THE LIMIT OF TRANSITIVITY OF A
SUBSTITUTION GROUP*

BY

MARIE J. WEISS†

1. Jordan published his final contribution to the problem of the limit of transitivity of a substitution group which does not contain the alternating group in Liouville's Journal of 1895.‡ Here he gave the interesting inequalities between \( n \), the degree, and \( t \), the multiplicity of transitivity, of a \( t \)-ply transitive group, stated in the following theorem:

Let \( n \) be the degree of a \( t \)-ply transitive group \( G \) of class \( >3 \). Then if \( t \geq 8 \), 
\[
 n - t \geq 2^a, \quad a \text{ an integer } \geq k - 3 - \log k/\log 2, \text{ and } k \text{ an integer such that } 5 \leq k \leq t; \text{ or}
\]
\[
 n - t \geq t^\delta \{ \delta!(t-\delta)! \}, \quad \delta \text{ being the greatest integer less than the quantity } \frac{t - (t-k+1)}{\log 2/(k+\log 2)}.
\]

This paper followed Bochert's study of the problem in the Mathematische Annalen of 1887 and 1889. Among the results of this study was the inequality

\[
 \log n \geq a(t \log t)^{1/2},
\]

where the constant \( a \) may be taken to be \( (1/8)(\log 2)/8 \)\(^{1/2} \), if \( t \geq 8 \).

In recent years Miller has given still another relation between \( n \) and \( t \), namely, \( n \geq (4/25)(t+2)^2.\)|

In this paper inequalities similar to those of Jordan will be established. To obtain these it was necessary to study separately the \( t \)-ply transitive group whose subgroup that fixes \( t \) letters is of order a power of two and the \( t \)-ply transitive group whose subgroup that fixes \( t \) letters has its order divisible by an odd prime. The methods used in the study of the former group, though suggested by Jordan's 1895 paper, are new, while those used in the investigation of the latter group follow to a certain extent those of Jordan. The following theorems give the chief results of this study:

Let the subgroup that fixes \( t \) letters of a \( t \)-ply transitive group of degree \( n \) and class \( >3 \) be of order \( 2^m \). Then if \( t \geq 8 \),
\[
 (n - t)/2 \geq t^\beta \{ \beta!(t - \beta)! \},
\]

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† National Research Fellow.
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where \( \beta \) is an integer chosen such that \( t/2 < \beta \leq t - 3 \), or chosen to be one unit less than the least value of \( \beta \) which satisfies the inequality

\[
[(\beta + 1)/2]! \geq 2t/\{\beta!(t - \beta)!\};
\]

or

\[
n - t \geq [(t + 1)/2]!;
\]

or

\[
n - t \geq t!/[(t/2)!^2].
\]

The symbol \([s]\) denotes the integral part of \( s \).

Let the order of the subgroup that fixes \( t \) letters of a \( t \)-ply transitive group of degree \( n \) and class \( > 3 \) be divisible by an odd prime. Then if \( t \geq 8 \),

\[
(n - 2t + 1)/2 \geq 2^\alpha, \alpha \text{ an integer } > 3 \text{ and } \alpha \geq t - 3 - \log t/\log 2;
\]

or

\[
(n - 2t + 1)/2 \geq 2^{\gamma t}, \gamma \text{ an integer } > 3 \text{ and } \gamma \geq k - 3 - \log k/\log 2,
\]

where \( k \) is an integer such that \( 5 \leq k \leq t - 1 \); or

\[
(n - 2t + 1)/2 \geq t!/\{\delta!(t - \delta)\},
\]

where \( \delta \) is the greatest integer less than the quantity \( t - (t - k + 1) \log 2/(k + \log 2) \).

It may be of interest to see how Jordan's, Miller's, and the author's results compare for a few values of \( t \). In the following table the minimum value of \( n \) has been calculated for a given \( t \) from the above formulas:

<table>
<thead>
<tr>
<th>( t )</th>
<th>8</th>
<th>16</th>
<th>25</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jordan</td>
<td>16</td>
<td>32</td>
<td>50</td>
<td>1,275</td>
<td>161,800</td>
</tr>
<tr>
<td>Miller</td>
<td>16</td>
<td>52</td>
<td>117</td>
<td>433</td>
<td>1,665</td>
</tr>
<tr>
<td>Author</td>
<td>31</td>
<td>271</td>
<td>849</td>
<td>6,499</td>
<td>409,799</td>
</tr>
</tbody>
</table>

Bochert's inequality is of no interest for these small values of \( t \).

2. The first step in the study of the \( t \)-ply transitive group \( G \) is the proof of the following lemma:

**Lemma.** If \( t > 5 \), the subgroup that fixes \( t \) letters of a \( t \)-ply transitive group \( G \) fixes exactly \( t \), \( t + 1 \), or \( t + 2 \) letters.

Let \( G_x \) be the subgroup that fixes \( x \) letters of a \( t \)-ply transitive group \( G \). Suppose that \( G_t \) fixes the \( t + r - 1 \) letters \( b_1, b_2, \ldots, b_t, a_1, a_2, \ldots, a_{r-1} \), \( r > 1 \). Now consider the largest subgroup of \( G_{t-1} \) in which \( G_t \) is invariant. Its order is \( rg_t \), where \( g_t \) is the order of \( G_t \), and it has a regular constituent of degree \( r \) on the letters \( b_1, a_1, a_2, \ldots, a_{r-1} \). Since all the subgroups similar to \( G_t \) found in \( G_{t-1} \) are conjugate, all the subgroups similar to \( G_t \) in any of the preceding groups \( G_{t-i} \), \( i = 2, 3, \ldots, t \), \( G_0 = G \), form one complete set of conjugates. Then the largest subgroup of \( G_{t-2} \) in which \( G_t \) is invariant has a doubly transitive constituent of degree \( r + 1 \) on the letters \( b_2, b_1, a_1, a_2, \ldots, a_{r-1} \).
Finally, the largest subgroup of $G$ in which $G_t$ is invariant has a $t$-ply transitive constituent of degree $r+t-1$ and order $r(r+1)\cdots(r+t-1)$ on the letters that $G_t$ fixes. Now Jordan has shown that such a non-alternating $t$-ply transitive group does not exist unless $t \leq 3$, or $t=4$ and $r=8$, or $t=5$ and $r=8$. Hence if $t>5$, the $t$-ply transitive constituent is alternating or symmetric and $r=2$, or 3.

In the remaining part of the paper it will be assumed that $t>5$. The study of the problem will be divided into two parts. In the first part the special case $G_t$ of order $2^m$ will be studied, while in the second part the more general case $G_t$ of order $p^aq$, $p$ an odd prime, will be investigated. It will be found that the former case leads to relations between $n$ and $t$ which give more favorable results than those derived from the more general case. Thus when $t$ is sufficiently large, the former relations may be neglected.

$G_t$ of order $2^m$

3. Suppose that $G_t$ is of order $2^m$ and that it contains a transitive subgroup. Let $M$ of degree $2^r$ be its largest transitive subgroup. Now $G_t$ cannot have two transitive subgroups on different sets of letters, for $2^r \geq u$, the class of $G$, and $u \geq (n-1)/2$.‡ Then let $G_{t-1}$ be of degree $2^r+q$. From the theory of primitive groups with transitive subgroups of lower degree,$§$ it is known that $q$ divides $2^r$, and that $G_{t-1}$ is imprimitive with systems of imprimitivity of $q$ letters. Hence $G_t$ fixes exactly $t+1$ letters, for $q$ would be odd in case $G_t$ fixed $t$ or $t+2$ letters, the only other possibilities by the above lemma. Now let the letters displaced by $G_{t-1}$ but fixed by $M$ be $a_1', a_2', \ldots, a_q'$; the letters displaced by $G_{t-1}$ but fixed by $G_t$ be $a_1'$ and $a_2'$; the letters introduced by $G_{t-2}, G_{t-3}, \ldots, G$, respectively, be $b, b_1, \ldots, b_{t-2}$. Then the letters $a_1', a_2', \ldots, a_q'$ form one system of imprimitivity of $G_{t-1}$, and let $a_1, a_2, \ldots, a_t$ be the letters of another system of imprimitivity of $G_{t-1}$.

