ON COMMUTATION FORMULAS IN THE ALGEBRA OF QUANTUM MECHANICS*

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INTRODUCTION

It is the purpose of this paper to make a study of commutation formulas in the algebra of quantum mechanics. The theories of quantum mechanics introduced by Heisenberg† and Dirac‡ are different in their conception and formulation but both make use of a non-commutative algebra. In Heisenberg's theory, the elements of the algebra are infinite matrices; in Dirac's they are abstract "q-numbers." Schrödinger's§ theory, although mathematically equivalent to Heisenberg's, does not make explicit use of this algebra. However, the operators which Schrödinger uses satisfy the same commutation formulas as Heisenberg's matrices.

We shall consider the algebra of the quantum mechanics from the matric standpoint although the results obtained do not depend upon the form of the variables. The variables enter in pairs as in classical mechanics and we shall use the expression "conjugate quantum variables" or simply "conjugate variables" in analogy with the idea of canonically conjugate variables in the classical theory. For a single pair of conjugate variables the properties of the algebra are determined by the fundamental commutation rule,‖

$$pq - qp = cI, \tag{1}$$

where q and p are matrices representing the coördinate and momentum respectively, c is a real or complex number and I is the unit matrix. In the quantum mechanics $c = h/(2\pi i)$, although the algebra does not depend upon the particular value assigned to c. The symbol I will be omitted but will be understood wherever a real or complex number occurs alone in a matric equation.

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† Heisenberg, W., Zeitschrift für Physik, vol. 33 (1925), pp. 879–893; Born and Jordan, Zeitschrift für Physik, vol. 34 (1925), pp. 858–888.

‡ Dirac, P. A. M., Proceedings of the Royal Society, (A), vol. 109 (1925), pp. 642–653.

§ Various papers in the Annalen der Physik beginning with vol. 79 (1926), pp. 361–376.

‖ Born and Jordan, loc. cit., p. 871.

For more than a single pair of conjugate quantum variables the relation (1) is replaced by the following:*

$$
\begin{aligned}
p_r q_s - q_s p_r &= c\delta_{rs}, \\
p_r p_s - p_s p_r &= 0, \qquad (2)\\
q_r q_s - q_s q_r &= 0.
\end{aligned}
$$

We shall be interested primarily in polynomials in a certain number of these variables. The results obtained in this paper are rigorously established for polynomials only, although they are formally correct for infinite series and thus give valid results if the series are convergent.

Let f and g be any two polynomials in the conjugate quantum variables, p and q, subject to the condition (1). In general $fg \neq gf$, but the relation (1) makes it possible to compute $fg-gf$ when f and g are given. This type of calculation arises frequently as, for example, in taking time derivatives. In order to simplify the calculations, the following formulas were derived:†

$$pg - gp = c\frac{\partial g}{\partial q}, \tag{3}$$

$$gq - qg = c\frac{\partial g}{\partial p}, \tag{4}$$

where g is a polynomial in p and q. We shall find it convenient to consider these equations as defining differentiation. It follows that the usual formulas for differentiating polynomials hold.‡

The obvious advantages of the formulas (3) and (4) suggest the desirability of obtaining a formula for computing $fg-gf$ directly, where f and g are arbitrary polynomials. Such a formula has been found. This formula reduces to equations (3) or (4) if one of the functions is taken as p or q, and it becomes identical with condition (1) if $f=p$, $g=q$.

This general commutation formula has been extended to functions of any number of quantum variables and applied to some special problems.§

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* Dirac, loc. cit., p. 649; Born, Heisenberg and Jordan, Zeitschrift für Physik, vol. 35 (1926), p. 573.

† Born, Heisenberg and Jordan, loc. cit., p. 563.

‡ Care must be taken to preserve the order of factors, that is,

$$\frac{\partial(fg)}{\partial q} = f\frac{\partial g}{\partial q} + \frac{\partial f}{\partial q}g.$$

§ A summary of some of the results obtained has been published in Proceedings of the National Academy of Sciences, vol. 15, pp. 200–202, March, 1929.

In the second part of this paper we extend these commutation formulas to the case of vectors in quantum mechanics. Born and Jordan* introduced the idea of using vectors whose components are functions of the quantum variables and Pauli† proved its usefulness in the theory of the hydrogen atom.

