ON LINEAR UPPER SEMI-CONTINUOUS COLLECTIONS OF BOUNDED CONTINUA†

BY

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1. Introduction and notation. If \( T = \{t\} \) denotes a closed set of points and with each point \( t \) there is associated a unique bounded continuum \( X(t) \) in such a way that (a) \( X_t \cdot X_{t'} = 0 \) if \( t \neq t' \), and (b) at each point \( t = \tau \) of \( T \) the upper closed limit of \( X_t \) as \( t \to \tau \) is a part of \( X_{\tau} \), we say that \( X = f(t) \) is an upper semi-continuous function in \( T \). The collection of continua \( \{X\} \) is also known as an upper semi-continuous collection of continua. We say that \( X = f(t) \) is a minimal upper semi-continuous function in \( T \) if there exists no upper semi-continuous function \( Y = g(t) \) such that at every point \( t, Y \subset X \) and at some point \( Y \neq X \). Examples of this concept are given elsewhere.

The following notation will be convenient. If \( X = f(t) \) in \( T \) and \( M = \sum [X] \), we write \( M = F(T) \). If \( T \) is a bounded continuum and \( f(t) \) is upper semi-continuous, it is obvious that \( M \) is a bounded continuum; in this case we say that \( X \) is an element of \( M \). If \( T \) is a simple arc \( ab \), \( M = F(ab) \) will be called a generalized arc, or simply an arc if no confusion is caused. This may be denoted by \( X_a X_b \) and the elements \( X_a \) and \( X_b \) will be called the ends. Likewise, \( M = (X_a + X_b) \) is called a (generalized) open arc and denoted by \( X_a X_b^* \). The meaning of \( X_a X_b^* \) and \( X_a^* X_b \) is apparent. The terms "upper limit" and "limit" of a system of continua are used in the sense of the closed limits of Hausdorff (Grundzüge der Mengenlehre, p. 236) as extended by L. S. Hill (Properties of certain aggregate functions, American Journal of Mathematics, vol. 49, pp. 420-421). If \( K \) is the upper limit of \( X_t \) as \( t \to \tau \), we write \( K = \limsup_{t \to \tau} X_t \); if the domain of definition \( T \) is a linear set and we restrict \( t \) to points at the right or left of \( \tau \), we write \( R \limsup_{t \to \tau} X_t \) or \( L \limsup_{t \to \tau} X_t \), respectively.

In the article referred to above various properties of upper semi-continuous functions were derived and in particular the following theorem was proved: 

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Let the aggregate \( \{t\} \) be a simple arc \( ab \), let \( X = f(t) \) be a minimal upper semi-continuous function defined over \( ab \), let \( M = \sum |X| \) lie in a plane, and let no element of \( M \) separate \( X_a \) from \( X_b \). Then the continuum \( M \) is irreducible between \( X_a \) and \( X_b \).

The last two hypotheses are needed in the particular proof of the theorem that is given, but there is no evidence that they are essential for the validity of the conclusion. It is the purpose of this article to discuss this particular question and to give some properties of generalized arcs which lie in a plane and have one or more elements separating the ends.

2. Theorem. Let \( M = F(ab) \) be a generalized bounded arc in a plane \( Z \). Let \( t_1, t_2 \), and \( t_3 \) be points of \( ab \), let \( t_1 < \tau < t_2 \), and let \( X_1 = f(t_1) \) and \( X_2 = f(t_2) \). Then, if \( t' < \tau < t'' \), \( X_\tau \) separates \( X' = f(t') \) from \( X'' = f(t'') \).

Let \( R \) and \( S \) be the components of \( Z - X \), containing \( X_1 \) and \( X_2 \), respectively. Since \( X_aX_b^* \) is connected and contains \( X_1 \), it lies in \( R \). For a like reason \( X_1X_b^* \) lies in \( S \). But \( X' \subseteq X_aX_\tau^* \) and \( X'' \subseteq X_\tau^*X_b \). Hence the theorem is proved.