Now consider the largest group $I$ of $G$ in which $G_t$ is invariant. It transforms $M$ into itself, for $M$ is the largest transitive subgroup of $G_t$. If $I$ did not transform $M$ into itself, it would transform $M$ into a conjugate $M'$ which, as has been seen in the above paragraph, is not entirely free from

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‡ W. A. Manning, these Transactions, vol. 31 (1929), p. 648.
W. A. Manning, these Transactions, vol. 7 (1906), pp. 499–508.
W. A. Manning, Primitive Groups, 1921, Chap. VI. The reader is referred to this last reference for a brief yet complete account of the theory.
the letters of $M$. The transitive groups $M$ and $M'$ then would generate a transitive subgroup of degree or order greater than $M$.

Further $I$ has a symmetric constituent on the letters $a'_i$, $a'_2$, $b$, $b_1$, \ldots, $b_{t-2}$. Hence it contains the substitution

$$T = (b_1b_2)(a'_1)(a'_2)(b_3) \cdots (b_{t-2}) \cdots .$$

It may be assumed that $T$ fixes $a_1$, for if $T$ displaces the letter $a_1$, the product of $T$ and a properly chosen substitution from $M$ gives a substitution fixing $a_1$. Since the set of letters $a'_1$, $a'_2$, \ldots, $a'_t$ and the set of letters $a_1$, $a_2$, \ldots, $a_q$ are the letters of two systems of imprimitivity of $G_{t-1}$, $T$ permutes the letters of each of these sets among themselves. Proper powers of $T$ will reduce it to a substitution of order three. It will be shown that this substitution occurs in a conjugate of $G_t$. Now it cannot displace all the letters $a_2$, $a_3$, \ldots, $a_q$ and all the letters $a'_1$, $a'_2$, \ldots, $a'_t$, for then $q-2 \equiv 0$, mod 3, and $q-1 \equiv 0$, mod 3. Thus this substitution fixes another letter $a_{s}$. Hence it occurs in the $G_t$ which fixes $b_2$, $b_3$, \ldots, $b_{t-2}$, $a'_1$, $a'_2$, $a_1$, and $a_2$. The following theorem has then been proved:

**Theorem 1.** If $t > 5$, and if the subgroup which fixes $t$ letters of a $t$-ply transitive group $G$ is of order $2^m$, $G$ contains no transitive subgroup of degree $< n - t + 1$.

4. Let $I'$ be that subgroup of $G_t$ in which $G_{t-1}$ is invariant, which has an alternating constituent on the $t$ letters $b_1$, $b_2$, \ldots, $b_t$ that $G_{t-1}$ fixes. Let $a'$ be the letters that $G_t$ displaces. Further assume that $I'$ permutes among themselves the letters of each transitive constituent of $G_t$. Then each substitution $(b_1b_2b_3)(b_4)(b_5) \cdots (b_t)(a_1) \cdots$ of order three of $I'$ fixes at least one letter of each transitive constituent of $G_t$. For each such transitive constituent is of degree a power of two. Hence $G_t$ has exactly two transitive constituents, for if it had three or more, these substitutions would occur in the $G_t$ which fixed the three or more $a'$'s of the transitive constituents fixed and the letters $b_4$, $b_5$, \ldots, $b_t$. The case when $G_t$ has only one transitive constituent falls under the previous discussion ($\S 3$). Moreover $G_t$ fixes exactly $t$ ($> 5$) letters, for if it fixed more, the substitution $(b_1b_2b_3)(b_4)(b_5) \cdots (b_t)(a_u)(a_v) \cdots$ of order three would occur in the $G_t$ which fixed the letters $b_4$, $b_5$, \ldots, $b_t$, $a_u$, $a_v$, $x$, where $a_u$ and $a_v$ are the letters of $G_t$ that such a substitution fixes and $x$ is the additional letter fixed by $G_t$.

If $G_{t-1}$ is imprimitive it has systems of imprimitivity of degree $2^r+1$, 

* If $t = 5$, $G_t$ (see lemma, $\S 2$) may fix 5, 6, 7, or 12 letters. However in the last three cases the substitution $(b_1b_2b_3)(b_4) \cdots (b_t)(a_u)(a_v) \cdots$ of order three again fixes more than $t$ letters. Thus $G_t$ also fixes exactly $t$ letters if $t = 5$. 

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where $2\tau$ is the degree of one of the transitive constituents of $G_t$. Let the other transitive constituent of $G_t$ be of degree $2^{\alpha}$. Then $2\tau+1$ divides $2\tau+1+2^{\alpha}$, the degree of $G_{t-1}$. This division is obviously impossible, and hence $G_{t-1}$ is primitive.

Now consider the subgroup $F$ of $G_t$ which fixes one letter of one of the transitive constituents of $G_t$. If $F$ reduces to the identity, $G_t$ has a regular constituent simply or multiply isomorphic to the second transitive constituent. If there is a multiple isomorphism between the constituents, let $F$ be the subgroup which fixes one letter from the constituent of smaller order. If the transitive constituents are in simple isomorphism, each constituent is regular or one is not. The former case will be considered later. In the latter case take $F$ to be the subgroup which fixes one letter of the non-regular constituent. Thus, unless $G_t$ is a simple isomorphism between two regular constituents, a subgroup $F$ of $G_t$ may be found which does not reduce to the identity.

Now let $X$ and $Y$ be the two transitive constituents of $G_t$. Further say that the letter $a_1$ of $G_t$ that $F$ fixes is found in the constituent $X$. Then $F$ fixes $2^{\alpha}+1$ letters of $X$. Assume that $F$ fixes none of the letters of $Y$. In $G_t$ the largest group $N$ in which $F$ is invariant has a regular constituent on the $2^{\alpha}$ letters fixed by $F$. Since $G_{t-1}$ is primitive and since $G_t$ has exactly two transitive constituents, $G_{t-1}$ contains a substitution* $S = (b_1 a_1) \cdots$ which transforms $F$ into itself and hence permutes among themselves the remaining $2^{\alpha}-1$ letters of $X$ that $F$ fixes. Consequently the group $\{N, S\}$ has a doubly transitive constituent of order $2^{\alpha}(2^{\alpha}+1)$ on the $2^\alpha$ letters of $X$ and the letter $b_1$, for the subgroup that fixes $b_1$ of $\{N, S\}$ is in $G_t$, and hence is $N$.

The $G_{t-1}$ which fixes $b_1$ but displaces $b_t$ likewise contains a substitution $S' = (b_2 a_1) \cdots$ which transforms $F$ into itself. The group $\{N, S, S'\}$ has a 3-ply transitive constituent of order $2^{\alpha}(2^{\alpha}+1)(2^{\alpha}+2)$. Thus in the $t$-ply transitive group $G$ a group $N' = \{N, S, S', S'', \cdots\}$ can be found which has a $(t+1)$-ply transitive constituent of degree $2^\alpha+t$ and of order $2^\alpha(2^\alpha+1)(2^\alpha+2) \cdots (2^\alpha+t)$. However, Jordan has shown that such a multiply transitive constituent is alternating or symmetric if $t \geq 4$. Thus $2^\alpha = 2$, and the multiply transitive constituent of $N'$ is symmetric.

Now consider a substitution $(b_1 b_2 b_3)(b_4)(b_5) \cdots (b_1)(a_1)(a_2) \cdots$ of order three of $N'$, $a_1$ and $a_2$ being the two letters fixed by $F$. Since this substitution also transforms $G_t$ into itself, it must also fix at least one letter, $a_{2x}$, of the constituent $Y$ of $G_t$. Thus it occurs in the $G_t$ which fixes $a_1, a_2, a_{2x}, b_4, b_5, \cdots$, $b_t$.

* W. A. Manning, these Transactions, vol. 29 (1927), p. 815, §§1 and 5, Corollary II.
Hence $F$ fixes some letters from each transitive constituent of $G_t$. First, it will be shown that the transitive constituents of $G_t$ are simply isomorphic. Suppose them multiply isomorphic. Choose $F$ as the subgroup which fixes one letter of the constituent of smaller order. Then, contrary to the above statement, $F$ does not fix letters from each transitive constituent of $G_t$.

Further it will be shown that the two constituents are of the same degree. Let $F$ be the subgroup which fixes one letter of the constituent of smaller degree. The order of this constituent will be $w g_1$, where $w$ is the degree of the constituent and $g_1$ is the order of $F$. Since $F$ also fixes letters of the second constituent of $G_t$, the order $g_2$ of the subgroup that fixes one letter of this constituent is at least $g_1$. Let $m$ be the degree of the second constituent. Then $w g_1 = m g_2$, where $g_2 \geq g_1$ and $m \geq w$. Hence $g_1 = g_2$ and $w = m$.