Let $A=(A_1, A_2, A_3)$ and $B=(B_1, B_2, B_3)$ be any two vectors. Scalar and vector products are defined by

$$A\cdot B = A_1B_1 + A_2B_2 + A_3B_3,$$

and

$$A \times B = (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1).$$

These are the ordinary definitions except that here the order A, B must be preserved throughout. It follows that $A\cdot B \neq B\cdot A$ and $A\times B \neq -B\times A$. This constitutes a difference from ordinary vector analysis. General commutation formulas for these scalar and vector products have been obtained.

I. Commutation formulas in quantum algebra

1. **Transformations of identities.**‡ We shall establish two theorems on transformations of identities in the single pair of quantum variables, p and q.

Theorem I. *Any identity remains an identity if p, q, c are replaced by p', q', Δc respectively, where p', q' are obtained from p, q by the non-singular linear transformation*

$$p' = \alpha p + \beta q,$$
$$q' = \gamma p + \delta q$$

of determinant Δ, the coefficients α, β, γ, δ being real or complex numbers.

This theorem is readily established by noting that $p'q' - q'p' = \Delta(pq - qp) = \Delta c$. Thus p', q' have an algebra equivalent to that of p, q with c replaced by Δc. This theorem was stated by Dirac§ for the special transformation given by $\alpha=\delta=0$, $\beta=\gamma=1$, which is equivalent to the interchange of p and q.

The following theorem was also given by Dirac‖ but without detailed proof.

Theorem II. *Any identity remains an identity if the order of all factors is reversed and c replaced by $-c$.*

* Loc. cit.

† Zeitschrift für Physik, vol. 36 (1926), pp. 336–363.

‡ The word "equation" will be frequently used to apply to an identity. We shall consider no equations of condition among the quantum variables.

§ Dirac, Proceedings of the Royal Society, (A), vol. 110 (1926), p. 565.

‖ Ibid.

The meaning of this theorem will be made clear by an example. Consider the identity

$$\frac{1}{pq+c}qp = \frac{1}{qp+2c}(pq-c).$$

According to the theorem,

$$pq\frac{1}{qp-c} = (qp+c)\frac{1}{pq-2c},$$

which may easily be verified by reducing it to the form

$$pq\frac{1}{qp-c} = pq\frac{1}{qp-c}\cdot$$

An identity of this last type which merely states the equivalence of two expressions of the same form will be called a formal identity.

Any identity may be reduced to a formal identity by means of the following transformations:

(a) $$pq = qp + c,$$

(b) $$qp = pq - c,$$

together with algebraic reduction. By reversing this process it follows that any identity may be generated from a formal identity by using these same transformations. We shall consider identities which are thus obtained from formal identities in a finite number of steps.

Theorem II is evident for formal identities. Hence we have only to show that if the theorem is true for a given identity it is true for those obtained from it by the transformations given above.

If ϕ is any expression in p and q we shall use $\bar{\phi}$ to denote the expression obtained from ϕ by reversing the order of all factors and changing the sign of c. Let $f=0$ be any identity for which $\bar{f}=0$ also. If we replace any factor of the form pq in a single term of the expression f by $qp+c$, the result will be an identity which will be denoted by $g=0$. The expression g differs from f in only one term and there only in that pq has been replaced by $qp+c$. Thus $\bar{g}$ differs from $\bar{f}$ only in that qp has taken the place of $pq-c$ and it follows that $\bar{g}=0$. The argument follows in a similar manner if a factor of the form qp is replaced by $pq-c$. The proof is completed by noting that if the theorem is true for a given identity it is true for any identity obtained from this one by algebraic manipulation.

Theorems I and II are extended without difficulty to identities in more than one pair of variables.

2. **A general commutation formula for functions of one pair of quantum variables.** We shall prove the following theorem:

THEOREM III. *If f and g are arbitrary polynomials in p and q, then* (a)

$$fg - gf = \sum_{s=1} \frac{c^s}{s!}\left[\frac{\partial^s g}{\partial q^s}\,\frac{\partial^s f}{\partial p^s} - \frac{\partial^s f}{\partial q^s}\,\frac{\partial^s g}{\partial p^s}\right], \tag{5}$$

and (b)

$$fg - gf = \sum_{s=1} \frac{(-c)^s}{s!}\left[\frac{\partial^s g}{\partial p^s}\,\frac{\partial^s f}{\partial q^s} - \frac{\partial^s f}{\partial p^s}\,\frac{\partial^s g}{\partial q^s}\right], \tag{6}$$

the sum in each case being taken over all non-null derivatives of f and g.

