ON THE DEGREE OF CONVERGENCE OF EXPANSIONS IN AN INFINITE INTERVAL*

BY

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In a recent investigation† of the degree of convergence of the Gram-Charlier series in the infinite interval $-\infty < x < \infty$ the writer has shown that in general the convergence is very much less rapid than in the case of a Fourier series under similar circumstances. The question arises whether the slow rate of convergence of this particular series is a characteristic of all such expansions on an infinite interval, or whether there may exist series of orthogonal functions for which the rate of convergence is as rapid as that of the Fourier series in a finite interval. A study of this question not only is of some theoretical interest, but may conceivably serve a very practical end by leading to the discovery of series better adapted to the representation of frequency functions than is the slowly convergent Gram-Charlier series.

This paper is devoted to those expansions on the infinite interval which are associated with the differential equation

$$d^2u/dx^2 + [\lambda - q(x)]u = 0, \tag{1}$$

in which $q(x)$ is real and continuous for all values of x, and

$$\lim_{x=\pm\infty} q(x) = +\infty. \tag{2}$$

The differential equation above is a special case of equations investigated on the infinite interval by Weyl,‡ Hilb,§ Gray,‖ and Milne.¶ It has been shown that there exists an infinite set of critical values of λ, $\lambda_0, \lambda_1, \lambda_2, \cdots$ with limit point at $+\infty$ only, corresponding to which equation (1) has solutions $U_0(x)$, $U_1(x)$, $U_2(x)$, $\cdots$, satisfying the conditions

$$\lim_{x=\pm\infty} U_n(x) = \lim_{x=\pm\infty} U_n'(x) = 0. \tag{3}$$

* Presented to the Society, August 27, 1929; received by the editors May 17, 1929.

† These Transactions, vol. 31 (1929), pp. 422–443.

‡ Weyl, Mathematische Annalen, vol. 68 (1910), p. 220, and Göttinger Nachrichten, 1910, p. 442.

§ Hilb, Mathematische Annalen, vol. 76 (1915), p. 333.

‖ Gray, American Journal of Mathematics, vol. 50 (1928), p. 431.

¶ Milne, these Transactions, vol. 30 (1929), p. 797.

The solution $U_n(x)$ vanishes exactly n times, and the integrals

$$\int_{-\infty}^{\infty} U_n^2(x)dx, \quad \int_{-\infty}^{\infty} U_n'^2(x)dx, \quad \int_{-\infty}^{\infty} q(x)U_n^2(x)dx$$

all exist. Aside from the solutions $U_n(x)$ (and constant multiples of them) the equation (1) possesses no solution whose square is integrable from $-\infty$ to $+\infty$.

The functions $U_n(x)$ have the orthogonal properties

$$\int_{-\infty}^{\infty} U_m(x)U_n(x)dx = 0, \qquad m \neq n,$$

so that for an arbitrary function $f(x)$, such that

$$\int_{-\infty}^{\infty} f^2(x)dx$$

exists, we have the formal expansion

$$f(x) = A_0U_0(x) + A_1U_1(x) + A_2U_2(x) + \cdots, \tag{4}$$

in which

$$A_n = \int_{-\infty}^{\infty} f(x)U_n(x)dx \Big/ \int_{-\infty}^{\infty} U_n^2(x)dx. \tag{5}$$

The Gram-Charlier series is a particular case of such an expansion where the function $q(x)$ is a polynomial of the second degree in x having the coefficient of x^2 positive.

It will be shown that the degree of convergence of the series (4) depends upon the function $q(x)$, and that by a suitable choice of $q(x)$ a degree of convergence may be obtained which is only slightly less than that of the Fourier series. It is further shown that the *same* degree of convergence as in the case of the Fourier series is not to be expected.

1. Before we can deal with the series (4) it is necessary to assemble a number of facts regarding the solutions of equation (1). For this purpose we impose some additional restrictions on $q(x)$, and make the following assumptions:

(a) The function $q(x)$ has continuous derivatives of the first three* orders for all values of x.

(b) $$q''(x) \geqq 0, \quad -\infty < x < \infty.$$

* The existence of a continuous third derivative is not necessary, but is convenient for purposes of proof.

