ON COMPOSITION OF SINGULARITIES

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1. Introduction. Our purpose will be to establish several theorems on composition of singularities analogous to Hadamard’s multiplication theorem.† In each of these we consider essentially an integral of the form

\[ F(z) = \frac{1}{2\pi i} \int_{C} f(x) \cdot g(q(z, x)) \cdot dx/x. \]

(In the proof of Hadamard’s theorem (1) is used, where \( q(z, x) = z/x \).) One of our theorems (§2) is derived with \( q(z, x) = \log z/\log x \); another (§3), which is derived with \( q(z, x) = (ax + bx + c)/(a_1x + b_2x + c_1) \), is a generalization of Hadamard’s theorem. Further, the theorem of §3 is extended to functions of two complex variables (§4) in a way analogous to the extension by Haslam-Jones‡ of the one-variable Hadamard’s theorem to functions of two variables. Finally, a theorem is derived (§5) making use of (1), with \( q(z, x) \) a more general function than the one employed in §3.

2. The case where \( q(z, x) = \log z/\log x \). We shall denote the closed region of the complex plane, consisting of all those points \( x \) for which \( \alpha \leq \arg x \leq 2\pi - \alpha \) (\( 0 < \alpha < \pi \)), by \( W(\alpha) \). Let \( f_1(x_1) \) be a uniform analytic function whose singularities \( \alpha'_1, 0 < |\alpha'_1| \leq |\alpha'_2| \leq \cdots \), form an isolated set, and let \( g_1(x_1) \) be another uniform analytic function with a single singularity \( \beta' \), \( |\beta'| > 0 \). By simple transformations \( f_1(x_1) \) can be changed to \( f(x) \), a uniform analytic function, whose singularities \( \alpha_1, 1 < |\alpha_1| \leq |\alpha_2| \leq \cdots \), form an isolated set such that none of them belong to \( W(\phi) \) (\( 0 < \phi < \pi \)). Since the \( \alpha'_i \) are isolated, such a region \( W(\phi) \) can be found. Let \( \phi_1 \) and \( \beta \) be defined by the following relations:

\[ \pi/\beta = \phi_1, \quad \phi < \phi_1 < \pi. \]

Then transform \( g_1(x_1) \) so that the new function \( g(x) \) is uniform, analytic and has only one singularity \( \beta \). By (1), \( \beta > 1 \) and is real. Without any loss of generality we may consider \( f(x) \) and \( g(x) \) in place of \( f_1(x_1) \) and \( g_1(x_1) \).

Consider in the \( x \)-plane a region \( G \) consisting of the portion of the plane

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bounded by a circle $C_R$, which has radius $R$ and center $x=0$, and which does not pass through any of the $a_i$, the points of $W(\phi)$ and the neighborhood of the origin being excluded, the latter by means of a small circle $C_0$ with $x=0$ for the center and radius $r_0$. For $R$ sufficiently large a number of the points $a_i$ will lie within the circle $C_R$. Surround each of these by small circles $C_i$, with these points for centers and with radii $r_i$, respectively. Join each $C_i$ with $C_R$ by means of a Jordan curve $l_i$. The interiors of the $C_i$ do not belong to $G$. Choose the $r_i$ sufficiently small so that $C_i$ has no points in common with $C_R$, $C_R$, $C_0$ or the boundary of $W(\phi)$. Also the $l_i$ can be taken so that $l_i$ has no points in common with $l_i$, $C_i$, $C_0$ or the boundary of $W(\phi)$. The region $G$, thus defined, is evidently a simply connected region. Its boundary $C$ consists of some of the circumferences $C_i$, the curves $l_i$, of those portions of the circumferences $C_R$, $C_0$ which are left when the points of $W(\phi)$ are excluded from them, and of the part of the boundary of $W(\phi)$ interior to $C_R$ and exterior to $C_0$ (this part consists of portions of straight lines through the origin).

Consider the transformation

\[ z = x^\theta \]

and apply it to the contour $C$. A circumference $C_i$ is given by

\[ x = a_i + r_i e^{i\theta} \quad (0 \leq \theta \leq 2\pi). \]

Its transformation in the $z$-plane will be given by a contour $C_i^\theta$,

\[ z = (a_i + r_i e^{i\theta})^\theta = a_i^\theta (1 + r_i e^{i\theta}/a_i)^\theta, \]

\[ z - a_i^\theta = r_i a_i^{\theta - 1} e^{i\theta} [1 + r_i \phi(\theta)] \quad (0 \leq \theta < 2\pi), \]

where $\phi(\theta)$ is of period $2\pi$ in $\theta$ and is bounded so that with $r_i$ sufficiently small $0 < 1 - \varepsilon < |1 + r_i \phi(\theta)| < 1 + \varepsilon$. Thus it follows from (4) that $C_i^\theta$ is a simple closed curve containing $z=a_i^\theta$ and with radii

\[ (1 - \varepsilon)r_i \beta |a_i^{\theta - 1}| \quad \text{and} \quad (1 + \varepsilon)r_i \beta |a_i^{\theta - 1}|, \]

respectively. When $r_i$ is made to approach zero, the contour $C_i^\theta$ will reduce to the point $z=a_i^\theta$.

The portion of the circumference $C_0$ belonging to $C$ consists of points $x$ given by

\[ x = r_0 e^{i\theta}, \quad -\phi < \theta < \phi. \]

Its transformation $C_0^\theta$,

\[ z = r_0^\theta e^{i\theta} = r_\theta e^{i\theta_1}, \quad -\beta \phi < \theta_1 < \beta \phi, \]

* Compare the procedure in this section with the proof of Hadamard's theorem, as given by Mandelbrojt; S. Mandelbrojt, The Rice Institute Pamphlet, vol. 14 (1927), No. 4, pp. 242–245.
is an arc of a circle in the $z$-plane. By (1), $\beta \phi < \pi$. Similarly, the transformation of the part of $C_R$ belonging to $C$ is an arc of a circle $C_{R\beta}$. This arc consists of the points

$$z = Re^{i\theta}, \quad -\beta \phi < \theta < \beta \phi < \pi. \tag{6}$$

The part of the boundary of $W(\phi)$ belonging to $C$,

$$x = |x|e^{\pm i\phi}, \quad r_0 \leq |x| \leq R, \tag{7}$$

is transformed into

$$z = |x|e^{\pm i\phi}, \quad r^\beta \leq |x| \leq R^\beta, \tag{8}$$
or

$$\arg z = \pm \beta \phi, \quad r^\beta \leq |z| \leq R^\beta. \tag{9}$$

The Jordan curves $l_i$ are transformed into Jordan curves $l_{i\beta}$, where $l_{i\beta}$ joins $C_{i\beta}$ with $C_{R\beta}$. Denote the transformation of $C$ by $C_{\beta}$. With the $\rho_i$ sufficiently small, $C_{\beta}$ is the boundary of a simply connected region $G_{\beta}$. This boundary consists of points $z$, satisfying (5), (6), (9), and of the curves $C_{i\beta}$ and $l_{i\beta}$.