It will be seen that formulas (5), (6) reduce to equations (3), (4) in the special cases $f=p, f=q$. While the theorem is stated for polynomials only it is formally correct for other analytic functions. It will give valid results if one of the functions is a polynomial and the other is any function whose first and higher derivatives exist under the definition given by equations (3) and (4).

An outline of the method of proof of the first part of this theorem is as follows.* The validity of formula (5) in the special cases $f=p, f=q$ has already been noted. We shall show later that if formula (5) is true for a given polynomial f with g an arbitrary polynomial it remains so if f is replaced by pf or by qf. Since any term of a polynomial may be obtained by starting with either p or q and multiplying by p or q on the left in the proper order, it follows that equation (5) is true if f is any term of a polynomial. Furthermore this equation is valid for the sum of two functions if it is for each separately and thus the function f may be any polynomial.

For convenience we shall introduce the notation

$$H^{(s)}(g,f) = \frac{\partial^s g}{\partial q^s}\,\frac{\partial^s f}{\partial p^s}\,.$$

Equation (5) may then be written in the form

$$fg - gf = \sum_{s=1} \frac{c^s}{s!}[H^{(s)}(g,f) - H^{(s)}(f,g)]. \tag{5'}$$

* Another proof of formula (5) where f and g are arbitrary functions subject to the condition that they be in the normal form for quantum mechanics has been given by M. Coulomb, Comptes Rendus, vol. 188, No. 20, May 13, 1929. Obviously no such restriction is made in the proof given here.

The following auxiliary relations may now be verified:

$$H^{(s)}(qf,g) = qH^{(s)}(f,g) + sH^{(s-1)}\left(f,\frac{\partial g}{\partial p}\right); \tag{7}$$

$$H^{(s)}(g,qf) = qH^{(s)}(g,f) + cH^{(s)}\left(\frac{\partial g}{\partial p},f\right); \tag{8}$$

$$H^{(s)}(pf,g) = pH^{(s)}(f,g); \tag{9}$$

$$H^{(s)}(g,pf) = pH^{(s)}(g,f) - cH^{(s)}\left(\frac{\partial g}{\partial q},f\right) + sH^{(s-1)}\left(\frac{\partial g}{\partial q},f\right). \tag{10}$$

Consider the last of these relations. Making use of the fact that

$$\frac{\partial^s(pf)}{\partial p^s} = p\frac{\partial^s f}{\partial p^s} + s\frac{\partial^{s-1}f}{\partial p^{s-1}},$$

we have

$$\begin{aligned} H^{(s)}(g,pf) &= \frac{\partial^s g}{\partial q^s}p\frac{\partial^s f}{\partial p^s} + s\frac{\partial^s g}{\partial q^s}\frac{\partial^{s-1}f}{\partial p^{s-1}} \\ &= p\frac{\partial^s g}{\partial q^s}\frac{\partial^s f}{\partial p^s} - c\frac{\partial^{s+1}g}{\partial q^{s+1}}\frac{\partial^s f}{\partial p^s} + s\frac{\partial^s g}{\partial q^s}\frac{\partial^{s-1}f}{\partial p^{s-1}} \end{aligned}$$

by equation (3). This is equation (10) in a different notation. The other relations may be verified in a similar manner.

We now assume that the function f satisfies equation (5′) with g an arbitrary polynomial, and shall consider the effect of replacing f by pf. We wish to show that

$$pfg - gpf = \sum_{s=1} \frac{c^s}{s!}\left[H^{(s)}(g,pf) - H^{(s)}(pf,g)\right].$$

By using relations (9), (10), this may be written in the form

$$pfg - gpf = \sum_{s=1} \frac{c^s}{s!}\left[pH^{(s)}(g,f) - cH^{(s)}\left(\frac{\partial g}{\partial q},f\right) + sH^{(s-1)}\left(\frac{\partial g}{\partial q},f\right) - pH^{(s)}(f,g)\right]. \tag{11}$$

In order to verify this equation, multiply equation (3) by f on the right and equation (5′) by p on the left and add. We get the result

$$pfg - gpf = p\sum_{s=1} \frac{c^s}{s!}[H^{(s)}(g,f) - H^{(s)}(f,g)] + c\frac{\partial g}{\partial q}f. \tag{12}$$

Subtract equation (11) from equation (12). We obtain

$$c\left\{ \sum_{s=1} \frac{c^s}{s!}H^{(s)}\left(\frac{\partial g}{\partial q},f\right) - \sum_{s=1} \frac{c^{s-1}}{(s-1)!}H^{(s-1)}\left(\frac{\partial g}{\partial q},f\right) + \frac{\partial g}{\partial q}f\right\}$$
$$= c\left\{ \sum_{s=1} \frac{c^s}{s!}H^{(s)}\left(\frac{\partial g}{\partial q},f\right) - \sum_{s=1} \frac{c^s}{s!}H^{(s)}\left(\frac{\partial g}{\partial q},f\right) - \frac{\partial g}{\partial q}f + \frac{\partial g}{\partial q}f\right\} = 0,$$

which establishes the equivalence of these equations. Since equation (12) is known to be correct, we have thus established the validity of equation (11).