There is then no loss of generality in assuming that the origin is so chosen that

$$q(0) = q'(0) = 0.$$

We assume further that

$$\text{(c)} \qquad \begin{aligned} &q'(x) > 0, \quad q'''(x) \geqq 0, \quad \text{if} \quad x > 0, \\ &q'(x) < 0, \quad q'''(x) \leqq 0, \quad \text{if} \quad x < 0; \end{aligned}$$

$$\text{(d)} \qquad \lim_{x=\pm\infty} q''(x)[q'(x)]^{-4/3} = 0.$$

Now let $u_1(x)$ and $u_2(x)$ denote two particular solutions of (1) satisfying the conditions

$$\text{(6)} \qquad \begin{aligned} u_1(0) &= \lambda^{-1/4}, \qquad & u_2(0) &= 0, \\ u_1'(0) &= 0, \qquad & u_2'(0) &= \lambda^{1/4}, \end{aligned}$$

and form the one-parameter family of solutions

$$\text{(7)} \qquad u(x,\theta) = u_1(x)\cos\theta + u_2(x)\sin\theta.$$

The equation of the envelope of the family (7) is

$$\text{(8)} \qquad y_{\text{en}} = \pm\,(u_1^2 + u_2^2)^{1/2}.$$

The equation of the locus of the extrema of $u(x, \theta)$ is

$$\text{(9)} \qquad y_{\text{ex}} = \pm\,(u_1'^2 + u_2'^2)^{-1/2}.$$

Likewise the equations of the envelope and locus of extrema of $u'(x, \theta)$ are respectively

$$\text{(10)} \qquad y_{\text{en}}^{(1)} = \pm\,(u_1'^2 + u_2'^2)^{1/2},$$

and

$$\text{(11)} \qquad y_{\text{ex}}^{(1)} = \pm\,(u_1^2 + u_2^2)^{-1/2}.$$

Let us for sake of brevity write

$$u = u(x,\theta), \qquad g^2 = \lambda - q(x),$$

in the interval in which

$$\lambda > q(x),$$

and form the two quadratic expressions*

$$\begin{aligned} Q_1 &= gu^2 + g^{-2}g'uu' + g^{-1}[1 - E]u'^2, \\ Q_2 &= g^{-1}u'^2 + g^{-2}g'uu' + g[1 + E]u^2, \end{aligned}$$

* The use of such forms was suggested by Watson in connection with Bessel functions. Watson, Proceedings of the London Mathematical Society, (2), vol. 16 (1917), pp. 170–171.

in which

$$E = -g^{-3}g''/2 + g^{-4}g'^2.$$

By differentiation and substitution from (1) we have

$$Q_1' = -u'^2(d/dx)(g^{-1}E), \qquad Q_2' = u^2(d/dx)(gE).$$

When the values of these derivatives are calculated in terms of $q(x)$ and reference is made to assumptions (b) and (c), it turns out that Q_1' is negative and Q_2' is positive in the interval for which $x>0$, $\lambda>q(x)$. Therefore if ξ_i denotes a positive root of u' in this interval (and consequently an extremum of u) we have the inequality

$$gu^2(\xi_i) < Q_1(0).$$

Using (6) and (7), together with the equations $q(0)=q'(0)=0$, we find that

$$\begin{aligned} Q_1(0) &= 1 - q''(0)\sin^2\theta/(4\lambda^2), \\ &= 1 + O(\lambda^{-2}). \end{aligned}$$

We shall adopt the notation 1_k to designate any expression of the form $1-O(\lambda^{-k})$. Using this notation and referring to (9) and the definition of $g(x)$, we arrive at the useful result

$$u_1'^2 + u_2'^2 > 1_2[\lambda - q(x)]^{1/2}. \tag{12}$$

Continuing this line of reasoning, we set $u=0$ in Q_1 and then $u'=0$ in Q_2 and again $u=0$ in Q_2 and finally by the aid of (9) and (11) establish three more inequalities

$$u_1'^2 + u_2'^2 < 1_2[\lambda - q(x)]^{1/2}[1+E], \tag{13}$$

$$u_1^2 + u_2^2 < 1_2[\lambda - q(x)]^{-1/2}, \tag{14}$$

$$u_1^2 + u_2^2 > 1_2[\lambda - q(x)]^{-1/2}[1-E]. \tag{15}$$

From (13) and (14) we see that

$$\begin{aligned} |u(x,\theta)| &\leqq 1_2[\lambda - q(x)]^{-1/4}, \\ |u'(x,\theta)| &\leqq 1_2[\lambda - q(x)]^{1/4}[1+E]^{1/2}, \end{aligned}$$

not merely at extrema but for all intermediate values. When $\lambda=\lambda_n$ there is a value of θ, $\theta=\theta_n$, for which

$$u(x,\theta_n) \equiv U_n(x),$$

so that we have at once

$$|U_n(x)| \leqq 1_2[\lambda_n - q(x)]^{-1/4}, \tag{16}$$

$$|U_n'(x)| < 1_2[\lambda_n - q(x)]^{1/4}[1+E]^{1/2}. \tag{17}$$

The formulas (12)–(17) have been derived for positive values of x. They evidently hold for negative values also as long as $\lambda > q(x)$.