Remarks. It should be noted that, if some sub-continuum \( K \) of \( M \) separates \( X_a \) from \( X_b \), at least one element \( X \) has this property. (See reference under §1, p. 601.) As will be seen later, however, \( X \) need not be a part of \( K \), nor \( K \) a part of \( X \).

3. Theorem. Let \( M = F(ab) \) be a generalized bounded arc in a plane \( Z \). Let \( X_1 = f(t_1) \) and \( X_2 = f(t_2) \) separate \( X_a \) from \( X_b \) and \( t_1 < t_2 \). Then \( X_1 \) separates \( X_a \) from \( X_2 \), and \( X_2 \) separates \( X_1 \) from \( X_b \).

Let the components \( R_1 \) and \( S_1 \) of \( Z - X_1 \) contain \( X_a \) and \( X_b \), respectively. Then \( R_1 \supset X_aX_1^* \) and \( S_1 \supset X_1^*X_b \). Let \( R_2 \) and \( S_2 \) be likewise defined for \( X_2 \). Then \( R_2 \supset X_2X_1^* \) and \( S_2 \supset X_2^*X_b \). Since \( t_1 < t_2 \), \( X_1 \subseteq X_aX_2^* \subseteq R_2 \) and \( X_2 \subseteq X_1^*X_b \subseteq S_1 \). This proves the theorem.

As a consequence of this theorem, it follows that, if the elements of \( M \) which separate \( X_a \) and \( X_b \) are arranged in the order of the corresponding values of \( t \), each such element may be said to lie within all those which follow it and to lie without all those which precede it, or vice versa.

4. Theorem. Let \( M = F(ab) \) be a generalized bounded arc in a plane. Let \( t' \) be any point of \( ab \) for which \( X' = f(t') \) separates \( X_a \) from \( X_b \). Then the set of points \( \{t'\} \) together with \( a \) and \( b \) form a closed set.

Let this set be \( T' \) and let \( \tau \) be a limiting point of \( T' \) different from \( a \).
and b. Then there is a sequence \( \{ t_n' \} \) of points of \( T' - (a + b) \) converging to \( \tau \). Let \( X'_\tau = f(t'_\tau) \). Since \( f(t) \) is upper semi-continuous, \( X'_\tau \supset \limsup_{n \to \infty} X'_n \). By hypothesis each \( X'_n \) separates \( X_a \) from \( X_b \). Since \( (X_a + X_b) \cdot X'_\tau = 0 \), \( \limsup_{n \to \infty} X'_n \) separates \( X_a \) from \( X_b \). Hence \( \tau \) lies in \( T' \). As \( a + b \in T' \), this shows \( T' \) is closed.

5. **Theorem.** Let \( X_t = f(t) \) be a minimal upper semi-continuous function defined over the interval \( ab \). Let \( M = F(ab) \) be bounded and lie in a plane. Let the set of points \( T'' = \{ t' \} \) for which \( X' = f(t') \) separates \( X_a \) from \( X_b \) be void or totally disconnected. Then \( M \) is irreducible between \( X_a \) and \( X_b \).

The case that \( T'' \) is void is merely the theorem cited in §1. In the alternative case set \( T' = T'' + a + b \). Then \( T' \) is a nowhere dense closed set and \( ab - T' \) is a finite or enumerable set of open intervals whose end points are points of \( T' \).

Let \( K \) be a sub-continuum of \( M \) irreducible between \( X_a \) and \( X_b \). Let \( c^*d^* \) be any one of the open intervals whose sum is \( ab - T' \). Set \( g(t) = f(t) \) for \( c < t < d \); \( g(c) = R \limsup_{t \to c} f(t); \) and \( g(d) = L \limsup_{t \to d} f(t) \). Then \( g(t) \) is a minimal upper semi-continuous function in \( c^*d^* \) and it is clear that \( G(c,d) = X^*_cX^*_d \). But for no \( t \) in \( c^*d^* \) does \( g(t) \) separate \( g(c) \) from \( g(d) \) by the hypothesis regarding \( cd \) and §2. Then the theorem quoted in §1 shows that \( X^*_cX^*_d \) is irreducible between \( g(c) \) and \( g(d) \) and therefore between \( X_aX_c \) and \( X_dX_b \). But \( K \) contains a sub-continuum irreducible between these arcs and \( K \subset M \). Hence \( K \supset X^*_cX^*_d \).