Thus $F$ fixes $2^q$ letters from each transitive constituent of $G_t$ and $N$ has two simply isomorphic regular constituents on the letters which $F$ fixes. If none of the substitutions $S, S', \cdots$ connect the letters of the two regular constituents of $N$, the group $N'$ may be formed and the reasoning of the above paragraphs may be applied to its multiply transitive constituent. As before $2^q = 2$ and in this case the substitution $(b_1 b_2 b_3)(a_1 \cdots) \cdots$ of order three of $N'$ fixes $b_1, b_2, \cdots, b_4$ and the 4 letters which $F$ fixes. Hence it occurs in a conjugate of $G_t$. Then at least one $S$ connects the sets of letters in question. Assume this substitution to be $S$. Then the group $\{N, S\}$ has a primitive constituent of degree $2^q + 1$ and of class $2^q + 1$ on the letters that $F$ fixes and the letter $b_1$. Hence this constituent is of degree $p^q$, $p$ a prime, and contains a characteristic elementary subgroup.*

As before, form the group $N'$. Since the substitution $(b_1 b_2 b_3)(b_4)(b_6) \cdots (b_i)(a_1) \cdots$ of order three of $N'$ transforms both $G_t$ and $F$ into itself, it permutes among themselves the letters of each set of $2^q$ a's that $F$ fixes. Hence it fixes exactly one a from each of these sets, for if it fixed more than one letter from each set, this substitution would occur in a conjugate of $G_t$. Thus $2^q = 1, \mod 3$, and $p = 3$. The group $N'$ cannot have so much as a 5-ply transitive constituent in the letters in question, for let $T = (a_1 a_2)(a_3 a_4) \cdots$ be a substitution of order two of $N$ on these letters. Then in $N'$ there exist the substitutions

$$V = (b_1 a_1 a_2 a_3 a_4 b_4 \cdots) \quad \text{and} \quad V^{-1} T V = (b_1) (b_2 b_3) (b_4 b_5) (a_2 a_4) \cdots.$$ 

Note that the substitution $V^{-1} T V$ transforms the elementary group of degree $3^q$ into itself. Now the partial substitution $U = (a_2 a_4) \cdots$ fixes 5 letters of

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* G. Frobenius, Berliner Sitzungsberichte, 1902, pp. 455-459.
the constituent of $N$ in question and transforms the elementary subgroup of \( \{S, N\} \) of degree $3^n$ into itself. This is obviously impossible. Hence $t \leq 4$, contrary to the present hypothesis.

Then the subgroup $F$ reduces to the identity and $G_t$ is of class $n-t$. Since $G_{t-1}$ is primitive, it is of degree $p^a$, $p$ a prime, and contains a characteristic elementary subgroup. Further since $G_t$ is of class $n-t$, each of its transitive constituents is of degree $2^r$. Now $2^r = 1 \mod 3$, for the substitution $(b_1b_2b_3)(b_4) \cdots (b_t) \cdots$ of order three of $I'$ fixes only one $a$ from each of the transitive constituents of $G_t$. Then $2^r + 2^r + 1 = 0 \mod 3$, and $p = 3$. Apply the reasoning of the previous paragraph to the groups $G$, $G_{t-1}$, and $G_t$, and it is seen that $t \leq 4$.

Then since $t > 5$, $I'$ permutes some, and hence $t$ or more, transitive constituents of $G_t$. Thus the following theorem has been proved:

**Theorem 2.** If $G_t$, the subgroup that fixes $t$ letters of a $t$-ply transitive group $G$, is of order $2^m$, the largest subgroup in which $G_t$ is invariant permutes $t(t>5)$ or more transitive constituents of $G_t$.

5. At this point it is necessary to assume $t \geq 8$, and this assumption will be held throughout the remaining part of this paper. It has been shown that the substitutions of $I'$ permute some transitive constituents of $G_t$. Hence $I'$ contains an imprimitive constituent, whose systems of imprimitivity are transitive constituents of $G_t$ and which is multiply isomorphic to the alternating group of degree $t$. Consider the group $H$ whose letters are the systems of imprimitivity of the imprimitive constituent just described. It is simply isomorphic to the alternating constituent $A$ of degree $t$ in $I'$. Let $A_1$ be the subgroup of $A$ which is simply isomorphic to $H_1$, the subgroup which fixes one letter of $H$. Now $A_1$ has either one transitive constituent of degree $> \frac{1}{3}t$, or all of its transitive constituents are of degree $\leq \frac{1}{3}t$. Let the former be true, and call the transitive constituent of degree $\beta > \frac{1}{3}t$, $B$. Let $B$ contain the alternating group of degree $\beta$ of $A_1$ which fixes $t-\beta$ letters of $A$.

Under this hypothesis, it will be shown that $A_1$ cannot be transitive. Note that if it is transitive it is alternating. Let $A_1$ be transitive and of degree $t-1$. Then $H$ is of degree $t$ and is the alternating group. Thus a substitution of degree and order three of $A_1$ is associated with a substitution of degree and order three of $H_1$. Since $t \geq 8$, this substitution fixes more than three letters of $H$ and in $I'$ this substitution which has a cycle of three letters in $A_1$ permutes among themselves the letters of more than three transitive constituents of $G_t$. Now since the transitive constituents of $G_t$ are of degree

* Note that Theorem 2 holds also for $t = 5$.

† M. J. Weiss, these Transactions, vol. 30 (1928). See footnote on p. 337.
a power of two, this substitution fixes more than three letters of \( G_t \). Hence it fixes \( t \) or more letters of \( G \), and consequently \( G_t \) contains a substitution of order three, contrary to the present hypothesis.

If \( A_1 \) is of degree \( t-2 \) and transitive, \( H \) is of degree \( p-t \) and its substitutions record the permutations of the \( p-t \) sequences of two letters by the alternating group of degree \( t \). Now a substitution of degree and order three of the alternating group of degree \( t \) fixes exactly \( (t-3)(t-4) \) sequences. Hence a substitution of \( H_1 \) which is associated with a substitution of degree and order three of \( A_1 \) fixes exactly \( (t-3)(t-4) \) transitive constituents of \( G_t \). Then a substitution of \( I' \) which has a cycle of three letters in \( A_1 \) fixes at least \( (t-3)(t-4) > 2 \) letters of \( G_t \). Hence such a substitution fixes \( t \) or more letters of \( G \).

Now \( A_1 \) cannot be transitive of degree \( t-3 \) or less, for then it is invariant in a group of at least three times its order and hence \( H_1 \) fixes three or more transitive constituents of \( G_t \). Then in this case \( G_t \) also contains a substitution of order three. Thus \( A_1 \) is necessarily intransitive, containing an invariant alternating group of degree \( \beta \), which fixes \( t-\beta \) letters of \( A \), and a constituent, transitive or intransitive, of degree \( \leq t-\beta \).

Moreover the constituent \( B \) is of degree \( < t-2 \). If \( B \) is of degree \( t-2 \), \( A_1 \) is of order \( (t-2)! \). Then \( H \) is of degree \( (p-t)/2 \) and records the permutations of the \( (p-t)/2 \) transpositions of the alternating group of degree \( t \). A substitution of degree and order three of \( A \) fixes exactly \( (t-3)(t-4)/2 \) transpositions. Hence the substitution of \( H \) associated with it fixes \( (t-3)(t-4)/2 > 2 \) transitive constituents of \( G_t \). Thus unless \( B \) is of degree \( < t-2 \), \( G_t \) contains a substitution of order three.

It is necessary to determine the order of \( A_1 \). First it will be shown that \( A_1 \) contains all the substitutions of degree and order three of \( A \) which fix the letters of the constituent \( B \). For this purpose it is necessary to have in mind Dyck's theorem* on transitive groups simply isomorphic to a given group. Now choose substitutions \( S_j \) not in \( A_1 \) such that \( A \) can be written in the array \( A_1 S_j, j=1, 2, \ldots, t!/(2m) \), where \( m \) is the order of \( A_1 \). Denote the substitutions of \( A_1 \) by \( \sigma_i, i=1, 2, \ldots, m \). Multiply this array on the right by the substitutions \( \sigma_i \). Then by Dyck's theorem, to each substitution \( \sigma_k \) of \( A_1 \), there corresponds the substitution

\[
\left( \begin{array}{c}
A_1 S_j \\
A_1 S_j \sigma_k
\end{array} \right) \quad (j=1, 2, \ldots, t!/(2m)),
\]

and the substitutions of \( H \) may be regarded as written on the sets of letters

* W. Dyck, Mathematische Annalen, vol. 22 (1883), pp. 70–108.
Thus the substitution of \( H_1 \) associated with the substitution \( \sigma_k \) of \( A_1 \) is the above substitution.