The case f is replaced by qf may be disposed of similarly except for the necessity of showing that

$$c\left\{ f\frac{\partial g}{\partial p} - \frac{\partial g}{\partial p}f - \sum_{s=1} \frac{c^s}{s!}\left[H^{(s)}\left(\frac{\partial g}{\partial p},f\right) - H^{(s)}\left(f,\frac{\partial g}{\partial p}\right)\right]\right\} = 0.$$

This is true since the polynomial g in (5′) is arbitrary and thus may be replaced by $\partial g/\partial p$. This completes the proof of the first part of Theorem III.

The proof of the second part is as follows. Let $f=f(p, q)$, $g=g(p, q)$ be the functions considered. By equation (5) and Theorem I we know that

$$\begin{aligned}(13)\quad & f(q,p)g(q,p) - g(q,p)f(q,p)\\ &= \sum_{s=1} \frac{(-c)^s}{s!}\left[\frac{\partial^s g(q,p)}{\partial p^s}\,\frac{\partial^s f(q,p)}{\partial q^s} - \frac{\partial^s f(q,p)}{\partial p^s}\,\frac{\partial^s g(q,p)}{\partial q^s}\right].\end{aligned}$$

Since $f(p, q)$, $g(p, q)$ are arbitrary polynomials it follows that $f(q, p)$, $g(q, p)$ are also arbitrary polynomials. If we replace $f(q, p)$, $g(q, p)$ by $f(p, q)$, $g(p, q)$ in equation (13) we are led at once to formula (6).

3. **An application to expressions of the type $q^m p^{m'} q^n p^{n'}$.** As an important special case of formula (5) let $f=p^n$, $g=q^m$. We find

$$p^n q^m - q^m p^n = \sum_{s=1} c^s s!\binom{n}{s}\binom{m}{s} q^{m-s}p^{n-s}$$

or

$$p^n q^m = \sum_{s=0} c^s s!\binom{n}{s}\binom{m}{s} q^{m-s}p^{n-s}. \tag{14}$$

We shall interpret $\binom{a}{b}$ to be zero if $a<b$. The corresponding expression for $q^m p^n$ may be obtained by formula (6). The result is

$$q^m p^n = \sum_{s=0}(-c)^s s!\binom{n}{s}\binom{m}{s} p^{n-s}q^{m-s}. \tag{15}$$

The relations (14) and (15) were given by Born and Jordan.*

Let us consider the expansion of $q^m p^{m'} q^n p^{n'}$ in the form $\sum a_{ij}\, p^i q^j$. We get, by an application of (15),

$$q^m p^{m'} q^n p^{n'} = \sum_{s=0} (-c)^s s! \binom{m}{s} \binom{m'}{s} p^{m'-s} q^{m+n-s} p^{n'}.$$

Repeating the process with $q^{m+n-s}\, p^{n'}$, we find

(16) $q^m p^{m'} q^n p^{n'}$

$$= \sum_{s=0} \sum_{r=0} (-c)^{s+r} s! r! \binom{m'}{s} \binom{n'}{r} \binom{m}{s} \binom{m+n-s}{r} p^{m'+n'-s-r} q^{m+n-s-r}.$$

By making certain restrictions on the exponents, m, m', n, n', we may obtain some interesting results. Assume that the left member of equation (16) is isobaric in p and q, that is, $m+n=m'+n'$. Reverse the order of all factors, at the same time interchanging p and q. Since the right member is unchanged, while the left member becomes $q^{n'} p^n q^{m'} p^m$, we have

$$q^m p^{m'} q^n p^{n'} = q^{n'} p^n q^{m'} p^m.$$

It will be seen that in general any term isobaric in p and q will have an expansion similar to (16) which is unchanged by this transformation. We have then the following theorem.