Now set \( Y_t = K \cdot X_t = k(t) \). We have just seen that for any point \( t \) in \( ab - T' \), \( Y_t \equiv X_t \). Let \( t' \) be any point of \( T' \) except \( a \) or \( b \). Since \( T' \) is nowhere dense, there are two univariant sequences of points \( \{ s_n \} \) and \( \{ t_n \} \) lying in \( ab - T' \) and converging to \( t' \) such that \( s_n < t'_n < t_n \); let \( M_n \) denote the arc of \( M \) whose ends are \( f(s_n) \) and \( f(t_n) \). Since \( f(s_n) = k(s_n) \) and \( f(t_n) = k(t_n) \), it is evident that \( K \cdot M_n \) is a continuum. Obviously \( X_{t'} = \prod_{t} [K \cdot M_n] \). Then \( Y_{t'} = K \cdot X_{t'} = \prod_{t} [K \cdot M_n] \), and is therefore a continuum. Similar reasoning shows that \( Y_a \) and \( Y_b \) are continua.

We now show that \( Y_t = k(t) \) is upper semi-continuous. This needs a proof only for points \( t' \) of \( T' \), since \( k(t) = f(t) \) in \( ab - T' \). As \( Y_t \subset K \) and \( Y_t \subset X_t \), \( \limsup_{t \to t'} Y_t \subset K \cdot \limsup_{t \to t'} X_t \subset K \cdot X_{t'} \). But \( Y_{t'} = K \cdot X_{t'} \) by definition. Hence \( \limsup_{t \to t'} Y_t \subset Y_{t'} \) everywhere.

Since \( f(t) \) is a minimal upper semi-continuous function and \( k(t) \subset f(t) \) everywhere, it follows that for every \( t \), \( Y_t = k(t) = f(t) = X_t \). Hence \( K = M \) and the theorem is proved.

\[\text{† For, if } A \text{ and } B \text{ are bounded continua; } \{ X_n \} \text{ is a sequence of bounded continua, each of which separates } A \text{ from } B; \ X = \limsup_{n \to \infty} X_n; \text{ and } X \cdot (A + B) = 0; \text{ then } X \text{ separates } A \text{ from } B.\]
Examples to which the theorem is applicable are to be found in the literature. Here is another. Let \( a = 0 \) and \( b = 1 \), and \( T' = \{ t' \} \) be any nowhere dense closed set in \( ab \). For each \( t' \) in \( T' \) let \( f(t') \) be a circumference whose center is \( X_0 \) and whose radius is \( t' \); let \( f(0) = X_0 \) and \( f(1) \) be a circumference with center \( X_0 \) and radius 1. Let \( t_1' \) and \( t_2' \) be any two points of \( T' \) which are the end points of one of the open intervals whose sum is \( ab - T' \). In the ring bounded by \( f(t_1') \) and \( f(t_2') \) place a spiral approaching the bounding circumferences asymptotically and for any \( t \) between \( t_1' \) and \( t_2' \) meeting the circumference of radius \( t \) and center \( X_0 \) in a single point, which we take as \( f(t) \).

Do this for each open interval of \( ab - T' \). Then \( f(t) \) is a minimal upper semi-continuous function and \( M = F(01) \) is irreducible between \( X_0 \) and \( X_1 \).

Let us now turn to the hypothesis that \( T' \) is totally disconnected. It is clear that, if \( f(t) \) is merely upper semi-continuous in \( ab \), there may be a sub-interval \( cd \) such that for every \( t \) in \( cd \), \( X_t \) separates \( X_a \) from \( X_b \). A set of concentric circumferences is an example, but in this case \( f(t) \) would not be a minimal function. If it is impossible for \( f(t) \) to have this quality and for \( T' \) at the same time to contain an interval, the hypothesis is redundant; on the other hand, the conclusion of the theorem may be true even if the hypothesis is not satisfied. Two examples bearing on these points will be given.