Now suppose that \( S \) is a substitution of degree and order three of \( A \) which fixes the letters of the constituent \( B \) and which is not found in \( A_1 \). The sets \( A_1S \) and \( A_1S^3 \) found in the above array are distinct, for if these sets had a substitution in common, that is, if
\[
\sigma_mS = \sigma_mS^2,
\]
then
\[
S = \sigma^{-1}_m\sigma_m,
\]
but \( S \) is not a substitution of \( A_1 \). Further let \( \sigma \) be a substitution of degree and order three of \( A_1 \) on the letters of the constituent \( B \) only. Then \( \sigma \) and \( S \) are commutative since they are written on different sets of letters. Hence to the substitution \( \sigma \) corresponds the substitution
\[
\begin{pmatrix}
(A_1S)_{j}
\end{pmatrix}
\quad (j = 1, 2, \ldots, t!/(2m)),
\]
which fixes three of the sets \( A_1S_{ij} \), namely, \( A_1S_1, A_1S, \) and \( A_1S^3 \), where \( S_1 = 1 \). Thus since the letters of \( H \) are also the transitive constituents of \( G_t \), the substitution of order three of \( I' \) which coincides with \( \sigma \) in its cycle on the letters of \( A_1 \) fixes at least three transitive constituents of \( G_t \), and hence is found in a \( G_t \).

Thus \( A_1 \) also contains the alternating group on the \( t-\beta \) letters not found in the constituent \( B \). Hence the minimum order of \( A_1 \) is \( \beta!(t-\beta)/4 \).

Now consider that subgroup \( D \) of \( A_1 \) which is the direct product of the alternating group of degree \( \beta \) and a substitution \( T \) of degree and order three on the letters of the constituent of degree \( t-\beta \). Since \( A \) and \( H \) are simply isomorphic groups, \( H \) contains a subgroup \( J \) which is simply isomorphic to the group \( D \). Further the subgroup of \( H_1 \) which is simply isomorphic to the alternating group of degree \( \beta \) displaces letters from every transitive constituent of \( H_1 \), for if it fixes the letters of one of the transitive constituents, a substitution of degree and order three of the alternating group of degree \( \beta \) fixes at least three transitive constituents of \( G_t \). The three transitive constituents fixed would be the one that \( H_1 \) fixes and the ones which are the letters of the transitive constituent of \( H_1 \) in question. Hence this substitution would fix at least three letters of \( G_t \) and \( G_t \) would contain a substitution of order three. Thus since the alternating group of degree \( \beta(\beta \geq 5) \) is simple, every transitive constituent of \( H_1 \) has a subgroup simply isomorphic to it and associated with it. Again for the above reason the substitution \( T \) displaces letters of every transitive constituent of \( H_1 \). Hence every transitive constituent of \( H_1 \) has a subgroup simply isomorphic to \( D \) and associated with \( D \).
Let $J'$ be one of the transitive constituents of $J$ simply isomorphic to $D$. Consider the subgroup $D_1$ of $D$ which is simply isomorphic to the subgroup $J'_1$ which fixes one letter of $J'$. It will be shown that $D_1$ cannot contain a substitution of degree and order three. First it cannot contain $T$, for $T$ is invariant in $D$. Thus $D_1$ is invariant in a group of at least three times its order and consequently $J'_1$ fixes at least three letters of $J'$. Then the substitution of $J'_1$ associated with a substitution of degree and order three of $D_1$ fixes at least three transitive constituents of $G_t$, and $G_t$ contains a substitution of order three.

Since $D_1$ contains no substitution of degree and order three a theorem by Bochert* may be applied to determine its maximum order. Its order thus $\leq 3\beta!/[(\beta + 1)/2]!$, where $[q]$ denotes the integral part of $q$. Then a transitive constituent of $J$ is of degree $\frac{1}{4}(\beta + 1)/2$ at least.

The value of $\beta$ must now be determined so that the degree of $H$ can be evaluated. Under the present hypothesis, the order of $A_1 \leq \beta!(t-\beta)!/2$, and hence the minimum degree of $H$ is $t!/\{\beta!(t-\beta)\}$. The maximum degree of $H$ is found by determining the minimum order of $A_1$. The latter has been seen to be $\beta!(t-\beta)!/4$ and thus the maximum degree of $H$ is $2t!/\{\beta!(t-\beta)\}$. When $t>8$, $H$ is simply transitive,† and $J$ must contain at least two transitive constituents. If $J$ and consequently $H_1$ contains only one transitive constituent, $H_1$ fixes exactly two letters of $H$, for it has been seen (§5, paragraph 3) that $H_1$ cannot fix so many as three letters. Then $H$ is imprimitive with systems of two letters. Its group in the systems is then either a doubly transitive group simply isomorphic to the alternating group of degree $t$ or the alternating group itself. The former case contradicts Maillet’s theorems. In the latter case a substitution of degree and order three of $A_1$ has a substitution of degree and order three associated with it in the group in the systems of $H$. Consequently the substitution of $H_1$ associated with a substitution of degree and order three of $A_1$ fixes all except 6 letters of $H$. Then a substitution of $G_t$ which has a cycle of three letters in $A_1$ fixes more than three transitive constituents of $G_t$. This has been shown previously to be impossible.

Now the least degree of $J$ must always be less than the maximum degree of $H$. Hence

\[(1) \quad [(\beta + 1)/2]! < 2t!/\{\beta!(t-\beta)\}, \quad t > 8.\]

Note that when $t=8$, $\beta$ is determined, for $t/2 < \beta \leq t-3$. When $t>8$ and $\beta < t-3$, inequality (1) will be used to determine $\beta$.

Since the degree of each transitive constituent of $G_t$ is at least two, $G_t$

has at most \((n-t)/2\) transitive constituents. Now the maximum number of transitive constituents of \(G_i\) must exceed or equal the minimum degree of \(H\), namely,

\[(n-t)/2 \geq t!/\{\beta!(t-\beta)!\}.\]  

It remains to determine a lower bound for \((n-t)/2\). The right hand member of inequality (2) decreases as \(\beta\) increases, \(\beta > t/2\). Now it has been seen that \(\beta\) must be chosen so that inequality (1) holds. Hence if \(\beta(<t-3)\) is chosen to be one unit less than the least value of \(\beta\) which satisfies the inequality

\[(\beta + 1)/2)! \geq 2t!/\{\beta!(t-\beta)!\}, \quad t > 8,\]  

inequality (2) holds for the value of \(\beta\) thus determined.

The above discussion may be summarized as follows:

If \(A_1\) has an alternating constituent of degree \(>t/2\), inequality (2) gives the relation between \(n\) and \(t\), with \(t/2 < \beta \leq t-3\) or \(\beta(<t-3)\) determined by inequality (3).