Throughout this section $\log u$ will denote the branch which reduces to zero for $u = 1$. The $\rho_i$ will be taken small enough and the $l_i$ be so chosen that $x = 1$ is not on $C$. Consider the following integral:

$$F(z) = \frac{1}{2\pi i} \int_{C} f(x) \cdot g \left( \frac{\log z}{\log x} \right) \frac{dx}{(x - 1)^\rho}, \tag{10}$$

where $\rho$ is a positive integer or zero. Since $x$ is on $C$, for $z$ interior to $C_{\beta}$, $\log z/\log x \neq \beta$. If it were otherwise it would follow that $z = x^\beta$.

and since $x$ is a point of $C$, $z$ would be on $C_{\beta}$. For $z$ within $C_{\beta}$ and $x$ on $C$, $f(x)$ is regular in $x$ and $g(\log z/\log x)$ is continuous in $z$ and $x$. Consequently $F(x)$ is continuous. Since $F^{(1)}(x)$ exists on account of the existence and continuity of $\partial g(\log z/\log x)/\partial z$, it follows that $F(z)$ is analytic within $C_{\beta}$. By letting $R$ approach infinity in such a way that the $\alpha_i$ are never on $C$, and by diminishing the $\rho_i$ indefinitely, the circular portion of the contour $C_{\beta}$, which belongs to $C_{R\beta}$, will be made to recede to infinity and the simple closed curves $C_{i\beta}$ will be made to close down on the points $\alpha_i^\beta$, respectively. The $l_i$ and consequently the $l_{i\beta}$ are arbitrary. Let $L(\phi\beta)$ denote the set complementary to $W(\phi\beta)$ and $C_{i\beta}$. In $L(\phi\beta)$, $F(z)$ is uniform and analytic with a possible exception of points of the form $\alpha_i^\beta$.

We shall show that when $z$ is restricted to a certain subregion $L$ of $C_{\beta}$,
the integrand of (10) is a uniform and analytic function of \( x \) in a region \( M \) consisting of the circumference of a small circle \( C' \) of radius \( r' \)

\[
x = 1 + r'e^{i\theta} \quad (0 \leq \theta < 2\pi)
\]

and the open region \( O \) bounded by \( C' \) and \( C \). For that purpose it is sufficient to prove the following lemma:

**Lemma I.** Let \( a, b, c \) be positive numbers defined in succession by the following conditions:

(i) For \( x \) in \( M \), \( \beta |\log x| > a \); (ii) \( b \) is sufficiently small, so that \( a^2 - b^2 > 0 \);

(iii) \( c^2 < a^2 - b^2 \).

Let a region \( L \) consist of the points

(iv) \( 1 \leq |z| < e^c, \quad |\arg z| \leq b, \)

which lie within \( C_\beta \) (such a region exists since there is a neighborhood of \( z = 1 \) interior to \( C_\beta \)).

Then

\[
|\log z/\log x| < \beta
\]

for all \( z \) in \( L \) and \( x \) in \( M \).

A number \( a \) satisfying (i) exists since there is a neighborhood of \( x = 1 \) which is not a part of \( M \). From (iv) it follows that

\[
0 \leq \log |z| < c.
\]

Hence

\[
|\log z| < (c^2 + \phi^2)^{1/2}, \quad \phi = \arg z.
\]

By (iv) this gives

\[
|\log z| < (c^2 + b^2)^{1/2}
\]

for all \( z \) in \( L \).

Thus we have

\[
|\log z| < (c^2 + b^2)^{1/2}, \quad a < \beta \cdot |\log x|.
\]

By (iii), \((c^2 + b^2)^{1/2} < a\). This gives

\[
|\log z| < \beta \cdot |\log x|
\]

or (11). This proves the lemma.

For \( z \) fixed within the region \( L \) defined by Lemma I, the integrand (10) is a uniform and analytic function of \( x \) in \( M \). The contour \( C \) can thus be deformed into \( C' \) without encountering any singular points. We have therefore
\[ F(z) = \frac{1}{2\pi i} \int_{C'} f(x) \cdot g \left( \frac{\log z}{\log x} \right) \frac{dx}{(x - 1)^p} \]

Let
\[ f(x) = \sum_{n=0}^{\infty} a_n \cdot (x - 1)^n, \]
where the power series converges within a circle
\[ x = (|\alpha_1| - 1)e^{i\theta} \quad (0 \leq \theta < 2\pi), \]
and let
\[ g(x) = \sum_{n=0}^{\infty} b_n \cdot x^n \quad (|x| < \beta). \]

For purposes of computation (12) can be written in the form
\[ F(z) = \frac{1}{2\pi i} \int_{C'} \sum_{n=0}^{\infty} a_n (x - 1)^n \cdot \sum_{n=0}^{\infty} b_n \left( \frac{\log z}{\log x} \right)^n \frac{dx}{(x - 1)^p}. \]

We have thus proved the following theorem:

**Theorem I.** Given the series
\[ \sum_{n=0}^{\infty} a_n (x - 1)^n, \quad \sum_{n=0}^{\infty} b_n x^n \]
representing the uniform and analytic functions \( f(x), g(x) \), respectively. Let the singularities \( \alpha_i \) of \( f(x) \) form an isolated set such that \( 1 < |\alpha_i| \) and that none of the \( \alpha_i \) are in \( W(\phi), \phi < \pi \). Let the only singularity of \( g(x) \) be \( \beta, \beta = \pi/\phi_1, \phi < \phi_1 < \pi \). With a possible exception of points of the form \( \alpha_i^\theta \), the function
\[ F(z) = \frac{1}{2\pi i} \int_{C'} f(x) \cdot g \left( \frac{\log z}{\log x} \right) \frac{dx}{(x - 1)^p} \]
(\( \phi \) a positive integer) is uniform and analytic in the region \( L(\phi\beta) \). This region consists of the set complementary to \( W(\phi\beta) \) and to a small circle around the origin. (\( C' \) is a small circle around \( x = 1 \).)

**Note.** Using (15) it can be shown that
\[ F(z) = \sum_{n=0}^{\infty} c_n (\log z)^n = \sum_{n=0}^{\infty} c_n (z - 1)^n, \]
where the power series converges within a circle having \( z = 1 \) for center and a radius \( \rho \) where \( \rho < 1 \) and \( \rho < |\alpha_i - 1| \) for all \( i \). The \( c_n \) depend on the \( a_i \) and the \( b_n \). The number \( \phi \cdot \beta \) can be taken arbitrarily near to \( \pi \).
3. An extension of Hadamard's theorem. We shall now consider the case where

\[ q(x, u) = \frac{au + bx + c}{axu + bxx + cx} \]

and shall proceed in a way analogous to that used by Haslam-Jones* in his extension of Hadamard's theorem to functions of two variables. Suppose that the series

\[ \sum_{n=0}^{\infty} a_n u^n, \quad \sum_{n=0}^{\infty} b_n u'^n \]

represent the uniform and analytic functions \( f(u) \) and \( g(u') \), respectively. Let the singularities \( \alpha_i \) of \( f(u) \) form an isolated set, \( |\alpha_i| \geq r_1 > 0 \). Similarly, let the singularities \( \beta_j \) of \( g(u') \) form an isolated set, \( |\beta_j| \geq r_2 > 0 \). Let \( u \) be within a Laurent ring \( L(r_0) \) defined by \( 0 < r_0 < |u| < r_1 \), and let \( x \) be in a region \( R_0 \) given by \( |x| < r \). Writing \( u' = (au + bx + c)/(axu + bxx + cx) \) we have