2. Our next object is the determination of a formula connecting λ_n with n. This can be accomplished by means of the fact that the function $U_n(x)$ corresponding to λ_n has exactly n roots $\rho_1, \rho_2, \cdots, \rho_n$, all in the interval for which $\lambda_n > q(x)$. We already know that $U_n(x)$ has exactly n roots, and we now show that $y = U_n(x)U_n'(x)$ does not vanish if $\lambda_n < q(x)$. For by differentiation and use of (1) we get

$$y' = U_n'^2(x) + [q(x) - \lambda_n]U_n^2(x),$$

so that y' is positive if $q(x) > \lambda_n$. Therefore, because of (3), y cannot vanish for any finite x such that $q(x) > \lambda_n$.

The function $U_n(x)$ may be expressed in the form

$$U_n(x) = (u_1^2 + u_2^2)^{1/2} \cos [\phi(x) - \theta_n],$$

where

$$\phi(x) = \arctan (u_2/u_1)$$

and

$$\phi'(x) = (u_1^2 + u_2^2)^{-1}.$$

As x increases from ρ_1 to ρ_n the function ϕ increases by the amount $(n-1)\pi$, so that

$$(n - 1)\pi = \int_{\rho_1}^{\rho_n} (u_1^2 + u_2^2)^{-1} dx.$$

This is the desired formula. By means of (14) and (15) we derive a pair of useful inequalities

$$(18) \qquad (n - 1)\pi > \int_{\rho_1}^{\rho_n} 1_2[\lambda_n - q(x)]^{1/2} dx,$$

$$(19) \qquad (n - 1)\pi < \int_{\rho_1}^{\rho_n} 1_1[\lambda_n - q(x)]^{1/2}[1 - E]^{-1} dx.$$

It is assumed here, and will be proved later, that $E < 1$ in the interval of integration provided λ_n is large.

3. In order to make profitable use of the inequalities just derived it is necessary to learn something about the location of the roots ρ_1 and ρ_n, the location of the largest and smallest roots, ξ_n and ξ_0, of $U_n'(x)$, and to determine the order of magnitude of E. Let h_1 be the negative and h_2 the positive root of the equation

$$q(x) = \lambda_n.$$

Then we know that

$$\rho_n < \xi_n < h_2,$$

and we wish to determine the magnitude of $h_2-\rho_n$.

If we use the notation

$$q_2' = q'(h_2), \qquad q_2'' = (1/2)q''(h_2), \qquad J_2 = q_2' \cdot [q_2']^{-4/3},$$

we see that the inequalities

$$\lambda_n - q(x) \leqq q_2'(h_2 - x),$$
$$\lambda_n - q(x) \geqq q_2'(h_2 - x) - q_2''(h_2 - x)^2$$

are valid for $x<h_2$ because of the assumptions made regarding $q(x)$. The change of variable

$$h_2 - x = [q_2']^{-1/3}t$$

transforms (1) into

$$d^2U_n/dt^2 + \gamma(t)tU_n = 0, \tag{20}$$

where, because of the inequalities above,

$$1 - J_2t \leqq \gamma(t) \leqq 1.$$

By assumption (d) it is possible to make J_2 as small as we please by taking λ_n sufficiently large.

Let t_0 be the value of t corresponding to $x=\rho_n$. Let t_1 be the first positive root of the solution of

$$d^2y/dt^2 + ty = 0 \tag{21}$$

for which $y'=0, y\neq 0$ at $t=0$. We find by comparing (21) and (20),

$$t_0 > t_1.$$

Also if t_2 is the first positive root of that solution of (21) for which $y'\neq 0$, $y=0$ at $t=0$, and if λ_n is so large that $1-J_2t_0>0$, then

$$t_0[1 - J_2t_0]^{1/3} < t_2.$$

If we also take λ_n so large that $J_2<1/(3)t_2$, then the last inequality assures us that

$$t_0 < 2t_2.$$

These results give us the inequalities

$$t_1[q_2']^{-1/3} < h_2 - \rho_n < 2t_2[q_2']^{-1/3}. \tag{22}$$

By numerical calculation it is found that $t_1=1.9\cdots$, $t_2=2.8\cdots$.