6. Now it may be assumed that the transitive constituent of degree \(\beta(>t/2)\) of \(A_1\) is not alternating. Let it be imprimitive. The largest imprimitive group of degree \(\beta\) is of order \(2(\beta/2)!^2\) or \(2(\beta-1)/2)!^2\), according as \(\beta\) is even or odd. Hence the order of \(A_1 \leq [\beta/2]!(t-\beta)!\) and \(H\) is of degree \(t!/\{2[\beta/2]!(t-\beta)!\}\) at least. Thus in this case the following inequality holds:

\[(n-t)/2 \geq t/\{2[\beta/2]!(t-\beta)!\}, \quad t/2 < \beta \leq t.\]  

Let the constituent of degree \(\beta\) be primitive. Since a non-alternating primitive group contains no substitution of degree and order three, the theorem of Bochert quoted above may be used to determine the order of the constituent of degree \(\beta\). The order of \(A_1 \leq \beta!(t-\beta)!/[\beta+1)/2)!\) and \(H\) is of degree \(t!/\{\beta+1)/2)!/[2\beta!(t-\beta)!\}\) at least. Thus if the constituent of degree \(\beta\) is primitive, the following inequality holds:

\[(n-t)/2 \geq t!/\{\beta+1)/2)!/[2\beta!(t-\beta)!\}, \quad t/2 < \beta \leq t.\]  

The minimum of each of the right hand members of inequalities (4) and (5) will now be determined. Denote them by \(m_1(\beta)\) and \(m_2(\beta)\), respectively. Recall Stirling's formula for log \(t!\), namely,

\[
\log t! = (t + \frac{1}{2}) \log t - t \\
+ \frac{1}{2} \log 2\pi + \theta/(12t), \quad 0 < \theta < 1.
\]

Then
\[ \log m_1(\beta) = \left( t + \frac{1}{2} \right) \log t + \frac{\theta}{12t} - \log 2 - \left( 2 \left[ \frac{\beta}{2} \right] + 1 \right) \log \left[ \frac{\beta}{2} \right] \\
+ 2 \left[ \frac{\beta}{2} \right] - \log 2\pi - \frac{\theta_1}{6 \left[ \frac{\beta}{2} \right]} - \left( t - \beta + \frac{1}{2} \right) \log (t - \beta) \\
- \beta - \frac{\theta_2}{12(t - \beta)} , \quad \frac{t}{2} < \beta \leq t - 1 , \quad 0 < \theta_i < 1. \]

The cases \( \beta \) even and \( \beta \) odd must be treated separately in order to apply Stirling's formula. However in both cases, the second derivative of \( \log m_1(\beta) \) is negative. Hence the minimum of \( \log m_1(\beta) \) occurs when \( \beta \) assumes its least even or odd value or when \( \beta \) assumes its greatest even or odd value, \( \lfloor t/2 \rfloor + 1 \leq \beta \leq t - 1 \). Consequently the minimum of \( m_1(\beta) \) occurs for some one of these values of \( \beta \). It is found that if \( t \) is odd, the minimum occurs when \( \beta = t - 1 \). If \( t \) is even, the minimum occurs when \( \beta = t \). Thus the minimum of \( m_1(\beta) \) is \( t!/(2 \lfloor t/2 \rfloor)! \) and the inequality (4) becomes

(6) \( (n - t)/2 \geq t!/(2 \lfloor t/2 \rfloor)! \).

Apply Stirling’s formula to \( \log m_2(\beta) \). Then

\[ \log m_2(\beta) = \left( t + \frac{1}{2} \right) \log t - \left( \beta + \frac{1}{2} \right) \log \beta - \left( \left[ \frac{\beta + 1}{2} \right] + \frac{1}{2} \right) \log \left[ \frac{\beta + 1}{2} \right] \\
- \log 2 - \left( t - \beta + \frac{1}{2} \right) \log (t - \beta) - \left[ \frac{\beta + 1}{2} \right] + \frac{\theta}{12t} - \frac{\theta_1}{12\beta} \\
+ \frac{\theta_2}{12 \left[ \frac{\beta + 1}{2} \right]} - \frac{\theta_3}{12(t - \beta)} , \quad \frac{t}{2} < \beta \leq t - 1 , \quad 0 < \theta_i < 1. \]

The cases \( \beta \) even and \( \beta \) odd must again be considered separately. However the second derivative of \( \log m_2(\beta) \) is again negative in both cases. Hence the minimum of \( \log m_2(\beta) \) occurs for the extreme values of \( \beta \), and consequently the minimum of \( m_2(\beta) \) occurs for some one of these extreme values of \( \beta \). Consider the values of \( \beta \) at the lower end of the interval. When \( \beta \) is even the least value of \( \beta \) is \( (t+1)/2, (t+3)/2, (t+2)/2, (t+4)/2 \), according as \( t=2k+1, 2k-1, 2k, \) or \( 4m, k \) odd. When \( \beta \) is odd the least value of \( \beta \) is \( (t+3)/2, (t+1)/2, (t+4)/2, (t+2)/2 \), according as \( t \) is of one of the above forms. For all these values of \( \beta \) except for \( \beta = (t+3)/2 \) and \( t=9 \), \( m_2(\beta) \geq m_1(\beta) \). However if \( t=9 \), the minimum of \( m_2(\beta) \) occurs for \( \beta = 9 \). Hence inequality (5) need not be considered for values of \( \beta \) at the lower end of the \( \beta \) interval.
Then only the values of $m_\alpha(\beta)$ for $\beta = t - 2$, $t - 1$, and $t$ need be considered. Now $m_\alpha(t)$ is less than $m_\alpha(t - 2)$ or $m_\alpha(t - 1)$. Thus inequality (5) may be replaced by the inequality

\[(n - t)/2 \geq \frac{1}{2}[(t + 1)/2]!.
\]

7. If no transitive constituent of $A_1$ is of degree $> t/2$, the order of $A_1 \leq \frac{1}{2}[t/2]^2$ and for this case

\[(n - t)/2 \geq t!/\lfloor t/2 \rfloor!.
\]

However since the right hand member of inequality (8) is greater than the right hand member of inequality (6), the former may be discarded.

8. Thus if $G_t$ is of order $2^m$ and $t \geq 8$, the following theorem has been proved:

**Theorem 3.** Let the subgroup that fixes $t$ letters of a $t$-ply transitive group of degree $n$ and class $> 3$ be of order $2^m$. Then if $t \geq 8$,

\[(n - t)/2 \geq t!/\{\beta!(t - \beta)!\},
\]

where $\beta$ is an integer chosen such that $t/2 < \beta \leq t - 3$, or chosen to be one unit less than the least value of $\beta$ which satisfies the inequality

\[\lfloor(\beta + 1)/2\rfloor! \geq 2t!/\{\beta!(t - \beta)!\};
\]

or

\[n - t \geq \lfloor(t + 1)/2\rfloor!;
\]

or

\[n - t \geq t!/\lfloor t/2 \rfloor!.
\]

The symbol $\lfloor s \rfloor$ denotes the integral part of $s$.

When $t$ is sufficiently great, it will be shown that the last inequality in Theorem 3 is the only one necessary to consider. Comparing the logarithms of the right hand members of the second and third inequalities, it is found by means of Stirling’s formula, that when $t \geq 16$, the former is always greater than the latter. Hence the former may be discarded when $t \geq 16$.

Further analyze the inequality (3) which limits $\beta$, and write it in the form

\[V(\beta) = [(\beta + 1)/2]!\beta!(t - \beta)!/(2t!) \geq 1.
\]

Then if $\beta$ is so chosen that $\log V(\beta) \geq 0$, the inequality (3) holds. Now by Stirling’s formula, it is found that $\log V(\beta)$ is positive for $\beta = t/2$ as soon as $t \geq 160$. Then for $t \geq 160$, $\beta$ may be chosen equal to $t/2$, but $\beta > t/2$ by hypothesis. Hence when $t \geq 160$ the first formula in Theorem 3 may also be discarded.
For later comparison purposes it is of interest to find the principal value of \( \log (n - t) \) when \( t \) becomes infinite. Now

\[
\log (n - t) \geq \log t! - 2 \log \lfloor t/2 \rfloor !.
\]

The right hand member of this inequality, when expanded by Stirling's formula, becomes

\[
(t + \frac{1}{2}) \log t - (t + 1) \log (t/2)
\]

\[
+ \frac{\theta/(12t) - \theta_1/(3t) - \frac{1}{2} \log 2\pi}{2}
\]

\[
= (t + 1) \log 2 - \frac{1}{2} \log t + \theta/(12t) - \theta_1/(3t) - \frac{1}{2} \log 2\pi,
\]

where \( 0 < \theta_i < 1, i = 1, 2 \), the principal value of which is \( t \log 2 \). Thus

\[
\log (n - t) \geq t! \log 2(1 + \epsilon),
\]

where \( \epsilon \) approaches 0 as \( t \) approaches \( \infty \).

\[G_t \text{ OF ORDER } p^m, \ p \ AN \ ODD \ PRIME\]

9. In the remaining part of this paper it will be assumed that the order \( g_t \) of the subgroup \( G_t \) which fixes \( t \) letters of the \( t \)-ply transitive group \( G \) is divisible by an odd prime \( p \). The analysis of this case is an extension of the method given by Jordan in his paper in Liouville's Journal of 1895.