THEOREM IV. *Let g be any term isobaric in p and q. Interchange p and q and reverse the order of all factors in g. If $\bar{g}$ denotes the resulting term, then $g=\bar{g}$.*

Since $q^m p^{m'} q^n p^{n'} = q^{n'} p^n q^{m'} p^m$, the coefficient of $p^{m+n-k}\, q^{m+n-k}$ must be the same in their expansions, and we are led to the identity

(17)
$$\sum_{s+r=k} s! r! \binom{m'}{s} \binom{n'}{r} \binom{m}{s} \binom{m+n-s}{r}$$
$$= \sum_{s+r=k} s! r! \binom{n}{s} \binom{m}{r} \binom{n'}{s} \binom{m+n-s}{r}.$$

If we divide by $m!m'!n!n'!\{k![(m+n-k)!]^2\}^{-1}$ this may be written in the form

(17′)
$$\sum_{s=0} \binom{k}{s} \binom{m+n-s}{n} \binom{m+n-k}{m'-s}$$
$$= \sum_{s=0} \binom{k}{s} \binom{m+n-s}{m'} \binom{m+n-k}{n-s}.$$

This identity may be proved directly by induction.

* Loc. cit., p. 873.

We may get a series of identities in a similar manner. If we consider the expansion of $q^m p^{m'} q^n p^{n'} q^l p^{l'}$ under the condition $m+n+l=m'+n'+l'$ we are led as above to the identity

$$
\begin{aligned}
&\sum_{s+r+t=k} \binom{k}{s}\binom{k-s}{r}\binom{m+n-s}{n}\binom{m'+n'-s-r}{n'-r}\binom{m+n+l-s-r}{l}\binom{m+n+l-k}{l'-t} \\
(18)\quad & \\
&= \sum_{s+r+t=k} \binom{k}{s}\binom{k-s}{r}\binom{l'+n'-s}{n'}\binom{l+n-s-r}{n-r}\binom{m+n+l-s-r}{m'}\binom{m+n+l-k}{m-t}.
\end{aligned}
$$

In the special case $n=n'=0$, it follows that $l+m=l'+m'$ and $r=0$. The resulting identity may be written in the form of (17′) with l replacing n.

4. **Taylor's series and the commutation formula.** It may be shown for polynomials without difficulty that

$$
(19)\qquad f(p+h, q+k) = \sum_{s=0}\sum_{\alpha=0}^{s} \frac{h^\alpha k^{s-\alpha}}{\alpha!(s-\alpha)!}\frac{\partial^s f(p,q)}{\partial p^\alpha \partial q^{s-\alpha}}
$$

where h and k are real or complex numbers. We thus have a Taylor's series expansion as in the case of functions of a real variable. For analytic functions other than polynomials, equation (19) is formally correct.

Some interesting formal results may be obtained from this expansion and the commutation formula (5). From (5) we have

$$
f(p,q)e^{nq} - e^{nq}f(p,q) = e^{nq}\sum_{s=1}\frac{c^s n^s}{s!}\frac{\partial^s f(p,q)}{\partial p^s} = e^{nq}[f(p+nc,q) - f(p,q)]
$$

by equation (19). From this it follows that

$$
(20)\qquad
\begin{aligned}
f(p,q)e^{nq} &= e^{nq}f(p+nc,q),\\
e^{nq}f(p,q) &= f(p-nc,q)e^{nq}.
\end{aligned}
$$

These results were first given by Dirac* and used in his theory of the hydrogen atom.

In the special case $f(p,q)=e^p$, $n=1$, the first of equations (20) reduces to

$$
e^p e^q = e^c e^q e^p.
$$

This result may also be obtained directly from relation (5) without the use of Taylor's series by placing $f=e^p$, $g=e^q$.

In a similar manner it may be shown that

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* Dirac, Proceedings of the Royal Society, (A), vol. 110 (1926), p. 566.

$$f(p,q)e^{p+q} - e^{p+q}f(p,q) = \sum_{s=1} \frac{c^s}{s!}\left[e^{p+q}\frac{\partial^s f(p,q)}{\partial p^s} - \frac{\partial^s f(p,q)}{\partial q^s}e^{p+q}\right]$$
$$= e^{p+q}[f(p+c,q) - f(p,q)] - [f(p,q+c) - f(p,q)]e^{p+q},$$

from which we get the interesting result

$$e^{p+q}f(p+c,q) = f(p,q+c)e^{p+q}.$$

5. **Extension of the general commutation formula to functions of n pairs of variables.** In the case of n pairs of conjugate quantum variables, the single commutation relation (1) is replaced by the set of relations (2). Equations (3) and (4) are replaced by the following:†

$$\text{(21)} \qquad p_r g - g p_r = c\frac{\partial g}{\partial q_r}, \quad g q_r - q_r g = c\frac{\partial g}{\partial p_r},$$

where now g is a function of all the variables.