\[ |u'| = \left| \frac{au + bx + c}{axu + bxx + cx} \right| = \left| \frac{a + (bx + c)/u}{a_1 + (b_1x + c_1)/u} \right| \]

\[ < \frac{|a| + (|b| r + |c|)/r_0}{|a_1| - (|b_1| r + |c_1|)/r_0} < r_2 \]

for \( u \) in \( L(r_0) \) and \( x \) in \( R_0 \), provided \( |a_1| > (|b_1| r + |c_1|)/r_0 \) and \( |a_1| \) is taken sufficiently great. On the other hand, if \( a_1 = 0 \) it will be supposed that \( c_1 \neq 0 \). We have in this case

\[ |u'| = \left| \frac{au + bx + c}{b_1x + c_1} \right| < \frac{|ar_1| + |br| + |c|}{|a_1| - |b_1r|} < r_2 \]

for \( u \) in \( L(r_0) \) and \( x \) in \( R_0 \), provided \( r < |c_1|/b_1 \) and \( r_2 \) is sufficiently great. Let \( C \) denote a contour in \( L(r_0) \). By (3), the series representing \( f(u) \), \( g(u') \) are uniformly convergent in \( u \) for \( u \) on \( C \).

Thus \( F(x) \) given by

\[ F(x) = \frac{1}{2\pi i} \int_C f(u) \cdot g\left( \frac{au + bx + c}{axu + bxx + cx} \right) \frac{du}{u} = \sum_{n=0}^{\infty} c_n x^n \]

is an analytic function at least in \( R_0 \). The \( c_n \) depend on the \( a_i \) and the \( b_k \). The integral (4) is analytic in \( x \) within a region \( R_0' \), more extensive than \( R_0 \), defined by the condition that none of the points in the \( u \)-plane given by

\[ u = \alpha_i, \quad (au + bx + c)/(axu + bxx + cx) = \beta_k, \quad u = 0, \quad u = \infty \]

shall lie within some assigned positive distance $\delta$ from $C$. This follows from the fact that for $u$ on $C$ the function $f(u)g(u')$ is analytic in $x$ for all $x$ in $R_0$, and that it is a continuous function of $x$ and $u$, when $x$ is in $R_0$ and $u$ on $C$. We observe further that, if $C_1$ is a second contour in the $u$-plane and is such that the region between $C$ and $C_1$ contains no points of the set $A$ defined by

$$u = \alpha_i, \quad (au + bx + c)/(a_1u + b_1x + c_1) = \beta_i, \quad u = 0, \quad u = \infty,$$

then

$$\int_C f(u)g\left(\frac{au + bx + c}{a_1u + b_1x + c_1}\right) du = \int_{C_1} f(u)g\left(\frac{au + bx + c}{a_1u + b_1x + c_1}\right) du.$$

Among the points $A$ it is not necessary to include the point $u$ given by $a_1u + b_1x + c_1 = 0$. If infinity is included among the values $\beta_i$, this value of $u$ is given by $(au + bx + c)/(a_1u + b_1x + c_1) = \beta_i$ for some $i$. On the other hand, if all the $\beta_i$ are finite, $g(u')$ is regular for $u' = \infty$, that is, $u$ as given by $a_1u + b_1x + c_1 = 0$ is not a singularity of $g(u')$.

If $x$ is in $R_0(|x| < r)$, $F(x)$ is analytic on account of the convergence of its Taylor's series (4). This function, however, exists in a more extended region $R_0$ which has with $R_0$ a neighborhood of the origin in common. As $x$ approaches the boundary of $R_0$ some of the singular points in the $u$-plane may approach $C$. To meet this situation $C$ is replaced by $C'$ by a continuous deformation of $C$, without passing over any of the points of the set $A$. The integral over $C_1$ is analytic for $x$ in $R_{C_1}$, that is, in a region including the neighborhood of $x = 0$. Thus this integral is an analytic continuation of the power series (4). This process of continuation fails only if points $A$ on opposite sides of $C$ tend to coincide. It follows therefore that $F(x)$ is analytic in every $x$-region which is such that for no point within that region do any two points of the set $A$ coincide. Accordingly, we shall proceed to find values of $x$ for which such coincidences may occur. When a point $u$ defined by $u' = \beta_i$ coincides with a point $\alpha$, we have

$$(a\alpha + bx + c)/(a_1\alpha + b_1x + c_1) = \beta_i.$$

When a point $u$, defined by $u' = \beta_i$, coincides with the point $u = 0$, we have $(bx + c)/(b_1x + c_1) = \beta_i$. Coincidences of points $u$, defined by $u' = \beta_i$, with $u = \infty$, would give the relation $a/a_1 = \beta_i$ which is contrary to (13); thus, no coincidences of this kind can occur. The coincidence of a $u$-point defined by

* $C$ can now be considered not necessarily restricted to $L(r_0)$; it does not pass through any of the $\alpha_i$ and it contains the origin in its interior.
$u' = \beta_i$ with a $u$-point defined by $u' = \beta_i (k \neq i)$ gives, as a result of solving for $x$ the equation

$$u' - \beta_i = u' - \beta_k = 0,$$

$x = - \frac{(c_0a - a_0c)}{(b_1a - a_1b)}$. Hence the singularities of $F(x)$ are all included among the points

$$x = \frac{a - a_1\beta_i}{b_1\beta_i - b} \cdot \alpha_r + \frac{c - c_1\beta_i}{b_1\beta_i - b}, \quad x = \frac{c - c_1\beta_i}{b_1\beta_i - b}, \quad x = - \frac{(c_0a - a_0c)}{(b_1a - a_1b)}.$$

We have proved the following theorem:

**Theorem II.** Let $q(x, u) = \frac{a_0 + bx + c}{(axu + bx + cx)}$, where $|a_1|$ is sufficiently great so that, for a positive $r_0$, $r_0 < r_1$, and a positive $r$, (3) is satisfied. If $a_1 = 0$ it will be assumed that $c_1 \neq 0$. In this case $r_1$ is supposed to be sufficiently great so that for a positive $r$, $r < |c_1/b_1|$, (3') is satisfied. Let the series $\sum a_n x^n$, with an isolated set of singularities $\alpha_i$, $|\alpha_i| \geq r_1$, and $\sum b_n x^n$, with an isolated set of singularities $\beta_i$, $|\beta_i| \geq r_2$, represent the uniform functions $f(x)$ and $g(x)$, respectively. Let $F(x)$ be defined by the integral (4). If $a_1 \neq 0$, 

(9) $F(x) = \sum c_n x^n = \sum a_n \cdot b_m \cdot m^p_n(x)$ \quad $q^m(x, u) = \sum_{i=0}^m m^p_i(x)/u^i; \quad |x| < r)$. If $a_1 = 0$,

(9') $F(x) = \sum c_n x^n = a_0 \sum b_0 \beta_i(x)$ \quad $q^m(x, u) = \sum_{i=0}^m m^p_i(x) \cdot u^i; \quad |x| < r)$. $F(x)$ possesses in the whole plane no other singularities than those given by

(10) $\frac{a - a_1\beta_i}{b_1\beta_i - b} \cdot \alpha_r + \frac{c - c_1\beta_i}{b_1\beta_i - b}, \quad \frac{c - c_1\beta_i}{b_1\beta_i - b}, \quad - \frac{(c_0a - a_0c)}{(b_1a - a_1b)}.$

There are no singularities for $|x| < r$.