6. Example I.† Let \( K_i \) and \( L_i \) denote two simple closed plane curves having one common point \( x_i \) and let \( K_i - x_i \) lie within \( L_i \). The union of \( K_i + L_i \) and the complementary domain whose frontier is this continuum we call a crescent and we denote it by \( C_i \). The point \( x_i \) will be called a cut point.

Let \( R \) be a circular ring bounded by two concentric circumferences \( K \) and \( L \), \( K \) being the smaller. It is easy to show that, if \( n \) is any integer, we can construct \( n \) crescents \( C_1, C_2, \ldots, C_n \) in \( R \) which satisfy the following requirements. No two of the sets \( C_i \) have common points. \( R - \sum |C_i| \) consists of \( n+1 \) rings (not necessarily circular) \( R_0, R_1, \ldots, R_n \) whose frontiers are \( K + K_1, L_1 + K_2, \ldots, L_{n-1} + K_n, L_n + L \), respectively, and each of these rings has in common with every radius from the common center \( E \) of \( K \) and \( L \) a segment of uniform length \( w \), which may be called the width of the ring. If \( E \) is taken for the pole of a system of polar coordinates, the vectorial angle \( \theta \) of each cut point \( x_i \) is \( 2\pi i/n \). Finally this construction can be repeated in each of the rings \( \{ R_i \} \).

Set up a system of polar coördinates \((r,\theta)\), where \( \theta \) is measured in terms

† This is a modification of a continuum previously described by the author: A curious irreducible continuum, Bulletin of the American Mathematical Society, vol. 32, pp. 679–681.
of $2\pi$ as a unit. Let $0 \leq t \leq 1$. Let $X_0 = f(0)$ and $X_1 = f(1)$ be the circumferences $r = 1$ and $r = 2$, respectively, and let $R_0$ be the ring bounded by these. As described in the previous paragraph, construct in $R_0$ two crescents $C_1$ and $C_2$, whose frontiers are $X_{0,1} = K_1 + L_1$ and $X_{0,2} = K_1 + L_2$ and which divide $R_0$ into three rings $R_{0,0}$, $R_{0,1}$, and $R_{0,2}$, each of which has the width $\epsilon_1$. Set $X_{0,1} = f(1/3)$ and $X_{0,2} = f(2/3)$ and let the cut points of $C_1$ and $C_2$ have $\theta$ equal to $1/2$ and $1$, respectively.

In $R_{0,0}$ construct four crescents $C_{0,1}$, $C_{0,2}$, $C_{0,3}$, and $C_{0,4}$, which divide $R_{0,0}$ into five rings $R_{0,0,0}$, $R_{0,0,1}$, $R_{0,0,2}$, $R_{0,0,3}$, $R_{0,0,4}$, each of width $\epsilon_2$, and whose cut points have $\theta$ equal to $1/4$, $1/2$, $3/4$, and $1$, respectively. Let their frontiers be $X_{0,0,1} = f(1/15)$, $X_{0,0,2} = f(2/15)$, $X_{0,0,3} = f(3/15)$, and $X_{0,0,4} = f(4/15)$. In the same way treat $R_{0,1}$ and $R_{0,2}$, getting $f(6/15)$, $f(7/15)$, $f(8/15)$, $f(9/15)$, $f(11/15)$, $f(12/15)$, $f(13/15)$, and $f(14/15)$.

In each of the 15 rings thus formed construct 8 crescents whose frontiers are $f(1/135)$, $f(2/135)$, \ldots, $f(8/135)$; $f(10/135)$, \ldots, $f(17/135)$; \ldots, $f(134/135)$. In each of these 15 rings let the cut points have $\theta$ equal to $1/8$, $1/4$, $3/8$, $1/2$, $5/8$, $3/4$, $7/8$, and $1$.