We are now in a position to estimate the magnitude of the largest maximum of $|U_n(x)|$ for positive values of x. This occurs at the point $x=\xi_n$ and it is therefore apparent from the fact that ξ_n is between ρ_n and h_2 that

$$\operatorname{Max} | U_n(x) | < (h_2 - \rho_n) | U_n'(\rho_n) | .$$

Using (17) and (22) we get

$$\operatorname{Max} | U_n(x) | < M_2[q_2']^{-1/6},$$

where

$$M_2 = 1_2(2t_2)^{5/4}[1 + E(\rho_n)]^{1/2}.$$

It remains to investigate E. We find

$$E = (3/8)(\lambda - q(x))^{-3}q'^2(x) + (1/4)(\lambda - q(x))^{-2}q''(x).$$

First of all it is apparent that for x in any fixed finite interval

$$E = O(\lambda^{-2}).$$

Next we note that, because of the hypotheses regarding $q(x)$ and the inequalities (22),

$$E(\rho_n) < (3/8)t_1^3 + \epsilon,$$

where ϵ approaches zero as λ becomes infinite. Therefore, a fortiori,

$$E(x) < (3/8)t_1^3 + \epsilon, \quad 0 < x < \rho_n.$$

All the results of this paragraph have been obtained for x positive. Entirely similar results may be obtained when x is negative. For example when x is negative

$$\operatorname{Max} | U_n(x) | < M_1[- q_1']^{-1/6}.$$

Therefore if δ denotes the smaller of the two quantities $-q_1'$ and q_2', we have

$$\operatorname{Max} | U_n(x) | < M\delta^{-1/6}, \quad -\infty < x < \infty. \tag{23}$$

4. Because of the inequalities just obtained for E it is clear that the integrals in (18) and (19) are of the same order of magnitude, and therefore either one will serve for the determination of the order of magnitude of λ_n in terms of n. Moreover by using the inequalities $\lambda_n - q(x) < q_2'(h_2 - x)$, and (22) we find that

$$\int_{\rho_n}^{h_2} (\lambda_n - q(x))^{1/2} dx < (2/3)(2t_2)^{3/2},$$

with a similar result for the integral from h_1 to ρ_1, so that for the determination of the order of magnitude of λ_n we may use the approximate equation

$$(n - 1)\pi = \int_{h_1}^{h_2} (\lambda_n - q(x))^{1/2} dx \qquad \text{(approx.)}. \tag{24}$$

It is interesting to test this equation for a problem in which the value of λ_n is known beforehand, namely where $q(x) = -1/2 + x^2/4$. Here we know that $\lambda_n = n$. For this case the value of the integral in (24) proves to be $\pi(\lambda_n + 1/2)$, giving us $\lambda_n = n - 3/2$, a rather surprisingly good approximation.

The integral in (24) may be transformed as follows: Let $x = h_2(z)$ be the inverse of $z = q(x)$ when x is positive and let $x = -h_1(z)$ be the inverse when x is negative. In the integral from h_1 to 0 we make the substitution $x = -h_1(\lambda_n s)$ and in the integral from 0 to h_2 the substitution $x = h_2(\lambda_n s)$, then integrate each term by parts, and obtain

$$(25) \qquad 2\pi(n-1) = \lambda_n^{1/2}\left\{ \int_0^1 \frac{h_2(\lambda_n s)ds}{(1-s)^{1/2}} + \int_0^1 \frac{h_1(\lambda_n s)ds}{(1-s)^{1/2}} \right\}$$

approximately. Since $h_1(\lambda_n s)$ and $h_2(\lambda_n s)$ become infinite with λ_n, we have from (25)

THEOREM I. *The order of magnitude of* λ_n *is less than* n^2.