The second series (9) can be derived as follows. Consider the integral (4). It is an analytic function of $x$ for $x$ in $R_0 (|x| < r)$. For $u$ in $L(r_0)$, that is, for $0 < r_0 < |u| < r_1$, and $x$ in $R_0$, letting

(11) $u^m = \left(\frac{au + bx + c}{a_1u + bx + c_1}\right)^m = \sum_{i=0}^m m^p_i(x)/u^i, \quad a_1 \neq 0$ \quad (here $|u' < r_2$),

it is observed that all the series involved in the integrand are absolutely and uniformly convergent. Thus, by multiplication and arranging $f(u) \cdot g(u')$ in powers of $u$, and noting that $F(x)$ is the term free of $u$, we obtain
\[
F(x) = \sum_{n,m} a_n b_m n! p_n(x) \quad (|x| < r; a_1 \neq 0).
\]

To derive the second series (9') we let
\[
\left(\frac{au + bx + c}{b_1 x + c_1}\right)^m = \sum_{i=0}^{m} s_i(x) \cdot u
\]
where the series converges and \( |u' | < r_2 \) for \( |x| < r \) and \( |u| < r_1 \). It is observed that \( F(x) \) is the term free of \( u \) in
\[
f(u) \cdot g(u') = \sum_{0}^{\infty} d_i(x) \cdot u^i.
\]
Hence
\[
F(x) = \sum_{0}^{\infty} d_i(x) = a_0 \cdot \sum_{i=0}^{\infty} b_i \cdot i! s_0(x) \quad (|x| < r; a_1 = 0).
\]

We see therefore that in the case when \( a_1 = 0 \) the singularities of the function \( F(x) \) are independent of the singularities of \( f(x) \). This case is consequently of no special interest. A priori it would seem to be possible to get Hurwitz's theorem as a particular case of Theorem II by setting \( a_1 = b_1 = c = 0, b = c_1 = 1, a = 1 \). Indeed introducing these values in formula (10) we get the singularities of the composed function of Hurwitz's theorem. Unfortunately, however, this function is not represented by (9') which is an entirely different function. While the integrals look alike, the contours of integration are different. It is not possible to choose the contour in Theorem II in such a fashion that both Hadamard's and Hurwitz's theorems come out as special cases.

If in Theorem II we let \( b = a_1 = 1 \) and \( a = c = b_1 = c_1 = 0 \), \( q(x, u) \) becomes \( x/u \) and (3) becomes \( r/r_0 < r_2 \). The latter inequality is seen to be satisfied for suitable values of \( r \) and \( r_0 \). The function \( F(x) \), as given by (9), will become \( F(x) = \sum a_n b_n x^n \). On the other hand, since there are no singularities of \( F(x) \) for \( |x| < r_1 \), we have from (10) the expressions \( \beta_{i} \alpha_{r} \) as the only ones among which all of the singularities of \( F(x) \) are found. Thus Theorem II contains Hadamard's theorem, as a particular case.

4. The extension of the theorem of §3 to functions of two variables.\dagger

Let \( f(u, v) \) be an analytic function of the complex variables \( u \) and \( v \), having a branch regular near \( u = 0, v = 0 \). Let for that branch
\[
f(u, v) = \sum a_{mn} u^m v^n.
\]
Similarly, let \( g(u', v') \) be another function for which
\[
(2) \quad g(u', v') = \sum \sum b_{mn} u'^m v'^n.
\]
Suppose that the singularities of \( f(u, v) \) and \( g(u', v') \) are given by \( p(u, v) = 0 \) and \( q(u', v') = 0 \), respectively. The functions \( p(u, v) \) and \( q(u', v') \) are analytic functions of type Z. A function \( s(u, v) \) is said to be of type Z if for any constant \( u_1, s(u_1, v) = 0 \) or has isolated zeros in the v-plane; and for any constant \( v_1, s(u, v_1) = 0 \) or has isolated zeros in the u-plane. Let \( f_1, f_2 \) and \( g_1, g_2 \) be positive constants such that (1) is convergent for \( |u| < f_1, |v| < f_2 \), and (2) is convergent for \( |u'| < g_1, |v'| < g_2 \). Then for \( |a_1| \) sufficiently large, positive constants \( u_1 \) and \( v_1 \) can be found, \( u_1 < f_1, v_1 < f_2 \), such that
\[
(3) \quad \frac{au + bx + c}{a_1 u + b_1 x + c_1} < g_1, \quad \frac{av + by + c}{a_1 v + b_1 y + c_1} < g_2
\]
for all \( u \) and \( v \) satisfying \( |u| > u_1, |v| > v_1 \) and for all \( x \) and \( y \) in a region \( R_0 \) defined by the relations \( u_1 < |u| < f_1, v_1 < |v| < f_2 \). As in (3) in §3, writing \( u' = (au + bx + c)/(a_1 u + b_1 x + c_1) \) and \( v' = (av + by + c)/(a_1 v + b_1 y + c_1) \), we have both inequalities
\[
(4) \quad |u'| = \frac{a + (bx + c)/u}{a_1 + (b_1 x + c_1)/u} \quad \text{and} \quad |v'| = \frac{a + (by + c)/v}{a_1 + (b_1 y + c_1)/v}
\]
satisfied, provided \( |a_1| \) is sufficiently great. Let \( u, v \) be within the region \( L \) defined by the relations \( u_1 < |u| < f_1, v_1 < |v| < f_2 \). In \( L \), both \( f(u, v) \) and \( g(u', v') \) have their respective Taylor’s series absolutely and uniformly convergent for all \( x, y \) in \( R_0 \). We shall consider now the following integral:
\[
(6) \quad F(x, y) = -\frac{1}{4\pi^2} \int_C \int_{C'} f(u, v) \cdot g \left( \frac{au + bx + c}{a_1 u + b_1 x + c_1}, \frac{av + by + c}{a_1 v + b_1 y + c_1} \right) \frac{dv}{v} \frac{du}{u},
\]
where \( C \) is a contour in the \( u \)-plane and \( C' \) a contour in the \( v \)-plane, both in \( L \). Substituting the power series (1) and (2) in (6), these series being absolutely and uniformly convergent on \( C \) and \( C' \), it is observed that multiplication term by term is possible. Thus for \( x, y \) in \( R_0 \)
\[
(7) \quad F(x, y) = \sum \sum c_{mn} x^m y^n
\]
where the \( c_{mn} \) depend on the \( a_{mn} \) and the \( b_{mn} \).

