$$

Let A be such that

$$|f(z) - 1| < 1 \tag{10}$$

for $x \leqq A$, and let $B > A$ be such that

$$\left|\frac{f(z)}{a_m e^{\alpha_m z}} - 1\right| < 1 \tag{11}$$

for $x \geqq B$. For any zero of $f(z)$, we have $A < x < B$.

Let S represent the sum of those zeros of $f(z)$ for which $u < y < v$. We assume that no zero of $f(z)$ lies on the lines $y = u$ or $y = v$. This assumption does not affect our results.

We have

$$2\pi i S = \int z \frac{f'(z)}{f(z)} dz, \tag{12}$$

the integration being performed in the positive sense around the rectangle of sides $x = A$, $x = B$, $y = u$, $y = v$.

Now

$$\int z \frac{f'(z)}{f(z)} dz = z \log f(z) - \int \log f(z) dz. \tag{13}$$

As $R(u, v)$ is the real part of S, we have to determine the imaginary part of the second member of (13).

We shall use that determination of $\log f(z)$ for which the coefficient of i at the point (A, v) is greater than $-\pi$ but not greater than π.

Let us determine first the variation of $z \log f(z)$ as z makes a circuit of the rectangle, starting from and returning to the point (A, v). Evidently $z \log f(z)$ is increased by $(A+vi)Ci$, where C is the variation in the amplitude of $f(z)$.

Because of (10), the variation of $\operatorname{amp} f(z)$ along the side $x=A$ is less than π. To get an idea of the variation along $y=u$ and $y=v$, we consider that*

$$\operatorname{amp} f(z) = \arctan \frac{Y}{X},$$

where X and Y are respectively the real and imaginary parts of $f(z)$. As z travels along a segment of the line $y=u$, for instance, $\operatorname{amp} f(z)$ cannot undergo a variation as great as π unless X is zero at some point on the segment. Hence the variation of $\operatorname{amp} f(z)$ along either horizontal side of our rectangle cannot exceed $\pi(p+1)$ where p is the number of zeros of X on such a side. On the line $y=u$, for instance,

$$X = 1 + b_1 e^{\alpha_1 x} + \cdots + b_m e^{\alpha_m x},$$

where the b's are real numbers depending upon u. It is known that a function like X cannot have more than n real zeros.† Hence the total variation of $\operatorname{amp} f(z)$ along $y=u$, $y=v$, $x=A$, is less than $(2n+3)\pi$.

In virtue of (11), the variation of $\operatorname{amp} f(z)$ along $x=B$ differs from the variation of the amplitude of $a_m e^{\alpha_m z}$ by less than π. The variation of the amplitude of $a_m e^{\alpha_m z}$ along $x=B$ is $\alpha_m(v-u)$. Hence the variation of $\operatorname{amp} f(z)$ as z goes around the rectangle differs from $\alpha_m(v-u)$ by less than $(2n+4)\pi$.

The change in $z \log z$ is thus of the form

$$(A+vi)[\alpha_m(v-u)+O(1)]i.$$

The coefficient of i in this variation is, since the $O(1)$ is real,

$$A[\alpha_m(v-u)+O(1)]. \tag{14}$$

We shall estimate the imaginary part of the integral of $\log f(z)$. We put upon A the further condition that $\log f(z)$ have, for $x \leqq A$, an expression as an absolutely convergent Dirichlet series

$$\log f(z) = c_1 e^{\rho_1 z} + c_2 e^{\rho_2 z} + \cdots,$$

where the ρ's are positive and increase indefinitely. We see immediately that

* Tamarkin, loc. cit., p. 27–28, or Wilder, *Expansion problems* etc., these Transactions, vol. 18 (1917), pp. 415–447; pp. 420–427.

† Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 49.

$$\int_{v+Ai}^{u+Ai} \log f(z)dz = O(1). \tag{15}$$

We now take the side $y=u$. The amplitude of $f(z)$ at (A,u), differing by less than π from the amplitude at (A,v), does not exceed 2π in absolute value. As the variation of the amplitude along $y=u$ is less than $(n+1)\pi$ the absolute value of amp $f(z)$ is less than $(n+3)\pi$ on $y=u$. Hence the imaginary part of the integral along $y=u$ is less in absolute value than

$$(n+3)\pi(B-A). \tag{16}$$

We need the integral along $x=B$ of the *real* part of $\log f(z)$. We put upon B the restriction that, for $x \geqq B$, $\log f(z)$ admit an absolutely convergent development

$$\log f(z) = \alpha_m z + \log a_m + d_1 e^{\sigma_1 z} + d_2 e^{\sigma_2 z} + \cdots,$$

where the σ's are negative and decrease indefinitely. Thus the coefficient of i in the integral along $x=B$ is

$$\alpha_m B(v-u) + (v-u)\log|a_m| + O(1). \tag{17}$$

Finally, we must have the integral along $y=v$ of the imaginary part of $\log f(z)$. Along $y=v$, we have

$$|\operatorname{amp} f(z) - \alpha_m(v-u)| < (2n+4)\pi.$$

Hence

$$\left| \int_{v+Bi}^{v+Ai} \operatorname{amp} f(z)dz - \alpha_m(v-u)(A-B) \right| < (2n+4)\pi(B-A). \tag{18}$$

Understanding now that A and B are fixed, we have $A=O(1)$, $B-A=O(1)$, and find, from (14), (15), (16), (17) and (18), the expression (9) for $R(u,v)$.

Columbia University,
New York, N. Y.