Let \( p \) be the greatest prime which divides the order of \( G_t \). The following slight extension of a theorem by Jordan* will be needed in the development of the theory for the case under discussion:

Consider a Sylow subgroup \( P \) of \( G_t \) of order \( p^r \). Then \( g_t = v p^a (r p + 1) \), where \( v p^a \) is the order of the largest subgroup of \( G_t \) in which \( P \) is invariant. Let \( a_1, a_2, \ldots \) be the letters that \( G_t \) displaces and \( b_1, b_2, \ldots, b_t \) the \( t \) letters that \( G_t \) fixes. Since \( G \) is \( t \)-ply transitive, there exists a substitution \( S = (b_1 b_2)(b_3 b_m)(a \cdots) \cdots \) which transforms \( G_t \) into itself, \( b_i, b_k, b_l, b_m \) being any four of the above \( t \) letters. In the group of order \( 2g_t \), thus obtained, let the order of the largest group in which \( P \) is invariant be \( v_1 p^a \). Then \( v_1 = 2v \) or \( v \) according as the new substitutions introduced do or do not transform \( P \) into itself. Sylow's theorem shows that \( v_1 = 2v \), for \( v_1 p^a (r p + 1) = 2v p^a (r p + 1) \), from which it is found that \( v_1 \equiv 2v \), mod \( p \). Then for each set of possible values of the subscripts \( i, k, l, m \) there exists a substitution \( S_i \) which transforms \( P \) into itself. Now form the group \( W = \{ P, S_1, S_2, \ldots \} \). It contains \( P \) invariantly and also \( P \)'s characteristatic elementary subgroup \( L \). Further \( W \) has an alternating constituent on the \( t \) letters that \( G_t \) fixes. The above discussion may be summarized as follows:

If the order of the subgroup which fixes \( t \) letters of a \( t \)-ply transitive group \( G \) is divisible by an odd prime \( p \), a subgroup \( W \) of \( G \) can always be found which contains an invariant elementary subgroup \( L \) of order \( p^\mu \) on the letters of \( G_t \) and which has an alternating constituent on the \( t \) letters that \( G_t \) fixes.

The substitutions of \( W \) are of the form \( ABPC \), where \( A \) denotes a substitution on the letters of the alternating constituent of degree \( t \), \( B \) a substitution which permutes the transitive constituents of \( L \), \( P \) the product of substitutions \( P_1, P_2, \ldots \), each of which permutes among themselves the letters of a transitive constituent of \( L \), and \( C \) a substitution on the letters of \( G_t \) not contained in \( L \). The substitutions \( P_1 \), say, on the \( p^\mu \) letters of a transitive constituent of \( L \) either do or do not generate a subgroup of the holomorph of the elementary group of degree \( p^\mu \) which has a quotient group simply isomorphic to the alternating group of degree \( t \). If the former be true, Jordan has shown that

\[
\mu \geq t - 3 - \log t/\log 2.
\]

Now the degree of a transitive constituent of \( L \) is \( <n-t\). Hence

\[ (10) \quad n - t > p^\mu, \quad p > 2, \quad \mu \geq t - 3 - \log t/\log 2. \]

10. Suppose that this inequality does not hold. Then none of the sets of substitutions \( P_i, i=1, 2, \ldots \), generate a subgroup of the holomorph of the elementary group of degree \( p^\mu \) which has a quotient group simply isomorphic to the alternating group of degree \( t \). Hence \( W \) contains a subgroup \( \overline{W} \), with an alternating constituent of degree \( t \), in which each of the sets of substitutions \( P_i \), except those forming the transitive constituents of \( L \) that are permuted in \( W \), reduces to the identity. Now note that the elementary group \( L \) is of degree \( \geq (n-1)/2, \) \( n \) being the degree of \( G \), for the degree of \( L \) must be equal to or greater than the class of \( G \). Then substitutions of \( \overline{W} \) must permute some transitive constituents of \( L \), for, otherwise, a substitution which has a cycle of three letters in the alternating group of degree \( t \) is of degree \( \leq 3+n-t-(n-1)/2<(n-1)/2 \), the class of \( G \).

Return to a consideration of the subgroup \( W \). Since the substitutions of \( \overline{W} \) permute some transitive constituents of \( L \), the group \( \overline{W} \), which contains \( \overline{W} \), also permutes some. The substitutions \( B \) together with those of \( L \) generate imprimitive groups whose systems of imprimitivity are the transitive constituents of \( L \), and which are multiply isomorphic to the alternating group of degree \( t \). Denote by \( B_1, B_2, \ldots \), respectively, the groups in the systems of these imprimitive groups, and by \( C_1, C_2, \ldots \) the groups generated by the substitutions \( C \). Now the groups \( B_i \) and \( C_i \) are either all simply

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isomorphic to the alternating group of degree \( t \) or some are not. Assume the latter to be true.

The groups \( B_t \) and \( C_t \), which are multiply isomorphic to the alternating group of degree \( t \) then have invariant subgroups which fix the \( t \) letters of the alternating constituent. Denote these invariant subgroups by \( R \). Now all the substitutions \( S_t = (b_1 b_2)(b_1 b_m)(a \cdot \cdot \cdot) \cdot \cdot \cdot \) of \( W \) transform \( R \) into itself. Then unless the order of \( R \) is a power of two, choose its Sylow subgroup \( P_1 \) of order \( p_1^\alpha \), \( p_1 \) an odd prime. The group \( P_1 \) reduces to \( R \) if \( R \) is of order a power of two. Then as in §9, paragraph 2, a group \( W_1 \) can be found in \( W \) which transforms \( P_1 \) into itself and which has an alternating constituent on the \( t \) letters that \( G_t \) fixes. The substitutions \( S_t, j = 1, 2, \cdot \cdot \cdot \), in this case are written on the letters of the alternating constituent of degree \( t \), on the transitive constituents of \( L \) regarded as single symbols, and on the letters of \( G_t \) itself.

Consider the characteristic elementary subgroup \( L_1 \) of \( W_1 \). The degree of none of the transitive constituents of \( L_1 \) exceeds \( (n-t)/p \) or \( (n-2t+1)/2 \), the former formula giving the maximum degree of an imprimitive group whose systems of imprimitivity are the transitive constituents of \( L \), the latter formula giving the maximum degree of a transitive constituent of \( W \) which displaces none of the letters of \( L \). Let \( Z \) be the greater of these two quantities. The substitutions of \( W_1 \) are of the form \( AB'B'PP'G \), where \( B \) denotes substitutions which permute transitive constituents of \( L \), but which with the substitutions of \( L \) generate imprimitive groups whose groups in the systems are simply isomorphic to the alternating group of degree \( t \); \( B' \) denotes substitutions which permute the transitive constituents of \( L_1 \); \( P \) denotes the product of substitutions \( P_1, P_2, \cdot \cdot \cdot \), each of which permutes among themselves the letters of a transitive constituent of \( L \); \( P' \) denotes the product of substitutions \( P'_1, P'_2, \cdot \cdot \cdot \), each of which permutes among themselves the letters of a transitive constituent of \( L_1 \); and \( C \) denotes substitutions on the letters of \( G_t \) not found in \( L \) or \( L_1 \). Recall that no one of the sets of substitutions \( P_i \) generates a group which has a quotient group simply isomorphic to the alternating group of degree \( t \).

If one of the sets of substitutions \( P'_1, P'_2, \cdot \cdot \cdot \), say, on the \( p_r \gamma \) letters of a transitive constituent of \( L_1 \), generates a subgroup of the holomorph of the elementary group of degree \( p_r \gamma \) which has a quotient group simply isomorphic to the alternating group of degree \( t \), Jordan's inequality gives

\[
Z \geq p_r^\gamma, \quad p_1 \geq 2, \quad \gamma \geq t - 3 - \log t/\log 2.
\]

If this inequality does not hold, it may be assumed that none of the above sets of substitutions \( P'_i \) generates a subgroup of the holomorph of an
elementary group which has a quotient group simply isomorphic to the alternating group of degree $t$. Then $W_1$ contains a subgroup $\overline{W}_1$ which has an alternating constituent of degree $t$ and in which each of the sets of substitutions $P_i'$, except those forming the transitive constituents of $L_1$ that are permuted by $W_1$, reduces to the identity. If $W_1$ fixes all the transitive constituents of $L_1$, the substitutions of $\overline{W}_1$ reduce to $ABC$. This case will be considered later. Then assume that the substitutions of $W_1$ permute some transitive constituents of $L_1$.