The extensions of the general commutation formulas (5) and (6) are made by the following theorem.

THEOREM V. *If f and g are polynomials in n pairs of variables, then* (a)

$$\text{(22)} \quad fg - gf = \sum_{s=1} \frac{c^s}{s!}[H^{(s)}(g,f) - H^{(s)}(f,g)]$$

where

$$H^{(s)}(g,f) = \sum_{\alpha+\beta+\cdots+\delta=s} \frac{s!}{\alpha!\beta!\cdots\delta!}\,\frac{\partial^s g}{\partial q_1^\alpha \partial q_2^\beta\cdots\partial q_n^\delta}\,\frac{\partial^s f}{\partial p_1^\alpha\partial p_2^\beta\cdots\partial p_n^\delta},$$

and (b)

$$\text{(23)} \quad fg - gf = \sum_{s=1} \frac{(-c)^s}{s!}[H^{*(s)}(g,f) - H^{*(s)}(f,g)]$$

where

$$H^{*(s)}(g,f) = \sum_{\alpha+\beta+\cdots+\delta=s} \frac{s!}{\alpha!\beta!\cdots\delta!}\,\frac{\partial^s g}{\partial p_1^\alpha\partial p_2^\beta\cdots\partial p_n^\delta}\,\frac{\partial^s f}{\partial q_1^\alpha\partial q_2^\beta\cdots\partial q_n^\delta}.$$

The following relations hold as in the case of one pair of variables:

$$H^{(s)}(q_1f,g) = q_1H^{(s)}(f,g) + sH^{(s-1)}\left(f,\frac{\partial g}{\partial p_1}\right),$$

$$H^{(s)}(g,q_1f) = q_1H^{(s)}(g,f) + cH^{(s)}\left(\frac{\partial g}{\partial p_1},f\right),$$

$$H^{(s)}(p_1f,g) = p_1H^{(s)}(f,g),$$

$$H^{(s)}(g,p_1f) = p_1H^{(s)}(g,f) - cH^{(s)}\left(\frac{\partial g}{\partial q_1},f\right) + sH^{(s-1)}\left(\frac{\partial g}{\partial q_1},f\right).$$

† Born, Heisenberg and Jordan, loc. cit., p. 574.

Consider the proof of the last of these expressions. We find

$$H^{(s)}(g,p_1f) = \sum_{\alpha+\beta+\cdots+\delta=s} \frac{s!}{\alpha!\beta!\cdots\delta!} \frac{\partial^s g}{\partial q_1^\alpha \partial q_2^\beta \cdots \partial q_n^\delta} p_1 \frac{\partial^s f}{\partial p_1^\alpha \partial p_2 \cdots \partial p_n^\delta}$$

$$+ \sum_{\alpha+\beta+\cdots+\delta=s} \frac{s!\alpha}{\alpha!\beta!\cdots\delta!} \frac{\partial^s g}{\partial q_1^\alpha \partial q_2^\beta \cdots \partial q_n^\delta} \frac{\partial^{s-1} f}{\partial p_1^{\alpha-1} \partial p_2^\beta \cdots \partial p_n^\delta}$$

$$= p_1 \sum_{\alpha+\beta+\cdots+\delta=s} \frac{s!}{\alpha!\beta!\cdots\delta!} \frac{\partial^s g}{\partial q_1^\alpha \partial q_2^\beta \cdots \partial q_n^\delta} \frac{\partial^s f}{\partial p_1^\alpha \partial p_2^\beta \cdots \partial p_n^\delta}$$

$$- c \sum_{\alpha+\beta+\cdots+\delta=s} \frac{s!}{\alpha!\beta!\cdots\delta!} \frac{\partial^{s+1} g}{\partial q_1^{\alpha+1} \partial q_2^\beta \cdots \partial q_n^\delta} \frac{\partial^s f}{\partial p_1^\alpha \partial p_2^\beta \cdots \partial p_n^\delta}$$

$$+ s \sum_{\alpha+\beta+\cdots+\delta=s} \frac{(s-1)!}{(\alpha-1)!\beta!\cdots\delta!} \frac{\partial^s g}{\partial q_1^\alpha \partial q_2^\beta \cdots \partial q_n^\delta} \frac{\partial^{s-1} f}{\partial p_1^{\alpha-1} \partial p_2^\beta \cdots \partial p_n^\delta}$$

by equations (21). This is the desired result. The other relations may be verified in a similar manner.

