ON GIBBS'S PHENOMENON FOR THE DEVELOPMENTS IN BESSEL'S FUNCTIONS*

BY

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Introduction. Several investigations have recently been published† concerning Gibbs's phenomenon in the case of the developments in Bessel's functions. As we should expect, the behavior of the developments in the neighborhood of the origin presented special difficulty, and the investigators limited themselves to very special types of functions‡ in dealing with this particular situation. It is the purpose of the present note to point out that the asymptotic formulas§ for the coefficients of the developments in Bessel's functions that have been obtained in two of my previous papers on this subject furnish a simple method of attacking the point in question. From these formulas the nature of Gibbs's phenomenon at the origin can be readily derived for general classes of functions. The same formulas also serve to establish the facts with regard to Gibbs's phenomenon at interior points of the interval (0, 1) and at the end point 1.

1. The cause of Gibbs's phenomenon at the origin. I will illustrate the application of these formulas to the discussion of Gibbs's phenomenon by dealing with the point $x=0$ and the class of functions considered in the earlier paper. This class includes functions which in the interval (0, 1) have a second derivative that is finite and integrable, except perhaps for a finite number of points at which the function itself or its first derivative has a finite jump. For such functions we have¶ for the coefficient of the general term of the development in Bessel's functions the following asymptotic

* Presented to the Society, April 7, 1928, and November 30, 1929; received by the editors in September, 1929, and February, 1930.

† Cf. R. G. Cooke, Proceedings of the London Mathematical Society, (2), vol. 27 (1928), pp. 171–192; J. R. Wilton, Journal für die reine und angewandte Mathematik, vol. 159 (1928), pp. 144–153.

‡ Cooke considers only the case where the function developed is a constant in the interval $(0\leqq x\leqq 1)$; Wilton requires that in the neighborhood of the origin the function developed be of the form $l+Ax^{\nu+1/2}+o(x^{\nu+1/2})$, where l and A are constants and ν is the index of the Bessel's function, and he further restricts his discussion to the case where ν does not exceed $3/2$.

§ Cf. these Transactions, vol. 12 (1911), p. 183, formula (8); vol. 21 (1920), p. 110, formula (15). The papers quoted here will be subsequently referred to as Transactions II and Transactions III respectively. An earlier paper (vol. 10 (1909), pp. 391–435) will be referred to as Transactions I.

¶ Loc. cit., Transactions II.

expression:

$$(1)\quad \sum_{m=1}^{m=k} \frac{M_m}{n^{1/2}} \cos\left[(n\pi + q)c_m - \frac{1}{4}(2\nu + 3)\pi\right] + \frac{Mf(1)}{n^{1/2}} \cos\left[n\pi + q - \frac{1}{4}(2\nu + 3)\pi\right] + \frac{\nu K f(0)}{\lambda_n} + r_n,$$

where the c's are the points of discontinuity of the function developed, r_n is the general term of an absolutely convergent series, and the M's, K, and q are constants. The precise value of K is readily determined. From (34) and (29) of Transactions II we have

$$K' = \int_0^\infty J_{\nu+1}(y)dy = 1.$$

From (38) of Transactions II we have $K = 2K'/A^2 = 2/(2/\pi) = \pi$, since from (44) of Transactions I we have $A = (2/\pi)^{1/2}$.

It was shown in Lemma 2 of Transactions II that the first and second terms of (1), multiplied by $J_\nu(\lambda_n x)$, yield series uniformly convergent in the neighborhood of the origin, and the same is obviously true of the fourth term. Hence the Gibbs's phenomenon cannot occur if the third term is missing, and therefore there will be none in case $\nu = 0$ or $f(0) = 0$. If neither of these conditions holds, the presence of Gibbs's phenomenon can be readily inferred from the lemma of the following section.

2. We prove the following lemma:

LEMMA. *The series whose general term is*

$$(2)\qquad a_n = \int_{\lambda_n}^{\lambda_{n+1}} \frac{J_\nu(xu)}{u} du - \frac{J_\nu(\lambda_n x)}{\lambda_n}(\lambda_{n+1} - \lambda_n)$$

is absolutely and uniformly convergent in the interval $(0 \leqq x \leqq 1)$.