The integral (6) is analytic for \( (x, y) \) in \( R_0 \) \( |x| < r, |y| < s \). Using (11) of §3, noting the absolute and uniform convergence of all of the series in-
volved and arranging \( f(u, v), g(u', v') \) in powers of \( v \) (\( u \) and \( v \) in \( L \)), we get from the product

\[
\sum_v v^r \left( \sum_{i} a_{i} u^i \right) \cdot \sum_v v^{-s} \left( \sum_{m,k,r} b_{mk} m p_r(x) k p_s(y) / u^r \right)
\]

the term free of \( v \)

\[
(8) \quad \frac{1}{2\pi i} \int_{C_1} f \cdot g \cdot \frac{dv}{v} = \sum_{r} \left( \sum_{r} a_{r} u^r \right) \left( \sum_{r} b_{r,s} \cdot \left( x, y \right) u^{-r} \right)
\]

where

\[
(9) \quad b_{r,s}(x, y) = \sum_{m,k} b_{mk} m p_r(x) k p_s(y).
\]

Multiplying \( (8) \) by \( du / (2\pi i u) \), and integrating with respect to \( u \) along the contour \( C \), and arranging \( (8) \) in powers of \( u \), we obtain the term free of \( u \)

\[
(10) \quad F(x, y) = \sum_{r,s} a_{rs} b_{r,s}(x, y) = \sum_{r,s,m,k} a_{rs} b_{mk} m p_r(x) k p_s(y).
\]

The series in the last member converges at least in \( R_0 \).

We shall now proceed to find all of the possible singular loci of the function \( F(x, y) \) whose analytic element in \( R_0 \) is given by \( (7) \). Denote the values of \( v \) satisfying \( p(u, v) = 0 \) by \( (\Gamma u) \). Let the \( d_i \) be the points of the \( u \)-plane for which \( p(u, v) = 0 \) for all \( v \), and which are therefore defined by the equation \( p(u, 0) = 0 \). The \( d_i \) form an isolated set. Denote the values of \( v \) satisfying \( q( (au + bx + c)/(a_1 u + b_1 x + c_1), (av + by + c)/(a_0 v + b_0 y + c_0)) = 0 \) by \( (\Gamma' u) \) and the isolated values of \( u \) satisfying \( q( (au + bx + c)/(a_1 u + b_1 x + c_1), 0) = 0 \) by \( \phi_i \). The \( (\Gamma' u) \) depend on \( u, x \), and \( y \); the \( \phi_i \) depend on \( x \). We shall now show that if \( (x, y) \) is in \( R_0 \) and if \( u \) describes \( C \) (\( C \) and \( C \) in \( L \)), the points \( v = (\Gamma' u) \) describe contours \( \Gamma' \) outside of \( C' \) and the points \( v = (\Gamma u) \) describe contours \( \Gamma' \) inside \( C' \), provided \( |a'| \) is large enough. For \( u \) on \( C \), \( |u| < f_1 \). If \( |v| = |(\Gamma u)| < f_2 \), there would be a contradiction to the assumption that \( (1) \) converges whenever \( |u| < f_1 \), \( |v| < f_2 \). Hence \( |(\Gamma u)| \geq f_2 \), and thus the \( \Gamma \) contours are outside of \( C' \). Similarly, when \( x \) and \( y \) are in \( R_0 \) the values \( v = (\Gamma' u') \) defined by \( q(u', v') = 0 \) satisfy the inequality \( |v'| \geq g_2 \). The points \( v = (\Gamma' u') \) satisfy the equation \( (av + by + c)/(a_0 v + b_0 y + c_0) = v' = (\Gamma' u') \). Thus

\[
(11) \quad v = (\Gamma' u) = \frac{(b_1 y + c_1)v' - (by + c)}{a - a_1 v'} \quad \text{where} \quad |v'| \geq g_2.
\]

We have

\[
(12) \quad |(\Gamma' u)| = \left| \frac{(b_1 y + c_1) - (by + c)/v'}{a - a_1 v'} \right|
\]

\[
< \frac{|b_1| s + |c_1| + (|b| s + |c|) / g_2}{|a_1| - |a| / g_2} \leq v_1,
\]
provided that $|a_i|$ is sufficiently large. The constant $|a_i|$ will be taken so that the inequalities (4), (5) and (12) are all satisfied. Having so chosen $|a_i|$, from (12) it follows that the contours $\Gamma'$ are all inside $C'$. Thus, for $C$ in $L$ and $(x, y)$ in $R_0$, the contour $C'$ in the $v$-plane separates the $\Gamma$ figure (consisting of the contours $\Gamma'$) from the $\Gamma'$ figure (consisting of the contours $\Gamma'$).

Introduce in the $v$-plane a contour $C_u$ depending on $u$ and consisting of a circle of a large radius $R$, $R > |(\Gamma'u)|$, with the center $v = 0$. Let it be indented to exclude those points $v = (\Gamma'u)$ which are inside of this circle, and let it contain all the points $(\Gamma'u)$. (In order that this should be possible, for $(x, y)$ not necessarily in $R_0$ and $u$ not necessarily in $L$, the conditions $(\Gamma'u) \neq \infty$, $(\Gamma) \neq (\Gamma'u)$ will be later introduced, the first to have a finite $R > |(\Gamma'u)|$, the second to make separation of the points $(\Gamma'u)$ from the points $(\Gamma'u)$ possible.) When $u$ is on $C$ and $(x, y)$ in $R_0$ the integrand of

$$\int_{C_u} f(u,v) \cdot g \left( \frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1} \right) \frac{dv}{v},$$

as a function of $v$, has no singular points in the region of the $v$-plane bounded by $C$ and $C_u$. Accordingly,

$$\int_{C_u} f(u,v) \cdot g(u',v') \frac{dv}{v} = \int_{C_u} f(u,v) \cdot g(u',v') \frac{dv}{v}$$

and

$$F(x, y) = -\frac{1}{4\pi^2} \int_C \int_{C_u} f(u,v) \cdot g(u',v') \frac{dv}{v} \frac{du}{u},$$

which holds for $(x, y)$ in $R_0$ and $C$ in $L$. Concerning this integral we shall prove the following lemma:

**Lemma II.** With $C$ any contour in the $u$-plane, the integral (15) is analytic in $x$ and $y$ for $(x, y)$ in a region $R_C$ defined by the condition that none of the points of the $u$-plane given by

$$q(u,v) = \begin{cases} \frac{au + bx + c}{a_1u + b_1x + c_1}, & \frac{av + by + c}{a_1v + b_1y + c_1} = 0, \\ \frac{a}{a_1} \end{cases}$$

or

$$q(u',v') = \begin{cases} \frac{au + bx + c}{a_1u + b_1x + c_1}, & a = 0, \\ \frac{a}{a_1} \end{cases}$$

shall lie within an assigned distance $\delta$, $\delta > 0$, from $C$.