Let this process be continued indefinitely; it is clear that the sequence $\epsilon_1$, $\epsilon_2$, \ldots converges to zero. This defines $X = f(t)$ for any rational $t < 1$ of the form $p/q$, where $p$ is relatively prime to $q$ and $q$ is one of the integers 3, 3·5, 3·5·9, 3·5·9·17, etc. Any other $t$ is the divisor of a decreasing sequence of intervals $d_1$, $d_2$, \ldots, of lengths $1/3$, $1/15$, $1/135$, etc., and to each of these corresponds a definite ring of width $\epsilon_1$ lying between the crescents corresponding to the end points of the interval. Let the divisor of such a sequence of rings be $f(t)$; it is obviously a simple closed curve.

The construction insures that $f(t)$ is upper semi-continuous. It is obvious that any continuum $P$ joining $X_0$ to $X_1$ contains all the cut points. As these are everywhere dense with respect to $\theta$ in each ring, they are everywhere dense in $M = \sum [X]$. Hence $P = M$ and $M$ is irreducible between $X_0$ and $X_1$. Hence also $f(t)$ is a minimal function.

Thus we have an example where $f(t)$ is a minimal upper semi-continuous function defined over an interval $ab$ and for every $t$ between $a$ and $b$ the corresponding element separates $f(a)$ from $f(b)$. That is, the hypothesis in the theorem of §5 that the set of points $T'$ be totally disconnected is not redundant and the conclusion of this theorem may be valid with this hypothesis omitted. We now proceed to construct an example which shows that this last situation does not always happen.

7. Example II. As a preliminary we state several facts regarding bounded continua which are either well known or easily demonstrated from known theorems.
(a) For the sake of brevity later a bounded plane continuum $F$ which has precisely two complementary domains $R$ and $S$ and is the frontier of both of them will be called a continuum of type $\alpha$. For such a continuum and for any $\epsilon > 0$, there exist simple polygons $P$ and $P'$ in $R$ and $S$, respectively, which have these properties: $R + F$ lies in the interior of $P'$; $P + P' \subset V_\epsilon(F)$, $F \subset V_\epsilon(P)$, $F \subset V_\epsilon(P')$; the rings between $P$ and $F$ and between $F$ and $P'$ contain no circle of diameter $\epsilon$; and the ring between $P$ and $P'$ contains no circle of diameter $2\epsilon$. (The notation $V_\epsilon(F)$ means the set of all points whose distances from $F$ are less than $\epsilon$.)

(b) It is possible to construct an indecomposable continuum $K$ of type $\alpha$ such that there is no continuum $C$ joining a point of $R$ to one of $S$ such that $C \cdot K$ is a proper sub-continuum of $K$. In other words $C \cdot K$ is always disconnected or identical with $K$. Such continua will be referred to later as of type $\beta$.†

(c) Let $\{F_i\}$ and $\{G_i\}$ be two sequences of continua of type $\alpha$ having the following properties. If $i < i'$, let $F_i$ lie in the interior of $F_{i'}$, i.e., in the bounded complementary domain. Let every $F_i$ be in the interior of every $G_i$. If $i < i'$, let $G_i$ lie in the interior of $G_{i'}$. For every $\epsilon > 0$ let there be an $i_0$ such that for every $i > i_0$, $F_i \subset V_\epsilon(G_i)$, $G_i \subset V_\epsilon(F_i)$, and the ring $H_i$ whose frontier is $F_i + G_i$ contains no circle of diameter $\epsilon$. Then, if $H = \prod_{i=1}^{\infty} [H_i]$, $H$ is a continuum of type $\alpha$ separating every $F_i$ from every $G_i$ and $H = \lim_{i \to \infty} F_i = \lim_{i \to \infty} G_i$.

(d) Let $P$ and $P'$ be simple polygons, let $P$ be in the interior of $P'$, and let $R$ be the ring between them. Then, for every $\epsilon > 0$, there is an integer $n$, such that there are $n$ polygons $\{P_i\}$ dividing $R$ into $n + 1$ rings, such that no ring contains a circle of diameter $\epsilon$, each ring is contained in an $\epsilon$-vicinity of each of the polygons which bound it, and, if we set $P_0 = P$ and $P_{n+1} = P'$, every $P_i \subset V_\epsilon(P_{i+1})$ and every $P_{i+1} \subset V_\epsilon(P_i)$, $i = 0, 1, 2, \ldots, n$.