We have

$$(3)\qquad a_n = \int_{\lambda_n}^{\lambda_{n+1}} \left[\frac{J_\nu(xu)}{u} - \frac{J_\nu(\lambda_n x)}{\lambda_n}\right] du.$$

By the law of the mean the integrand in (3) is equal to

$$(4)\qquad (u - \lambda_n)\left[\frac{uxJ_\nu'(xu) - J_\nu(xu)}{u^2}\right] \qquad (u = \lambda_n + \theta,\ 0 < \theta < \lambda_{n+1} - \lambda_n)$$

$$= (u - \lambda_n)\left[\frac{(\lambda_n + \theta)xJ_\nu'([\lambda_n + \theta]x)}{(\lambda_n + \theta)^2} - \frac{J_\nu([\lambda_n + \theta]x)}{(\lambda_n + \theta)^2}\right],$$

and by the asymptotic expansion for Bessel's functions the expression following the sign of equality does not exceed a constant multiple of $1/\lambda_n^{3/2}$ for $0<x\leqq 1$, the constant multiplier being independent of n, x and u.

3. The measure of Gibbs's phenomenon. We prove the following theorem:

THEOREM. *For functions of the class defined in* §1 *and their corresponding developments in Bessel's functions of order* $\nu>0$, *the height of the first wave in the approximation curve,* $y=S_n(x)$, *will for sufficiently large values of* n *be as near as we choose to the value**

$$\nu f(0)\int_0^{\gamma_1}\frac{J_\nu(t)}{t}\,dt, \tag{5}$$

where γ_1 *is the first positive root of the equation* $J_\nu(t)=0$.

As pointed out in §1, the developments in Bessel's functions of a function of the class considered will differ from a series whose general term is

$$\nu\pi f(0)\frac{J_\nu(\lambda_n x)}{\lambda_n} \tag{6}$$

by a series which is uniformly convergent in the neighborhood of the origin. Furthermore the latter series defines in this neighborhood a continuous function which vanishes at the origin. Therefore, for the purpose of studying Gibbs's phenomenon at the origin, the developments in Bessel's functions may be replaced† by the series whose general term is (6). It follows from Theorem I of Transactions I that this latter series may be replaced by the series whose general term is

$$\nu(\lambda_{n+1}-\lambda_n)\frac{J_\nu(\lambda_n x)}{\lambda_n}f(0). \tag{7}$$

From the lemma of §2 and the fact that the a_n there defined is for each n a continuous function vanishing at the origin, it follows that if we first choose x in a sufficiently small interval extending to the right of the origin,

* If the function $f(x)$ has an artificial singularity at the origin, we remove it by defining $f(0)=f(+0)$. In order to infer from (5) that there is actually a Gibbs's phenomenon, we need to be certain that the value of (5) exceeds $f(0)$. That this is the case may be inferred as follows. The sequence

$$\left|\int_{\gamma_n}^{\gamma_{n+1}}[J_\nu(t)/t]\,dt\right|,$$

where the γ_n's are the successive positive roots of $J_\nu(t)=0$, is steadily decreasing for increasing n (see Cooke, loc. cit., p. 178). Hence the value of $\int_0^{\gamma_1}[J_\nu(t)/t]dt$ exceeds the value, $1/\nu$, of $\int_0^\infty[J_\nu(t)/t]dt$.

† See Bôcher, Annals of Mathematics, (2), vol. 7 (1906), p. 130.

the sum of n terms of the series whose general term is (7) can by a proper choice of n for an x in this interval be made as near as we please to

$$(8)\qquad \nu f(0)\int_0^{\lambda_{n+1}}\frac{J_\nu(xu)}{u}du=\nu f(0)\int_0^{\lambda_{n+1}x}\frac{J_\nu(t)}{t}dt.$$

Our theorem follows from the fact that the maximum value of the integral $\int_0^y[J_\nu(t)/t]dt$ for values of $y\geqq 0$ is $\int_0^{\gamma_1}[J_\nu(t)/t]dt$ (see first footnote of this section).

4. **Extension to a more general class of functions and to other points of the interval (0, 1).** If we add to the conditions of Lemma 1 of Transactions III the requirement that in an interval including the origin $f(x)$ has a first derivative that is integrable (Lebesgue) and approaches a finite limit as x approaches $+0$, it may be shown by means of formula (15) of Transactions III that for $\nu=0$ or $f(0)=0$ the development in Bessel's functions of $f(x)$ converges uniformly in the neighborhood of the origin.* For $\nu>0$, $f(0)\neq 0$, the development is no longer uniformly convergent on account of the non-vanishing of the fourth term on the right hand side of the formula in question, and since this term is identical with the third term of (1), Gibbs's phenomenon in the neighborhood of the origin is seen to be of the same nature for the more general class of functions considered here.