Consider the imprimitive groups generated by the substitutions $B'$ and $L_1$. Denote their groups in the systems by $B_1'$, $B_2'$, $\cdots$, respectively, and the groups generated by the substitutions $C$, by $C_1$, $C_2$, $\cdots$, respectively. Again, unless all the groups $B_1'$ and $C_i$ are simply isomorphic to the alternating group of degree $t$, choose their invariant subgroups denoted by $R_i$ which fix the $t$ letters of the alternating group and repeat the above analysis, obtaining a group $W_2$ which contains an invariant elementary subgroup $L_2$. If all the groups $B_1'$ and $C_i$ are simply isomorphic to the alternating group of degree $t$, the analysis has been completed for the present purpose. The substitutions of $W_1$ then have the form $A\overline{B}\overline{B}'CQP'$, where $A$, $B$, $P$, $P'$ are defined as before; $\overline{B}'$ denotes substitutions which with those of $L_1$ generate imprimitive groups whose groups in the systems are simply isomorphic to the alternating group of degree $t$; and $\overline{C}$ denotes substitutions on the letters of $G_i$ not found in $L$ or $L_1$ which generate subgroups simply isomorphic to the alternating group of degree $t$. Note that none of the substitutions $P_i$ or $P_i'$ generate groups which have quotient groups simply isomorphic to the alternating group of degree $t$.

Now return to the case in which $W_1$ fixes all the transitive constituents of $L_1$, and consider the group $\overline{W}_1$, whose substitutions are of the form $ABC$. If the substitutions $C$ generate groups which are simply isomorphic to the alternating group of degree $t$, the analysis has been completed. If they do not, take their invariant subgroups denoted by $R_i$, which fix the $t$ letters of the alternating group, and apply the reasoning of the previous paragraphs, again obtaining a group $\overline{W}_2$ which contains an invariant elementary subgroup $L_2$.

Continue with this analysis until either a group $W_i$ or $\overline{W}_i$ of the desired type is obtained, or an inequality, such as inequalities (10) or (11), derived from this analysis holds. Obviously these two inequalities are the most unfavorable of any which might be obtained in carrying this analysis further. Consequently any further inequalities may be neglected. Then it remains to investigate the case when a group $W_1$ or $\overline{W}_1$ is obtained in which the substitutions are of the form
where the substitutions $\overline{B}, \overline{B}', \overline{B}'', \cdots$ denote substitutions which permute transitive constituents of $L, L_1, L_2, \cdots$, respectively, but which with the substitutions of the latter generate imprimitive groups whose groups in the systems are simply isomorphic to the alternating group of degree $t$; $\overline{C}$ denotes substitutions on the letters of $G_t$ not found in $L, L_1, L_2, \cdots$, which generate groups simply isomorphic to the alternating group of degree $t$; $P, P', P'', \cdots$ denote the product of substitutions $P_1, P_2, \cdots, P_1', P_2', \cdots, P_1'', P_2'', \cdots$, each of which permute the letters of a transitive constituent of $L, L_1, L_2, \cdots$, respectively, among themselves, but which do not generate groups which have quotient groups simply isomorphic to the alternating group of degree $t$.

11. Thus suppose that $W_i$ is the first group in the series $W_i$, $\overline{W_i}$, $i = 1, 2, \cdots$, of the desired type. Let the groups in the systems of the imprimitive groups generated by $\overline{B}^m$ and $L_m$ be denoted by $\overline{B}^m, v = 1, 2, \cdots$, and $m = 0, 1, \cdots, \overline{B}_v = \overline{B}_v$, and the groups generated by the substitutions $\overline{C}$, by $\overline{C}_v$. Recall that the groups $\overline{B}^m$ and $\overline{C}_v$ are all simply isomorphic to the alternating group of degree $t$. Consider a subgroup $K$ of $W_i$ which has an alternating constituent on the $k$ letters $b_1, b_2, \cdots, b_k$, where $5 \leq k \leq t$. First suppose that $K$ fixes more than one half the letters in each of the groups $\overline{B}^m$ and $\overline{C}_v$. Now all these letters or symbols fixed cannot reduce to the identity in $G_t$ for then the degree of a substitution of $W_i$ which has a cycle of three letters in the alternating constituent of degree $k$ is of degree $\leq 3 + (n-t)/2 < (n-1)/2$. Then according to the theorem by Manning quoted in §3, $G$ is alternating. Therefore corresponding to at least one symbol fixed and arising from the letters of $L, L_1, L_2, \cdots$, there exists a group simply or multiply isomorphic to the alternating constituent of degree $k$ which transforms an elementary group into itself. Now the degree of this group cannot exceed the degree of one of the systems of imprimitivity of an imprimitive group generated by the substitutions $\overline{B}^m$ and those of $L_m$. It has been shown that such a group permutes its systems according to a group simply isomorphic to the alternating group of degree $t$. Hence it has at least $t$ systems and the degree of a system is at most $(n-\delta)/t$. If this imprimitive group is a group generated by the substitutions $\overline{B}$ and those of $L, L_1, L_2, \cdots$, Jordan's inequality gives

\[(n-\delta)/t \geq p^\delta, \quad p > 2, \quad 5 \leq k \leq t - 1, \quad \delta \geq k - 3 - \log k/\log 2.\]

If the substitutions $\overline{B}$ reduce to the identity in $K$, let the group under discussion be a group generated by the substitutions $\overline{B}'$ and those of $L_1$. The degree of this imprimitive group is then at most $Z$. Hence the degree of one of its systems is at most $Z/t$. In this case Jordan's inequality gives
If the substitutions $\mathcal{B}'$ reduce to the identity in $K$, let the group in question be a group generated by the substitutions $\mathcal{B}''$ and those of $L_2$. Any further inequalities thus obtained evidently give more favorable bounds for the transitivity of $G$ and hence may be discarded.

12. It may now be assumed that the group $K$ displaces a half or more than half of the letters in at least one of the groups simply isomorphic to the alternating group of degree $t$. Let $Q$ be the particular group in which $K$ displaces a half or more than half of the letters. Let $\Gamma_1$ be the subgroup of the alternating constituent $\Gamma$ of degree $t$ which corresponds to the subgroup $Q_1$ that fixes one letter of $Q$. Now the degree of $Q$ is at most $(n-t)/p$ or $(n-2t+1)/2$, according as $Q$ has arisen from the letters of $L$ or from the letters of $G_t$ not found in $L$. On the other hand, the degree of $Q$ is $t!/2q$, where $q$ is the order of the subgroup $\Gamma_1$. Hence for this case the inequality

$$Z \geq t!/2q$$

must be studied.

Let $\gamma_1, \gamma_2, \ldots$ be the degrees of the transitive constituents of $\Gamma_1$. Then the order of each transitive constituent group of $\Gamma_1$ divides $\gamma_i!$, respectively. Hence the order $q$ divides $\gamma_1!\gamma_2! \ldots$.

Recall the present hypothesis that the group $K$ fixes at most one half the letters in the group $Q$. Let $Q_1$ have $v$ conjugates under $Q$. Then $\Gamma_1$ has $v$ conjugates under $\Gamma$. Now those conjugates of $Q_1$ which fix letters that $K$ fixes, but which fix no letters that $K$ displaces, are invariant under $K$. Hence at most one half of the conjugates of $Q_1$ are invariant under $K$ and consequently at most one half the conjugates of $\Gamma_1$ are invariant under the alternating group of degree $k$ on the letters $b_1, b_2, \ldots, b_k$. Using this hypothesis Jordan* has shown that if one of the numbers $\gamma_i$ is greater than the number $\delta$, defined to be the greatest integer less than the quantity $t-(t-k+1)\log 2/(k+\log 2)$, the corresponding transitive constituent group of $\Gamma_1$ cannot contain the alternating group of the same degree.

Thus there are the following three cases to consider:

I. All the numbers $\gamma_i \leq \delta$.

II. One number $\gamma_i > \delta$, and the corresponding transitive constituent of degree $\gamma_i$ of $\Gamma_1$ is imprimitive.

III. One number $\gamma_i > \delta$, and the corresponding transitive constituent of degree $\gamma_i$ of $\Gamma_1$ is primitive.