On the other hand if $q(x) = x^{2\kappa}$, we find directly from (25) that λ_n is of the order of magnitude of $n^{2-2/(\kappa+1)}$. By taking κ large enough we may make the exponent as nearly 2 as we please. In fact if $q(x)$ behaves like $e^{|x|}$ for x large, say $q(x) = \cosh x - 1$, we find that the order of magnitude of λ_n is at least as great as $n^2/\log^2 n$. Further refinements may be made indefinitely. Hence we have proved

THEOREM II. *By a suitable choice of* $q(x)$ *the order of magnitude of* λ_n *may be made as near* n^2 *as we please.*

5. It is necessary to obtain a lower bound for the integral in the denominator of (5). Now as we go farther from the origin in either direction the areas under the arches of the curve $y = U_n^2(x)$ increase, and there are in all $(n+1)$ arches. Hence the value of the integral will be greater than $(n+1)a_0$, where a_0 is the area of the smallest arch. Since the smallest maximum of $U_n^2(x)$ is greater than or equal to $1_2 \cdot \lambda_n^{-1/2}$, and the smallest distance between a maximum and a root is greater than $\pi/(2\lambda_n^{1/2})$, we find that $a_0 > \pi/(3\lambda_n)$. Therefore

$$(26) \qquad \int_{-\infty}^{\infty} U_n^2(x)dx > \pi(n+1)/(3\lambda_n).$$

6. We are now ready to take up the question of convergence of (4) and enunicate the first result as follows:

THEOREM III. *If $f(x)$ has a continuous derivative of bounded variation in the infinite intervals; if the integral*

$$\int_{-\infty}^{x} f(s)q(s)ds$$

exists and is bounded; and if the Stieltjes integral

$$I = \int_{-\infty}^{\infty} [1 + q(x)]^{1/4} |dF(x)|$$

exists, where

$$F(x) = f'(x) - \int_{-\infty}^{x} f(s)q(s)ds ;$$

then

$$R_n(x) = \sum_{i=n+1}^{\infty} A_i U_i(x) = O(\delta^{-1/6}\lambda_n^{-1/4}), \quad -\infty < x < \infty ,$$

$$= O(\lambda_n^{-1/2}), \quad \alpha < x < \beta.$$

Multiply the equation

$$U_n(x) = \lambda_n^{-1} q(x) U_n(x) - \lambda_n^{-1} U_n''(x) , \tag{27}$$

which is in effect equation (1), by $f(x)dx$, integrate from $-\infty$ to $+\infty$, and integrate each term on the right by parts. The result is

$$\int_{-\infty}^{\infty} f(x) U_n(x) dx = \lambda_n^{-1} \int_{-\infty}^{\infty} F(x) U_n'(x) dx,$$

since the integrated terms vanish at the limits because of (3). Now $U_n(x)$ is continuous and $F(x)$ is of bounded variation, so that we may integrate by parts between finite limits α and β, and obtain

$$\int_{\alpha}^{\beta} F(x) U_n'(x) dx = F(\beta) U_n(\beta) - F(\alpha) U_n(\alpha) - \int_{\alpha}^{\beta} U_n(x) dF(x) .$$

Because of the hypothesis regarding the existence of the Stieltjes integral on the infinite interval we may let α and β become infinite, and obtain finally

$$\int_{-\infty}^{\infty} f(x) U_n(x) dx = -\lambda_n^{-1} \int_{-\infty}^{\infty} U_n(x) dF(x).$$

We easily see that the right hand integral is less in absolute value than

$$\lambda_n^{-1} \operatorname{Max} | U_n(x) | \, [1 + q(x)]^{-1/4} I,$$

and because of (16) and (23) the whole expression is $O(\lambda_n^{-5/4})$. Referring now to (5) and (26) we have

$$A_n = O(\lambda_n^{-1/4} n^{-1}).$$

Therefore

$$\begin{aligned} A_n U_n(x) &= O(\delta^{-1/6}\lambda_n^{-1/4} n^{-1}), \quad -\infty < x < \infty, \\ &= O(\lambda_n^{-1/2} n^{-1}), \quad \alpha < x < \beta, \end{aligned}$$

and

$$\begin{aligned} R_n(x) &= O(\delta^{-1/6}\lambda_n^{-1/4}), \quad -\infty < x < \infty, \\ &= O(\lambda_n^{-1/2}), \quad \alpha < x < \beta. \end{aligned}$$

For the case in which $q(x) = x^{2\kappa}$ we have

$$\delta = O(\lambda_n^{1-1/\kappa})$$

and

$$\lambda_n = O(n^{2-2/(\kappa+1)}),$$

so that the above equations become

$$\begin{aligned} R_n(x) &= O(n^{(-5\kappa+1)/(6\kappa+6)}), \quad -\infty < x < \infty, \\ &= O(n^{-\kappa/(\kappa+1)}), \quad \alpha < x < \beta. \end{aligned}$$

For $\kappa = 1$ we have the same results that were secured for the Gram-Charlier series in the paper referred to. For large values of κ the exponents are approximately $-5/6$ and -1 respectively. In view of Theorem I it seems that we need hardly expect actually to attain these values for the class of expansions under discussion.