* (17) is obtained from $q(u',v') = 0$ by letting $v \to \infty$. 
When \( u \) is a fixed point on \( C \), \((\Gamma' u) \neq (\Gamma u)\) by (16); and \((\Gamma' u) \neq \infty\) by (17). Thus the contour \( C_u \) can be constructed with a finite \( R \), so that it separates the points \((\Gamma' u)\) and \((\Gamma u)\) in such a way that these points do not lie within some assigned positive distance from \( C_u \). Consequently, we can have a contour \( C_u \) which is such that \( C_u \) can be continuously deformed into \( C_u \), without encountering any of the \( v \)-singularities of the integrand. Then

\[
\int_{C_u} f(u, v) \cdot g(u', v') \, dv/v = \int_{C_u} f(u, v) \cdot g(u', v') \, dv/v,
\]

where the second integral (18) remains analytic in \((x, y)\). Thus, for \( u \) fixed on \( C \), and \((x, y)\) in \( R_C \), the first integral (18) is analytic in \( x \) and \( y \). We shall now prove that this integral is analytic in \( x \), \( y \) and \( u \), for \((x, y)\) in \( R_C \) and \( u \) within a small circle \( s \) with center at \( u \), \( u \) a point on \( C \); that is, that it is continuous in \( x \), \( y \) and \( u \) for \((x, y)\) in \( R_C \) and \( u \) on \( C \). This will complete the proof of the lemma. Choose \( s \) small enough so that the representations of \( s \) on the \( \Gamma \) and \( \Gamma' \) figures, \( \sigma \) and \( \sigma' \) respectively, have no points in common. If, for \( s \) no matter how small, \( \sigma \) and \( \sigma' \) had a point in common, (13) would hold for \( u \) on \( C \) and \((x, y)\) in \( R_C \). Let \( C' \) be a contour excluding \( \sigma \) and including \( \sigma' \). Then

\[
\int_{C'} f(u, v) \cdot g(u', v') \, dv/v = \int_{C'} f(u, v) \cdot g(u', v') \, dv/v.
\]

As in the case of the second integral (18), the second integral (19) is analytic in \( x \), \( y \) and \( u \) when \((x, y)\) is in \( R_C \) and \( u \) is in \( s \). Hence the same holds true of the first integral (19).

**Lemma III.** Given in the \( u \)-plane a contour \( C_1 \) such that \( C \) can be deformed into \( C_1 \) by a continuous deformation without passing over any of the points \( A \) defined by

\[
p(u, v) = q \left( \frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1} \right) = 0,
\]

\[
q \left( \frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{a}{a_1} \right) = 0,
\]

\[
\rho(u, 0) = 0,
\]

\[
u = 0, \quad \eta = \infty,
\]

then

\[
\int_{C} \int_{C_u} f(u, v) \cdot g(u', v') \frac{dv}{v} \frac{du}{u} = \int_{C} \int_{C_u} f(u, v) \cdot g(u', v') \frac{dv}{v} \frac{du}{u}.
\]
We denote by $D$ the region in the $u$-plane bounded by $C$ and $C_1$, the representations of $D$ on the $\Gamma$ and $\Gamma'$ figures by $\Delta$ and $\Delta'$, respectively, and the contour in the $v$-plane containing $\Delta'$ and excluding $\Delta$ by $C_\Delta$. If $\Delta$ and $\Delta'$ have no points in common

$$\frac{1}{u} \int_{C_u} f(u, v) \cdot g(u', v') \cdot \frac{dv}{v} = \frac{1}{u} \int_{C_\Delta} f(u, v) \cdot g(u', v') \cdot \frac{dv}{v}. \tag{25}$$

For $(x, y)$ fixed, the second, and therefore the first, integral is an analytic function of $u$ for all $u$ in $D$. When $\Delta$ and $\Delta'$ have points in common we may use precisely the procedure of Haslam-Jones.* Thus, the function of $u$, given by the first integral (25), can be integrated along $C$ giving the first integral (24). $C$ can be then deformed into $C_1$. This completes the proof of the lemma.

$F(x, y)$, which is analytic in $R_0$, is given there by the convergent power series (7). By Lemma II, $F(x, y)$ is analytic in the more extensive region $R_0$, its analytic expression there being given by (15). The region of validity of the representation of $F(x, y)$, by means of a double integral, can be further extended by deforming $C$ into $C_1$, utilizing Lemma III. It is impossible to continue this process when two points of the set $A$, lying on opposite sides of $C$ (the contour in the $u$-plane), coincide. Thus, $F(x, y)$ has a principal branch all of whose singularities are found among the relations expressing coincidence of two points of the set $A$. This set is defined by (20), (21), (22), (23). Using Hartogs' theorem that an analytic function of two variables cannot have isolated points for singularities, and a lemma by Haslam-Jones,† coincidences of the points of the set $A$ will be shown to yield no singularities of $F(x, y)$. Coincidences of two solutions of (21), of a solution of (21) with that of (22), of a solution of (22) with that of (23), may occur only for isolated values of $x$; the points $u$ defined by those equations being independent of $y$, there arise no singularities. There are no singularities of $F(x, y)$ in connection with (22), (23).


We quote the lemma:

Let $f(u, x, y)$ be an analytic function of $u, x, y$ which has continuous singularities $u=\alpha(x, y)$, and consider a contour $C$ in the $u$-plane. If, for $\phi(x, y)=0$, $\alpha=\alpha_0$ and the other singularities are such that $|\alpha_0-\alpha_i|>\delta>0$, then

$$F(x, y) = \int_C f(u, x, y)du$$

is regular for $\phi(x, y)=0$ provided that $f(u, x, y)=f_1(u, x, y)+f_2(u, x, y)$, where $f_1(u, x, y)$ is regular for $u=\alpha_0(x, y)$, and $f_2(u, x, y)$ is regular for $u=\alpha(x, y)$.
Since the case when \( F(x, y) \) is a constant is of no interest, it will be supposed that \( b \) and \( b_1 \) are not zero together and, if \( a = 0, b = 0 \), then \( c \neq 0 \). If \( a = 0 \) and \( b \neq 0 \) the values of \( x \) and \( y \) satisfying \( bx + c = by + c = 0 \) will, in the following, be excluded. If \( a \neq 0 \), it will be assumed that \( bc_1 - b_1 c \neq 0 \).

Suppose that a point \( u = \alpha_2(x, y) \), satisfying (20), coincides with a point \( u = \alpha_1 \), satisfying (22). Suppose \( |v| \to 0 \), as \( \alpha_2(x, y) \to \alpha_1 \). For \( v = 0 \), (20) becomes

\[
\rho(u, 0) = q\left( \frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{by + c}{b_1y + c_1} \right) = 0.
\]

We have the isolated set of values \( u \) defined by \( \rho(u, 0) = 0 \), \( u = d_* \), and from the second member, \( u = \epsilon_4(x, y) \). Therefore singularities may arise for \( d_* = \epsilon_4(x, y) \), that is, for \( x = n_4(d_*, y) \). Substituting this expression for \( x \) in (20), eliminating \( v \), and letting \( u = d_* \), we derive an isolated set of values \( y, y = n_4(d_*, y) \). By Hartogs’ theorem no singularities, due to this case, arise. Suppose \( |v| \) does not approach 0, as \( \alpha_2(x, y) \to \alpha_1 \). In (20), let \( u = \alpha_2(x, y) = \alpha_1 \). Assume that whenever \( |v| < \epsilon \),

\[
|\rho(\alpha_1, v)| \leq \eta,
\]

\[
\left| q\left( \frac{a\alpha_1 + bx + c}{a_1\alpha_1 + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1} \right) \right| \leq \eta \quad (\eta \to 0, \text{ as } \epsilon \to 0).
\]