In Case I, the order $q \leq 1/2(1-\delta)!$. In Case II recall that the largest imprimitive group of degree $\gamma_i$ is of order $2(\gamma_i/2)!$ or $2((\gamma_i-1)/2)!$ ac-
According as \(\gamma_1\) is even or odd. Thus \(q \leq [\gamma_1/2]^n(t-\gamma_1)!\) where \([s]\) denotes the integral part of \(s\). In Case III, Bochert’s theorem may be used to determine the order of the primitive constituent, and hence \(q \leq (t-\gamma_1)!\gamma_1!/[(\gamma_1+1)/2]!\). These three cases thus lead to the following inequalities:

\[
\begin{align*}
(14) & \quad Z \geq t!/\{\delta!(t-\delta)!\}, \\
(15) & \quad Z \geq t!/\{2[\gamma_1/2]!(t-\gamma_1)!\}, \\
(16) & \quad Z \geq t![(\gamma_1 + 1)/2]!/\{2\gamma_1!(t - \gamma_1)!\}.
\end{align*}
\]

It has been seen (§6) that the minimum of the right hand member of the inequality (15) occurs when \(\gamma_1 = t\), and that inequality (16) need be considered only for \(\gamma_1 = t\). Further \(Z\) may be taken equal to \((n-2t+1)/2\), for the greatest value of \((n-t)/p\) occurs when \(p = 3\), and the former number is always greater than the latter as soon as \(n \geq 4t - 3\). Now \(n \geq 4t - 3\), for the right hand members of the above inequalities are always \(\geq t\). Thus inequalities (14), (15), and (16) may be replaced by

\[
\begin{align*}
(17) & \quad (n - 2t + 1)/2 \geq t!/\{\delta!(t-\delta)!\}, \\
(18) & \quad (n - 2t + 1)/2 \geq t!/\{2[t/2]!\}, \\
(19) & \quad (n - 2t + 1)/2 \geq 1/[(t + 1)/2]!.
\end{align*}
\]

In the list of inequalities in §§9, 10, and 11 substitute the least possible values for \(p\), \(p_1\), and \(Z\). Then only the following inequalities need be considered:

\[
\begin{align*}
(20) & \quad n - t \geq 3^a, \quad \alpha \geq t - 3 - \log t/\log 2, \\
(21) & \quad (n - 2t + 1)/2 \geq 2^a, \\
(22) & \quad (n - t)/t \geq 3^b, \quad \beta \geq k - 3 - \log k/\log 2, \quad 5 \leq k \leq t - 1, \\
(23) & \quad (n - 2t + 1)/(2t) \geq 2^b.
\end{align*}
\]

In inequality (20), \(\alpha > 3\), for the holomorph of an elementary group of degree 9 or 27 obviously has no quotient group simply isomorphic to the alternating group of degree \(t(\geq 8)\). Likewise in inequality (21), \(\alpha > 3\). For a similar reason \(\beta > 3\) in inequalities (22) and (23), for in these cases the alternating group of degree \(t\) is replaced by one of degree \(k\). Although these same restrictions on \(\alpha\) and \(\beta\) might not hold if larger primes had been substituted for \(p\) and \(p_1\), similar considerations show that greater values for \(p\) and \(p_1\) give still more favorable values for \(n\).

Now write inequalities (22) and (23) in the forms \(n \geq 3^b t + t\) and \(n \geq 2^b + t + 2t - 1\), respectively. Since \(3^b \geq 2^b + 1 - 1/t\) for all values of \(\beta > 3\), inequality (22) may be discarded. Again write inequalities (20) and (21) in
the forms \( n \geq 3^a + t \) and \( n \geq 2^{a+1} + 2t - 1 \), respectively. Then if \( 2^{a+1} + t - 1 < 3^a \), inequality (20) may be discarded. Now \( t - 1 < 2^a, \alpha > 3 \). Since \( 2^a < 3^{a-1} \) for all values of \( \alpha > 3 \), the above inequality holds.

Further, compare the right hand members of inequalities (18) and (19) with the right hand member of inequality (21). The logarithm of the right hand member of inequality (18) is always greater than the logarithm of the right hand member of inequality (21). Hence inequality (18) may be discarded. Again the logarithm of the right hand member of inequality (19) is greater than the logarithm of the right hand member of inequality (21) except when \( t = 8, \alpha = 4 \). However, when \( t = 8 \), inequality (17) gives a still lower bound for \( n \) than inequality (19). Thus inequality (19) may be discarded for all values of \( t \).

13. The results of the study of the \( t \)-ply transitive group \( G \), the order of whose subgroup that fixes \( t \) letters is divisible by an odd prime, may be summarized in the following theorem:

**Theorem 4.** Let the order of the subgroup that fixes \( t \) letters of a \( t \)-ply transitive group of degree \( n \) and class \( > 3 \) be divisible by an odd prime. Then if \( t \geq 8 \),

\[
\frac{n - 2t + 1}{2} \geq 2^a, \text{ } a \text{ an integer } > 3 \text{ and } \geq t - 3 - \log t / \log 2;
\]
or

\[
\frac{n - 2t + 1}{2} \geq 2^n, \gamma \text{ an integer } > 3 \text{ and } \geq k - 3 - \log k / \log 2,
\]
where \( k \) is an integer such that \( 5 \leq k \leq t - 1 \); or

\[
\frac{n - 2t + 1}{2} \geq t! / \{ \delta! (t - \delta)! \},
\]
where \( \delta \) is the greatest integer less than the quantity \( t - (t - k + 1) \log 2 / (k + \log 2) \).

It may be of interest to find the principal value, when \( t \) becomes infinite, of \( (n - 2t + 1) / 2 \), or, which is more convenient, of \( \log ((n - 2t + 1) / 2) \). The logarithm of the right hand member of the first inequality in Theorem 4 obviously has the principal value \( t \log 2 \). It remains to find the principal values of the right hand members of the second and third inequalities in the above theorem. Denote the logarithm of the right hand member of the second inequality by \( f(k) \) and that of the third by \( g(k) \). Thus

\[
f(k) = (k - 3) \log 2 - \log k + \log t,
\]
and by Stirling's formula

\[
g(k) = (t + \frac{1}{2}) \log t - (\delta + \frac{1}{2}) \log \delta - (t - \delta + \frac{1}{2}) \log (t - \delta) - \frac{1}{2} \log 2\pi
\]
\[
+ \theta_i / (12t) - \theta_i / (12\delta) - \theta_i / (12(t - \delta)), \text{ where } 0 < \theta_i < 1, \quad i = 0, 1, 2.
\]
Now \( f(k) \) is a function which increases in value with \( k \), while \( g(k) \) is a function which decreases in value as \( k \) increases. First determine \( k \) so that the principal values of these functions are equal. Assume \( k \) and \( t \) of equal order. Then the principal value of \( t - \delta \) is \( \{ (t - k) \log 2 \}/k \). Now in \( g(k) \) replace \( \log \delta \) by

\[
\log t + \log \{ 1 - (t - \delta)/t \} = \log t - \log t - (t - \delta)^2/(2t^2) - \cdots .
\]

Then the principal value of \( g(k) \) is \( \{ (t - k) \log 2 \log t \}/k \), while the principal value of \( f(k) \) is obviously \( k \log 2 \). Setting these two values equal, it is found that

\[
k = - \frac{1}{k} \log t + \frac{1}{k} (4t \log t + \log^2 t)^{1/2},
\]

of which the principal value is \( (t \log t)^{1/2} \). Thus \( k \) is of order less than the order of \( t \), and using this value of \( k \), the principal value of \( \log ((n - 2t + 1)/2) \), derived from the second and third inequalities in the above theorem, is \( (t \log t)^{1/2} \log 2 \). This result is evidently less favorable than the one derived from the first inequality, and hence for the case \( G_t \), of order \( p^\theta m \), \( p \) an odd prime, the following relation holds:

\[
(24) \quad \log ((n - 2t + 1)/2) \geq (t \log t)^{1/2} \log 2(1 - \eta),
\]

where \( \eta \) approaches 0 as \( t \) approaches \( \infty \).

In §8, it was shown that for the case \( G_t \) of order \( 2^r \) the principal value of \( \log (n - t) \) is \( t \log 2 \). Evidently this result gives a more favorable relation between \( n \) and \( t \) than the one just established. Hence inequality (24) holds for all \( t \)-ply transitive groups of class \( >3 \). This inequality is similar to the one found by Jordan in his 1895 paper, namely,

\[
\log (n - t) \geq (t \log t)^{1/2} \log 2(1 - \epsilon),
\]

where \( \epsilon \) approaches 0 as \( t \) approaches \( \infty \).