By combining (1), or (15) of Transactions III, with the asymptotic expansion of the Bessel's function $J_\nu(\lambda_n x)$, it is readily shown that Gibbs's phenomenon at interior points of the interval (0, 1) is identical with the Gibbs's phenomenon of Fourier's cosine development of the same function in the interval (0, 1), the interval of periodicity of the function being of length 2. This will also be the case at the end point $x=1$, if the l of equation (1) of Transactions I is zero; this of course implies that there will be no Gibbs's phenomenon if $f(1)=0$, since the function $f(x)$, defined for all values of x by the periodic property, is then continuous at $x=1$. Also, if $l\neq 0$, there will be no Gibbs's phenomenon, since in this case the Bessel's development is uniformly convergent in the neighborhood of $x=1$, as was shown in Theorem I of Transactions II.

5. **Extension to series summed by Rieszian or Cesàro means.**† We find it more convenient to use in our discussion a Rieszian mean equivalent to the Cesàro mean of order $k(0<k\leqq 1)$, and to consider therefore sums of the type

* This result was contained in a paper presented to the Society in 1917. See Bulletin of the American Mathematical Society, vol. 24 (1918), p. 282.

† The only previous results in this connection are due to Wilton, who shows that in case the index ν lies in the interval (0,1/2) the ($C1$) sums do not exhibit Gibbs's phenomenon. (Loc.cit., p.152.)

$$\sum_{m=1}^{m=n-1}\left(1-\frac{\lambda_m}{\lambda_n}\right)^k u_m(x).$$

On account of the relationship between the two means in question, the results obtained hold equally well for either of them.

We begin by proving a lemma.

LEMMA. *The sum from $m=1$ to $m=n-1$ of terms of the form*

$$(9)\qquad a_m(x)=\int_{\lambda_m}^{\lambda_{m+1}}\left[\left(1-\frac{t}{\lambda_n}\right)^k\frac{J_\nu(xt)}{t}-\left(1-\frac{\lambda_m}{\lambda_n}\right)^k\frac{J_\nu(\lambda_m x)}{\lambda_m}\right]dt$$

approaches a limit uniformly in the interval $(0\leqq x\leqq 1)$ *as n becomes infinite.*

By the law of the mean the expression in the integrand of the above integral may be written in the form

$$(t-\lambda_m)\frac{\dfrac{-kv}{\lambda_n}\left(1-\dfrac{v}{\lambda_n}\right)^{k-1}J_\nu(xv)+vx\left(1-\dfrac{v}{\lambda_n}\right)^k J_\nu'(xv)-\left(1-\dfrac{v}{\lambda_n}\right)^k J_\nu(xv)}{v^2}$$
$$(\lambda_m<v<t<\lambda_{m+1}).$$

For $1\leqq m\leqq n-2$ it follows from the asymptotic expansion for Bessel's functions that the above expression is less in absolute value than a constant multiple of $1/\lambda_m^{3/2}$ for $0<x\leqq 1$, the constant multiple being independent of m, x, and t. For $m=n-1$ it is readily seen that if H represents the maximum absolute value of $J_\nu(u)$ for $u>0$, the term $a_m(x)$ is less in absolute value than $2H(\lambda_n-\lambda_{n-1})/\lambda_{n-1}$. Combining these results and the fact that $a_m(0)=0(m=1,2,3,\cdots)$ with the inequalities in (23) of Transactions I, we readily infer that for any positive ϵ we can so choose q_1 that $\left|\sum_{m=p}^{m=n-1}a_m(x)\right|<\frac{1}{3}\epsilon$ $(0\leqq x\leqq 1)$ for any $p\geqq q_1$.

By a similar application of the law of the mean to the integrand of

$$b_m(x)=\int_{\lambda_m}^{\lambda_{m+1}}\left[\frac{J_\nu(xt)}{t}-\frac{J_\nu(\lambda_m x)}{\lambda_m}\right]dt$$

and the use of the asymptotic expansion of $J_\nu(u)$, we readily infer that $b_m(x)$ is the general term of a series uniformly convergent in $(0<x\leqq 1)$ and hence, since $b_m(0)=0$, in $(0\leqq x\leqq 1)$. Therefore we may choose q_2 sufficiently large to make $\left|\sum_{m=p}^{\infty}b_m(x)\right|<\frac{1}{3}\epsilon$ in $(0\leqq x\leqq 1)$ for $p\geqq q_2$.

Taking q as the greater of q_1 and q_2 and holding q fixed, we can then choose a $p>q+1$ so that $\sum_{m=1}^{m=q-1}a_m(x)$ differs by less than $\frac{1}{3}\epsilon$ from $\sum_{m=1}^{m=q-1}b_m(x)$ for all $n\geqq p$. Thus our lemma is established.