Relation (22) may now be established as follows. Assume it to be true for polynomials in n pairs of variables. It remains valid if g is a polynomial in $n+1$ pairs of variables and f is a polynomial in n pairs of variables. Suppose, for convenience, that f is a polynomial in the variables $(q_2, q_3, \cdots, q_{n+1}, p_2, p_3, \cdots, p_{n+1})$. It may be shown as in the proof of Theorem III(a) that equation (22) is still valid if f is replaced by p_1f or by q_1f. It follows that the relation (22) is true for polynomials in $n+1$ pairs of variables. The case $n=1$ is disposed of directly by Theorem III and thus formula (22) is true for polynomials in any number of variables.

The proof of Theorem V(b) follows from Theorem V(a) and Theorem I as in the proof of Theorem III(b).

The expression for $H^{(s)}(g, f)$ assumes a rather simple form in the case of two pairs of variables. It may be written in the form

$$H^{(s)}(g,f) = \sum_{\alpha=0}^{s} \frac{s!}{\alpha!(s-\alpha)!} \frac{\partial^s g}{\partial q_1^\alpha \partial q_2^{s-\alpha}} \frac{\partial^s f}{\partial p_1^\alpha \partial p_2^{s-\alpha}}.$$

This is seen to be a symbolic expansion of

$$\left(\frac{\partial g}{\partial q_1}\frac{\partial f}{\partial p_1} + \frac{\partial g}{\partial q_2}\frac{\partial f}{\partial p_2}\right)^s.$$

The corresponding definitions of $H^{(s)}(g, f)$ in the case of more variables are the corresponding multinomial expansions.

6. **Extension of the theory to rational functions of the quantum variables.** We shall now discuss a few commutation problems involving rational functions of p and q. The existence of these functions is assumed.

From equations (3) and (4) it follows that

$$\frac{\partial(1/f)}{\partial p} = -\frac{1}{f}\frac{\partial f}{\partial p}\frac{1}{f},$$

if f is any polynomial. If $f=p$, we see that $\partial(1/p)/\partial p = -1/p^2$ as usual.

Let us consider the possibility of extending the general commutation formula (5) to rational functions. In the special case $f=1/p$, $g=1/q$ we find the formal result

$$\frac{1}{p}\frac{1}{q} - \frac{1}{q}\frac{1}{p} = \sum_{s=1}^{\infty} c^s s! \frac{1}{q^{s+1}}\frac{1}{p^{s+1}},$$

which is obviously divergent.

Suppose, however, that f is a polynomial while g is an arbitrary rational function of p and q. The proof of formula (5) is seen to include such cases. Since f is a polynomial we are assured of the termination of the right hand member. As f and g play similar rôles it is clear that we may also take f as the more general function if g is a polynomial. This argument applies also to the commutation formulas for functions of n pairs of variables.

An interesting special case arises when f is a polynomial and $g=1/q$. Then

$$f\frac{1}{q} - \frac{1}{q}f = \sum_{s=1}(-c)^s \frac{1}{q^{s+1}}\frac{\partial^s f}{\partial p^s}.$$

While the general commutation formulas do not give useful results when both f and g involve negative powers of p and q, a given expression can often be transformed into one to which the formulas can be applied. As an example let us calculate $(1/p^m)(1/q^n)-(1/q^n)(1/p^m)$. If this expression is represented by X it follows that

$$p^m X p^m = \frac{1}{q^n}p^m - p^m\frac{1}{q^n} = -\sum_{s=1}(-c)^s s!\binom{n+s-1}{s}\binom{m}{s}\frac{1}{q^{n+s}}p^{m-s}$$

or

$$X = -\sum_{s=1}(-c)^s s!\binom{n+s-1}{s}\binom{m}{s}\frac{1}{p^m}\frac{1}{q^{n+s}}\frac{1}{p^s}.$$

From this we may write the identity

$$\frac{1}{q^n}\frac{1}{p^m} = \sum_{s=0}(-c)^s s!\binom{n+s-1}{s}\binom{m}{s}\frac{1}{p^m}\frac{1}{q^{n+s}}\frac{1}{p^s}.$$

A different expansion may be obtained by considering $q^n X q^n$. We obtain in this way

$$\frac{1}{q^n}\frac{1}{p^m} = \sum_{s=0}(-c)^s s!\binom{m+s-1}{s}\binom{n}{s}\frac{1}{q^s}\frac{1}{p^{m+s}}\frac{1}{q^n}.$$

II. Commutation rules in vector analysis

7. **Introduction of vector notation.** In this part we shall consider vectors whose components are functions of one or more pairs of the quantum variables discussed previously.