We are now ready to proceed with our construction. Let $K$ be a continuum of type $\beta$, let $P'$ and $P''$ be simple polygons, let $P'$ be in the interior of $K$ and $K$ in the interior of $P''$, and let $\Gamma$ be the ring between $P'$ and $P''$.

Let $C$ be a circumference of center $E$ and radius 1. Let $0 < \epsilon_1 < 1/4$. Let $C_0 = E$ and $C_1, C_2, \ldots, C_n = C$ be a set of concentric circumferences, $\ldots$.

† It is not natural to infer that any indecomposable continuum of type $\alpha$ is necessarily of type $\beta$. That this is not true may be seen from the following modification of the continua of Wada, the idea for which is due to a suggestion by H. M. Gehman. Let $C'$ and $C''$ be two circumferences, $C'$ being within $C''$, and let $L$ be a segment joining $C'$ and $C''$. Let the simply connected region whose frontier is $C' + C'' + L$ be the "island" of K. Yoneyama (Tôhoku Mathematical Journal, vol. 12, pp. 60-62), the interior of $C'$ be the "lake of fresh water," and the exterior of $C''$ be the "sea." If "canals" are constructed as described by Yoneyama, the resulting indecomposable continuum is of type $\alpha$, but not of type $\beta$. For the constituent containing $L$ is accessible from both of the complementary domains.
each within the following and having a distance from the two adjacent ones
less than $\epsilon_1$. Put $\Gamma$ in a homeomorphism with each of the $n_i - 1$
circular rings thus determined and let $K_i$, $i = 1, 2, \ldots, n_i - 1$, be the image of $K_i$.  
Set $K_0 = E$ and $K_n = C$. For each $i < n_i$, $K_i \subset V_\eta(K_{i+1})$ and $K_{i+1} \subset V_\eta(K_i)$, 
where $\eta_i = 2\epsilon_i$. Moreover, the point $K_0 = E$ and the continua $K_1$, $K_2$, \ldots, $K_{n_i-1}$ divide the interior of $C$ into $n_i$ rings $R_0$, $R_1$, $R_2$, \ldots, $R_{n_i-1}$ no one of which contains a circle of diameter $\eta_i$. At the same time divide the interval $0 \leq t \leq 1$ into $n_i$ equal intervals $D_0$, $D_1$, $D_2$, \ldots, $D_{n_i-1}$ by the points $c_1$, $c_2$, \ldots, $c_{n_i-1}$ and set $c_0 = 0$ and $c_{n_i} = 1$. For each point $t = c_i$, set $f(c_i) = K_i$. We note that this also establishes a correspondence between the intervals $\{D_i\}$ and the rings $\{R_i\}$.

For some positive $\epsilon_2 < \frac{1}{2}\epsilon_1$ and some integer $n_2$ there can be constructed, in each ring $R_i$, $n_2$ simple polygons dividing it into $n_2 + 1$ rings in such a manner that, if all the $n_in_2$ polygons and the continua $\{K_i\}$ are taken in order starting from $K_0 = E$, each polygon or continuum is contained in an $\epsilon_2$-vicinity of the one preceding and the one following and no ring contains a circle of diameter $\epsilon_2$. This is a consequence of (a) and (d), above. Each ring $R_i$ contains $n_i - 1$ rings bounded by polygons. Put $\Gamma$ in a homeomorphism with each of these and let the images of $K$ be $\{K_{i,j}\}$, $j = 1, 2, \ldots, n_2 - 1$. Set $K_{i,0} = K_i$ and $K_{i,n_2} = K_{i+1} = K_{i+1,0}$. If $\eta_i = 2\epsilon_1$, each $K_{i,j}$ is in an $\eta_2$-vicinity of $K_{i,j+1}$ and of $K_{i,j-1}$. Moreover, the $n_in_2$ continua $\{K_{i,j}\}$, $i = 0$, 1, 2, \ldots, $n_i - 1$, $j = 0, 1, 2, \ldots, n_2 - 1$ divide the interior of $C$ into $n_in_2$ rings $R_{0,0}$, $R_{0,1}$, \ldots, $R_{n_i-1,n_2-1}$ no one of which contains a circle of diameter $\eta_i$, and the frontier of each $R_{i,j}$ is $K_{i,j} + K_{i,j+1}$. At the same time divide each of the intervals $D_i$ into $n_2$ equal intervals $\{D_{i,j}\}$, $j = 0, 1, 2, \ldots, n_2 - 1$, by the points $c_{i,j}$, $c_{i,j+1}$, \ldots, $c_{i,n_2-1}$. For each point $c_{i,j}$ set $f(c_{i,j}) = K_{i,j}$, $i = 0, 1, 2, \ldots, n_i - 1$, $j = 0, 1, 2, \ldots, n_2 - 1$. This also establishes a correspondence between the intervals $\{D_{i,j}\}$ and the rings $\{R_{i,j}\}$.