The two inequalities cannot hold simultaneously, since that would mean that \( v \to 0 \), as \( \alpha_2 \to \alpha_1 \). The inequality (27) holds since \( \rho(\alpha_1, 0) = 0 \). Therefore

\[
|q\left( \frac{a\alpha_1 + bx + c}{a_1\alpha_1 + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1} \right)| > \eta, \quad \text{whenever } |v| < \epsilon.
\]

Let \( C_{u_0} \) be a contour in the \( u \)-plane consisting of \( C_u \) and in addition indented to exclude the origin. We have then

\[
\int_{C_{u_0}} f(u, v) \cdot g \left( \frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1} \right) \frac{dv}{v} = \int_{C_{u_0}} f(u, v) \cdot g \left( \frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{av + by + c}{a_1v + b_1y + c_1} \right) \frac{dv}{v}
\]

\[
+ \frac{1}{2\pi i} f(u, 0) \cdot g \left( \frac{au + bx + c}{a_1u + b_1x + c_1}, \frac{by + c}{b_1y + c} \right).
\]

This relation follows from the following considerations. Restrict \((x, y)\) so that none of the \( u \)-solutions of (20) coincide with a solution of (22). Then
\( p(\alpha_2(x, y), 0) \neq 0 \), and \( f(u, v) \) is thus analytic for \( |u - \alpha_2| < \delta, |v| < \epsilon \) (for \( \delta, \epsilon \) sufficiently small). The function \( g(u', v') \) has \( u = \alpha_2(x, y), v = 0 \) for singular values only if

\[
q \left( \frac{a\alpha_2(x, y) + bx + c}{a_1\alpha_2(x, y) + b_1x + c_1}, \frac{by + c}{b_1y + c_1} \right) = 0.
\]

If this relation were true, one of the equations (20), \( g(u', v') = 0 \), where \( u \) is replaced by \( \alpha_2(x, y) \), would yield a solution \( v = 0 \). Another equation (20), \( p(\alpha_2(x, y), v) = 0 \), would become \( p(\alpha_2(x, y), 0) = 0 \), and this was shown to be impossible. Thus, \( g(u', v') \) is analytic for \( |u - \alpha_2| < \delta, |v| < \epsilon \), and the same is true of \( f(u, v) \cdot g(u', v') \). Hence \( f(u, v) \cdot g(u', v')/(2\pi i) \), with \( v = 0 \), is a function of \( u \) analytic in the neighborhood of \( u = \alpha_2(x, y) \). This function is the last term in (30) and is the residue at \( v = 0 \) of the integrand in (30). By (29), \( g(u', v') \) is analytic for \( |u - \alpha_1| < \delta, |v| < \epsilon \) and hence the integral in the second member of (30) is a function of \( u \) analytic in a neighborhood of \( u = \alpha_2 \).

Hence, by the lemma of Haslam-Jones, there are no singularities due to the case under consideration (with the above hypothesis on \( x \) and \( y \)).

If \( a = 0, b \neq 0 \), the preceding reasoning is neither applicable for \( x = -c/b \) nor for \( y = -c/b \). Consider the case \( a = 0, b \neq 0, y = -c/b \). With \( a = 0, b \neq 0 \), from (11) of §3 it can be shown that every \( m_{p_m(x)} \) contains \( (bx + c) \) to the \( m \)th power, as a factor, and that \( m_{p_r(x)} = 0 \) for \( r < m \). Thus letting \( y = -c/b \) in (9), and observing that \( p_m(-c/b) = 0 \), unless \( k = s = 0 \) \((p_0(-c/b) = 1)\), we get

\[
B_{r,s}(x, -c/b) = 0, \quad s \geq 1,
\]

and

\[
B_{r,0}(x, -c/b) = \sum_m b_{m,0} \cdot m_{p_r(x)}.
\]

Thus from (10)

\[
F(x, -c/b) = \sum_r a_{r,0} \cdot B_{r,0}(x, -c/b) = \sum_r a_{r,0} \cdot b_{m,0} \cdot m_{p_r(x)}.
\]

Since

\[
f(u, 0) = \sum_r a_{r,0} \cdot u^r, \quad g(u, 0) = \sum_m b_{m,0} \cdot u^r,
\]

by comparing (31) with (12) of §3, it can be seen that \( F(x, -c/b) \), as given by the methods of this section, is derivable from \( f(u, 0), g(u, 0) \) by means of Theorem II. This theorem is applicable since, by the hypothesis on \( p(u, v) \), \( g(u, v) \), the functions \( f(u, 0), g(u, 0) \) have isolated singularities. Thus \( F(x, -c/b) \) has only isolated singularities. Similarly, \( F(-c/b, y) \) has only isolated singularities. By Hartogs' theorem no singularities of \( F(x, y) \) arise for this case.
Singularities, due to the coincidences of a \( u \)-point of (20) with a point of (21) or a point of (23), are eliminated in the particular case treated by Haslam-Jones, by methods which cannot be readily extended to our case. Neither did we succeed in eliminating these singularities by any other method. Thus we assert the following.

All the singular loci of \( F(x, y) \) are found among the relations expressing coincidence of two \( u \)-points of (20), of a point of (20) with a point of (21), of a point of (20) with that of (23). The points (23) are \( u = 0, u = \infty \).

We have thus proved the following theorem:

**Theorem III.** Let \( f(u, v) \) be an analytic function of \( u \) and \( v \) having a branch regular at \( u = 0, v = 0 \), and having for that branch

\[
f(u, v) = \sum \sum a_{m,n,u} u^m v^n \quad \text{(convergent for } |u| < f_1, \ |v| < f_2)\]

let \( g(u', v') \) be a similar function for which

\[
g(u', v') = \sum \sum b_{m,n,u'} u'^m v'^n \quad \text{(convergent for } |u'| < g_1, \ |v'| < g_2)\]

and let \( f(u, v) \) and \( g(u', v') \) have singularities given by \( p(u, v) = 0 \) and \( q(u', v') = 0 \), respectively, where \( p(u, v) \) and \( q(u, v) \) are analytic functions of the type \( Z \). Let \( r(x, u) = (a_{ij} u + bx + c)/(a_{ij} u' + b_{ij} x + c_{ij}) \), where \( |a_{ij}| \) is sufficiently great, so that for positive \( r, s, u_1, v_1 < f_1, v_1 < f_2 \) the inequalities (4), (5) and (12) are all satisfied. Exclude the relation \( b = b' = 0 \). If \( a \neq 0 \), let \( b c_1 - b' c \neq 0 \).