We shall let f, g denote scalar functions and F, G vector functions of the quantum variables. We define

$$\operatorname{grad}_Q f = \left(\frac{\partial f}{\partial q_1}, \frac{\partial f}{\partial q_2}, \frac{\partial f}{\partial q_3}\right),$$

$$\operatorname{grad}_P f = \left(\frac{\partial f}{\partial p_1}, \frac{\partial f}{\partial p_2}, \frac{\partial f}{\partial p_3}\right).$$

Corresponding definitions will be assumed for div F, curl F, $\nabla^2 F$.

It is found that a large number of relations go over unchanged from the classical theory.* We shall consider some commutation formulas which of course do not ordinarily appear.

8. **Fundamental commutation relations.** Consider the two vectors $P=(p_1, p_2, p_3)$ and $Q=(q_1, q_2, q_3)$ where the components satisfy relations (2). It follows that

$$P\cdot Q - Q\cdot P = 3c,$$

and

$$P\times Q + Q\times P = 0.$$

Corresponding to previous relations of the type

$$p_r f - f p_r = c\frac{\partial f}{\partial q_r},$$

we find the following formulas which may be easily verified:

$$(24)\quad \begin{array}{lrcl}
\text{(a)} & Pf - fP &=& c\operatorname{grad}_Q f,\\
\text{(b)} & fQ - Qf &=& c\operatorname{grad}_P f,\\
\text{(c)} & P\cdot F - F\cdot P &=& c\operatorname{div}_Q F,\\
\text{(d)} & F\cdot Q - Q\cdot F &=& c\operatorname{div}_P F,\\
\text{(e)} & F\times P + P\times F &=& c\operatorname{curl}_Q F,\\
\text{(f)} & F\times Q + Q\times F &=& -c\operatorname{curl}_P F.
\end{array}$$

* Pauli, loc. cit.; Handbuch der Physik, Band XX, *Licht als Wellenbewegung*, Berlin, Springer, 1928.

9. General commutation formulas. The general commutation formulas developed above may be extended without difficulty to vector functions. If we define

$$\frac{\partial F}{\partial q_1} = \left(\frac{\partial F_1}{\partial q_1}, \frac{\partial F_2}{\partial q_1}, \frac{\partial F_3}{\partial q_1}\right),$$

Theorem V is still applicable if one of the functions is a scalar and the other a vector.

In the case of scalar and vector products we have the following theorem, the proof of which follows without difficulty from Theorem V.

THEOREM VI. *If F and G are vector functions of the quantum variables $(p_1, p_2, p_3, q_1, q_2, q_3)$, then* (a)

$$F\cdot G - G\cdot F = \sum_{s=1} \frac{c^s}{s!}\left[H^{(s)}(G\cdot F) - H^{(s)}(F\cdot G)\right] \tag{25}$$

where

$$H^{(s)}(G\cdot F) = \sum_{\alpha+\beta+\gamma=s} \frac{s!}{\alpha!\beta!\gamma!}\,\frac{\partial^s G}{\partial q_1^\alpha \partial q_2^\beta \partial q_3^\gamma}\cdot\frac{\partial^s F}{\partial p_1^\alpha \partial p_2^\beta \partial p_3^\gamma},$$

and (b)

$$F\times G + G\times F = -\sum_{s=1} \frac{c^s}{s!}\left[H^{(s)}(G\times F) + H^{(s)}(F\times G)\right] \tag{26}$$

where

$$H^{(s)}(G\times F) = \sum_{\alpha+\beta+\gamma=s} \frac{s!}{\alpha!\beta!\gamma!}\,\frac{\partial^s G}{\partial q_1^\alpha \partial q_2^\beta \partial q_3^\gamma}\times\frac{\partial^s F}{\partial p_1^\alpha \partial p_2 \,\partial p_3^\gamma}.$$

An alternate form for these expressions may be obtained as in the second part of Theorem V.

An interesting special case of formula (26) arises when $G=F$. Then

$$F\times F = -\sum_{s=1} \frac{c^s}{s!} H^{(s)}(F\times F).$$

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