For the class of functions considered in the previous section it was the presence of the term (6) in the general term of the development in Bessel's functions that caused the failure of uniform convergence and hence the appearance of Gibbs's phenomenon in the case $\nu>0, f(0)\neq 0$. Consequently, if we sum the development of a function of the same class by Cesàro means of positive order, the presence or absence of Gibbs's phenomenon and its amount will be determined by using the same method of summation on the series whose general term is (6). Since the term (6) differs from the term

$$\nu f(0)(\lambda_{n+1}-\lambda_n)\frac{J_\nu(\lambda_n x)}{\lambda_n} \tag{10}$$

by the general term of a series that is absolutely and uniformly convergent in the interval $(0\leqq x\leqq 1)$, it is evident that we need only to consider the nature of Gibbs's phenomenon in the following sum:

$$\sum_{m=1}^{m=n-1}\left(1-\frac{\lambda_m}{\lambda_n}\right)^k \nu f(0)(\lambda_{m+1}-\lambda_m)\frac{J_\nu(\lambda_m x)}{\lambda_m}\,. \tag{11}$$

But since the limit, $\sum_{m=1}^{\infty} b_m(x)$, approached by $\sum_{m=1}^{m=n-1} a_m(x)$ as n becomes infinite, is continuous for $x\geqq 0$ and equal to zero for $x=0$, it follows from our lemma that the sum (11) will for values of x sufficiently small be as near as we choose to

$$\nu f(0)\int_0^{\lambda_n}\left(1-\frac{t}{\lambda_n}\right)^k\frac{J_\nu(xt)}{t}\,dt=\nu f(0)\int_0^{\lambda_n x}\left(1-\frac{u}{\lambda_n x}\right)^k\frac{J_\nu(u)}{u}\,du,$$

when n is chosen sufficiently large.

Hence the presence or absence of Gibbs's phenomenon in the case of summation by Rieszian or Cesàro means of order $k>0$ will depend on the behavior of the function $\phi_k(\lambda)$ defined by

$$\phi_k(\lambda)=\int_0^{\lambda}\left(1-\frac{u}{\lambda}\right)^k\frac{J_\nu(u)}{u}du.$$

If for all positive values of λ, $\phi_k(\lambda)$ remains less than or equal to its limiting value as λ becomes infinite, $1/\nu$, there will be no Gibbs's phenomenon. If for certain positive values of λ it rises above this limit, there will be a Gibbs's phenomenon and the measure of this phenomenon will be determined by the maximum value of $\phi_k(\lambda)$.

Consider first the case $k=1$. We have

$$\phi_1(x)=\int_0^x\left(1-\frac{u}{x}\right)\frac{J_\nu(u)}{u}du=\int_0^1(1-t)\frac{J_\nu(tx)}{t}dt,$$

$$\phi_1'(x) = \int_0^1 (1-t)J_\nu'(tx)dt = \left[\frac{(1-t)J_\nu(tx)}{x}\right]_0^1 + \frac{1}{x}\int_0^1 J_\nu(tx)dt$$
$$= \frac{1}{x}\int_0^1 J_\nu(tx)dt = \frac{1}{x^2}\int_0^x J_\nu(u)du.$$

Since $\int_0^x J_\nu(u)du$ is positive for all positive values of x (see reference in first footnote of §3), it is apparent that $\phi_1(x)$ is a steadily increasing function of x for such values and that therefore *there is no Gibbs's phenomenon when we sum the developments in Bessel's functions by Rieszian or Cesàro means of order* 1.

Consider now values of k in the interval $0<k<1$. The function $\phi_k(x)$ is a continuous function of x and k for $x>0$, $k\geqq 0$. Hence, for sufficiently small values of k it will oscillate about its limiting value, $1/\nu$, just as $\phi_0(x)$ does. Therefore *we have the Gibbs's phenomenon for values of k near zero.*

That this is not the case for values of k near 1 we prove as follows. We have

$$\phi_k'(x) = \int_0^1 (1-t)^k J_\nu'(tx)dt = \left[\frac{(1-t)^k J_\nu(tx)}{x}\right]_0^1 + \frac{k}{x}\int_0^1 (1-t)^{k-1}J_\nu(tx)dt$$

(12)

$$= \frac{k}{x^2}\int_0^x \left(1-\frac{u}{x}\right)^{k-1} J_\nu(u)du.$$

It can readily be shown that for values of $k>\frac{1}{2}$ the integral on the right hand side of (21) approaches $\int_0^\infty J_\nu(u)du = 1$ as x becomes infinite.