The function \( F(x, y) \), given by the series

\[
F(x, y) = \sum \sum c_{m,n,x} x^m y^n = \sum \sum r_{r,s,m,k} \cdot m p_r(x) \cdot n p_s(y)
\]

converging for \( |x| < r, \ |y| < s \), has a branch which is regular near \( x = 0, y = 0 \). It has all of its singular loci included among the \((x, y)\)-relations expressing coincidence of two \( u \)-solutions of

\[
p(u, v) = q\left(\frac{a_{ij} u + bx + c}{a_{ij} u' + b_{ij} x + c_{ij}}, \frac{a_{ij} v + by + c}{a_{ij} v' + b_{ij} y + c_{ij}}\right) = 0,
\]

of a solution of (20) with that of \( p(u, 0) = 0 \), of a solution of (20) with \( u = 0 \), of a solution of (20) with \( u = \infty \).

5. An extension of Theorem II. Consider the integral

\[
\frac{1}{2 \pi i} \int f(u) g(q(x, u)) du/u,
\]

where
\[ q(x,u) = \frac{e_0 + e_1 u + \cdots + e_n u^n}{g_0 + g_1 u + \cdots + g_n u^n}, \]

\[ e_i = e_{i0} + e_{i1} x + \cdots + e_{im} x^m, \]

\[ g_i = g_{i0} + g_{i1} x + \cdots + g_{im} x^m. \]

The constants \( e_{ik}, g_{ik} \), with the exception of \( g_{n0} \), satisfy the inequalities

\[ |e_{ik}|, \ |g_{ik}| \leq s. \]

Also \( e_n \neq 0, g_n \neq 0 \). Concerning the functions \( f(u), g(u') \) we make the assumptions of \( \S3 \). For \( u \) within a Laurent ring \( L(r_0) \), defined by \( 0 < r_0 < |u| < r_1 \), and \( x \) within a region \( R_0 \), defined by \( |x| < r \), the following holds:

\[ |g_i(x) - g_{i0}| < s_m(r). \]

The inequalities (3) do not include \( g_{n0} \); however, \(|g_n(x) - g_{n0}| < s_m(r)\).

Also,

\[ |g_n + g_{n-1} u^{-1} + \cdots + g_0 u^{-n}| = |g_{n0} + (g_n - g_{n0}) + g_{n-1} u^{-1} + \cdots + g_0 u^{-n}| > |g_{n0}| - s_m(r)(1 + r_0^{-1} + \cdots + r_0^{-m}) > 0, \]

provided \(|g_{n0}|\) is sufficiently great. Thus writing \( u' = q(x, u) \), it follows that

\[ |u'| = |q(x, u)| = \frac{|e_n + e_{n-1} u^{-1} + e_{n-2} u^{-2} + \cdots + e_0 u^{-n}|}{g_n + g_{n-1} u^{-1} + g_{n-2} u^{-2} + \cdots + g_0 u^{-n}} < s_m(r)(1 + r_0^{-1} + \cdots + r_0^{-m}) < r_2, \]

provided that \(|g_{n0}|\) is sufficiently great and \( x \) is in \( R_0 \) and \( u \) in \( L(r_0) \).

Let \( C \) denote a closed contour in \( L(r_0) \). The series representing \( f(u) \) and the series representing \( g(u') \) converge absolutely and uniformly when \( x \) is in \( R_0 \) and \( u \) on \( C \). The latter series converges since, by (4), \(|u'| < r_2\).

Thus \( F(x) \), given by

\[ F(x) = \frac{1}{2\pi i} \int_C \left( \sum a_n u^n \right) \cdot \left( \sum b_n q^n(x, u) \right) \cdot du/u \]

(5)

\[ = \sum c_n x^n = \sum_{n, m} a_n \cdot b_m \cdot m s_n(x), \]

is an analytic function. The last two representations (5) converge at least in \( R_0 \). The second of the two representations is derived by noting that

\[ q^n(x, u) = \sum_i m s_i(x)/u^i \]
is absolutely and uniformly convergent for \( x \) in \( R_0 \) and \( u \) in \( L(r_0) \), and by proceeding in the way similar to the one used in deriving (12) of §3.

The integral (5) is analytic within a wider region \( R_C \), defined by the condition that none of the points \( A \) in the \( u \)-plane, given by

\[
 u = \alpha_i, \quad q(x, u) = \beta_k, \quad u = 0, \quad u = \infty,
\]

shall lie within some assigned positive distance from \( C \). The contour \( C \) is now not necessarily in \( L(r_0) \), it contains \( u = 0 \) in its interior, and does not pass through any of the \( \alpha_i \). Analyticity of (5) follows from analyticity of \( f(u) \cdot g(q(x, u)) \) in \( x \), for all \( x \) in \( R_C \), and \( u \) any fixed point on \( C \); and from the continuity of this product in \( x \) and \( u \), for \( x \) in \( R_C \) and \( u \) on \( C \). Further, if \( C_1 \) is a second contour in the \( u \)-plane containing \( C \) and there are no points of the set \( A \) in the region bounded by \( C \) and \( C_1 \) (nor on \( C_1 \)), then

\[
\int_C f(u) \cdot g(u') \cdot du/u = \int_{C_1} f(u) \cdot g(u') \cdot du/u.
\]

Among the points \( A \) it is not necessary to include the points \( u \) defined by

\[
g_0 + g_1 \cdot u + \cdots + g_n u^n = 0.
\]

Any such point gives \( q(x, u) = u' = \infty \). If infinity is included among the values \( \beta_i \), it has already been given by \( q(x, u) = \beta_i \) for some \( i \). If the \( \beta_i \) are bounded as a set, \( g(u') \) is regular for \( u' = \infty \), that is, it is regular for any \( u \) satisfying (9). By repeating the reasoning of §3, it is found that all of the singularities of \( F(x) \) are included among the values \( x \) for which two of the points of the set \( A \) coincide. Coincidences of a \( u \)-solution of \( q(x, u) = \beta_k \) with \( u = \alpha_i, u = 0, u = \infty \) give

\[
q(x, \alpha_i) = \beta_k, \quad q(x, 0) = 0, \quad q(x, \infty) = 0,
\]

respectively. Coincidences of two \( u \)-solutions, whether they do belong to the same equation of the set of equations \( q(x, u) = \beta_k \), or not, give also possible singularities of \( F(x) \). We are able thus to state the following theorem:

**Theorem IV.** Let \( q(x, u) \) be a function defined by (1) and (2). Let \( |g_{00}| \) be sufficiently great so that, for a positive \( r_0, r_0 < r_1 \), and a positive \( r \), (4) is satisfied. Let the series \( \sum a_n x^n \) with an isolated set of singularities \( \alpha_i, \ |\alpha_i| \geq r_1 \), and \( \sum b_n x^n \) with an isolated set of singularities \( \beta_i, \ |\beta_i| \geq r_2 \), represent the uniform functions \( f(x) \) and \( g(x) \), respectively. The function

\[
F(x) = \sum_n c_n x^n = \sum_{n, m} a_n \cdot b_m \cdot m S_n(x), \quad q^m(x, u) = \sum_i m S_i(x)/u^i,
\]
where both series converge at least for all $|x| < r$, possesses in the whole plane singularities which are all included among the values of $x$ satisfying

$$q(x, \alpha) = \beta_k, \quad q(x, 0) = 0, \quad q(x, \infty) = 0$$

and the values of $x$ for which two $u$-solutions of $q(x, u) = \beta$ coincide.

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