THEOREM IV. *If $f(x)$ has a continuous second derivative of bounded variation on the infinite interval; if the Stieltjes integral*

$$I_1 = \int_{-\infty}^{\infty} [1 + q(x)]^{3/4} \,|\, dF_1(x) \,|$$

exists, where

$$F_1(x) = q(x)f(x) - f''(x);$$

and if

$$\lim_{x=\pm\infty} F_1(x) = 0;$$

then

$$\begin{aligned} R_n(x) &= O(\delta^{-1/6}\lambda_n^{-3/4}), \quad -\infty < x < \infty, \\ &= O(\lambda_n^{-1}), \quad \alpha < x < \beta. \end{aligned}$$

Multiply the equation (27) by $f(x)dx$, and integrate from $-\infty$ to $+\infty$, integrating the second term on the right by parts twice. Then

$$\int_{-\infty}^{\infty} f(x) U_n(x) dx = \lambda_n^{-1} \int_{-\infty}^{\infty} F_1(x) U_n(x) dx.$$

As in the proof of Theorem III we again integrate by parts to obtain

$$\int_{-\infty}^{\infty} f(x)U_n(x)dx = -\lambda_n^{-1}\int_{-\infty}^{\infty}\left[\int_0^x U_n(s)ds\right]dF_1(x).$$

By exactly the same process as was used in the paper on Gram-Charlier series* it may be shown that

$$\int_0^x U_n(x)dx = O(\lambda^{-5/4})$$

in any fixed finite interval and is bounded in all intervals. With this fact at hand we may complete the proof in substantially the same manner as in Theorem III.

For the case of a function $f(x)$ with a continuous kth derivative of bounded variation in the infinite interval, the generalization of Theorems III and IV is easily accomplished. For we have

$$\int_{-\infty}^{\infty} f(x)U_n(x)dx = \lambda_n^{-m}\int_{-\infty}^{\infty} F_m(x)U_n(x)dx,$$

where

$$\begin{aligned} F_1(x) &= q(x)f(x) - f''(x), \\ F_2(x) &= q(x)F_1(x) - F_1''(x), \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

and where $m=k/2$ if k is even, and $m=(k-1)/2$ if k is odd. For the case where k is odd the proof is completed as in Theorem III, while if k is even it is completed as in Theorem IV. The formulation and proof of this generalized theorem may be left to the reader.

The foregoing arguments establish the uniform convergence of the series (4) in the infinite interval but do not of themselves prove that $f(x)$ is equal to the sum of the series. This fact has been established by Weyl† on the assumption that

$$\int_{-\infty}^{\infty}[q(x)f(x) - f''(x)]^2 dx$$

exists.

In cases where Weyl's result is not directly applicable, as for example in the case of Theorem III, it is still easy to draw the desired conclusion. For either by following Weyl's reasoning or by applying Hilbert's theory of

* Loc. cit., paragraph 4, pp. 427–429.

† Göttinger Nachrichten, 1910, pp. 449–450.

integral equations* directly to our particular problem we may show that the set of functions $U_0(x)$, $U_1(x)$, $U_2(x)$, $\cdots$ is *closed*. The convergence to $f(x)$ is therefore a consequence of uniform convergence.

In conclusion it should be pointed out that the results of this paper may be extended to cases where $q(x)$ does not fulfil the conditions (b) and (c) in the entire infinite interval, provided these conditions hold for $|x|$ sufficiently large, say $|x| > X$. For when λ_n is sufficiently large we have an asymptotic representation of $U_n(x)$ of the form

$$U_n(x) = \lambda_n^{-1/4} \sin \lambda^{1/2}(x - x_0) + O(\lambda^{-3/4}),$$

valid in the interval $-X < x < X$, from which upper and lower bounds for the number of zeros in this interval may be obtained, together with upper bounds for $|U_n(x)|$, $|U_n'(x)|$ and $\int_0^x U_n(x)dx$. Outside of this interval a suitable modification of the treatment given here may be applied.

Having thus determined the orders of magnitude of λ_n, $|U_n(x)|$, $|U_n'(x)|$ and $\int_0^x U_n(x)dx$, we repeat the proofs of Theorems III and IV without change.

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* Hilbert, Göttinger Nachrichten, 1904, pp. 73–74 and pp. 221–222.

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