We divide the interval of integration into four parts $(0, l)$, $(l, x/2)$, $(x/2, x-1)$, $(x-1, x)$ and write

$$\int_0^x \left(1-\frac{u}{x}\right)^{k-1} J_\nu(u)du = \int_0^l + \int_l^{x/2} + \int_{x/2}^{x-1} + \int_{x-1}^x = I_1 + I_2 + I_3 + I_4.$$

Applying the second law of the mean to I_2 and I_3 and the first law of the mean of I_4, we obtain

$$I_2 = 2^{1-k}\int_y^{x/2} J_\nu(u)du \ (y > l), \quad I_3 = x^{1-k}\int_\xi^{x-1} J_\nu(u)du \qquad (\xi > \tfrac{1}{2}x),$$

$$I_4 = \frac{J_\nu(x-\theta)x^{1-k}}{k} \qquad (0 < \theta < 1).$$

I_2 obviously approaches zero as l becomes infinite, and it follows from the asymptotic expansion of $J_\nu(u)$ that, if $k>\frac{1}{2}$, I_3 and I_4 approach zero as

x becomes infinite. If we now allow x and l to become infinite in such a manner that x/l becomes infinite, I_1 approaches $\int_0^\infty J_\nu(u)du$, and therefore we have

$$\lim_{x\to\infty}\int_0^x\left(1-\frac{u}{x}\right)^{k-1}J_\nu(u)du=\int_0^\infty J_\nu(u)du=1 \qquad (k>\tfrac{1}{2}).$$

It follows from the above relationship that we can find an X such that $\int_0^x(1-u/x)^{k-1}J_\nu(u)du$ remains positive for $x\geqq X$ when $k>\frac{1}{2}$. Moreover, for values of $x<X$, we can choose k sufficiently near 1 to make $\int_0^x(1-u/x)^{k-1}\cdot J_\nu(u)du$ differ from $\int_0^x J_\nu(u)du$ by as small a quantity as we please throughout the interval $0<x<X$. But $\int_0^x J_\nu(u)du$ is positive for all $x>0$ and approaches 1 as x becomes infinite. Hence it has a lower limit $\rho>0$ for all $x\geqq\delta>0$. Consequently, for k near enough to 1, $\int_0^x(1-u/x)^{k-1}J_\nu(u)du$ will also remain positive for $x\geqq\delta$, where δ is any given positive quantity. But for a δ less than the first positive root of $J_\nu(u)=0$, this integral is obviously positive for $0<x<\delta$ and any $k>0$. Thus *for values of k sufficiently near 1 the function $\phi_k(x)$ steadily increases with x and there is no Gibbs's phenomenon.*

We wish finally to show that there is a value r between 0 and 1 such that Gibbs's phenomenon occurs for $k<r$ and does not occur for $k\geqq r$. Let us set

$$M_k(x,n)=\sum_{m=1}^{m=n-1}\left(1-\frac{\lambda_m}{\lambda_n}\right)^k(\lambda_{m+1}-\lambda_m)\frac{J_\nu(\lambda_m x)}{\lambda_m}\cdot$$

Combining the lemma of this section with the fact that the $a_m(x)$ defined by (9) is zero for $x=0$, we obtain

$$h(k)=\limsup_{\substack{x\to 0\\ n\to\infty}} M_k(x,n)=\limsup_{x>0}\phi_k(x).$$

But if we write $M_{k+\delta}(x,n)$ in the form

$$\sum_{m=1}^{m=n-1}\left(1-\frac{\lambda_m}{\lambda_n}\right)^\delta\left(1-\frac{\lambda_m}{\lambda_n}\right)^k(\lambda_{m+1}-\lambda_m)\frac{J_\nu(\lambda_m x)}{\lambda_m},$$

it follows from Abel's lemma that $h(k+\delta)\leqq h(k)$. Hence $h(k)$ is a monotonic decreasing function of k in the interval $0<k<1$. Moreover, since $\phi_k(x)$ is a continuous function of x and k for $x>0$, $k\geqq 0$, $h(k)$ is a continuous function of k for $k\geqq 0$. But $h(k)=1/\nu$ for values of k near 1 and $h(0)>1/\nu$. Consequently there is a value r of k between 0 and 1 such that $h(k)=1/\nu (r\leqq k\leqq 1)$ and $h(k)>1/\nu$ $(0<k<r)$. Thus *Gibbs's phenomenon will occur for $k<r$ and will not occur for $k\geqq r$.*

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