Now take $\epsilon_3 < \frac{1}{2}\epsilon_2$ and continue this process indefinitely. If $t$ is one of the division points $c_{i,j}$, $k$, $c_{i,j+k}$, etc. set $f(t)$ equal to the corresponding $K_{i,j+k}$, $K_{i,j+k}$, etc. Any other point $t$ is the divisor of a decreasing sequence of intervals $D_i$, $D_{i,j}$, $D_{i,j+k}$, etc., to which corresponds a decreasing sequence of rings $R_{i,j}$, $R_{i,j}$, $R_{i,j+k}$, etc., whose divisor is a continuum. This continuum we set equal to $f(t)$. It is clear from the construction that, if $M = \sum [f(t)] = F(01)$ and $C'$ denotes the sum of $C$ and its interior, $M = C'$.

Let us now consider the nature of $f(t)$. Since $\epsilon_n \rightarrow 0$, it follows from the construction and (c) above that $f(t)$ is not only upper semi-continuous, but continuous for each $t$ not a division point $c_{i,j}$, $c_{i,j+k}$, etc. This is also true at the division points, for any such point, as $c_{i,j+k}$, can be regarded as the divisor of the sequence of intervals $c_{i,j+k}$, $c_{i,j+k}$, \ldots, $c_{i,j+k}$, $c_{i,j+k}$, etc.
\[c_{i,j,k}, \text{ etc.},\] and the sequence of rings bounded by the pairs of continua corresponding to the end points of these intervals with their frontiers satisfy the hypotheses of (c). It also follows from (c) that, if \( t \) is different from 0 and 1, \( f(t) \) separates \( f(0) \) from \( f(1) \).

Let \( g(t) \leq f(t) \) and be upper semi-continuous in \((0,1)\) and let \( N = \sum [g(t)] \). The sets of continua \( K_i, K_{i,j}, K_{i,j,k}, \text{ etc.} \) were constructed so as to be everywhere dense in \( M \). Hence, if \( M \neq N \), there is some \( K \) containing points not in \( N \). Let this \( K \) be \( f(\tau) \); then \( g(\tau) \neq f(\tau) \). This gives us a continuum \( N \) joining the interior of \( f(\tau) \) to the exterior and having in common with \( f(\tau) \) a proper sub-continuum \( g(\tau) \) of \( f(\tau) \). This is impossible by (b) above since each \( K \) is of type \( \beta \). Therefore \( M = N \); i.e., \( g(t) = f(t) \) at every point and \( f(t) \) is a minimal upper semi-continuous function. As \( M \) fills a part of the plane, it is not an irreducible continuum.

8. Remarks. As already stated, the construction of Example II shows that, if \( f(t) \) is a minimal upper semi-continuous function defined over an interval \( ab \), \( M = F(ab) \) need not be irreducible between \( X_a \) and \( X_b \), even though \( M \) is a plane continuum. The example can be easily modified to give a similar construction in space. Whether such an example can be constructed in space without the use of cyclic continua is an open question. An example of a continuous decomposition of an irreducible continuum into elements none of which are points, communicated to the author by Dr. Bronislaw Knaster, makes it seem probable that a decomposition similar to that in Example II can be arrived at without the use of indecomposable continua. Finally, this example justifies the last sentence in §2.

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