A CANONICAL FORM OF GREEN'S PROJECTIVE ANALOGUE OF THE GAUSS DIFFERENTIAL EQUATIONS*

BY

V. G. GROVE

1. Introduction

The projective differential geometry of a surface in space of three dimensions may be founded upon Green's projective analogue of the Gauss differential equations.† In this paper we assume for the parametric curves any non-asymptotic system, and derive a canonical form for Green's equations. The method used is an adaptation of the method used by Lane in deriving a canonical form‡ for the defining differential equations of a conjugate net. By specializing the parametric curves to be conjugate, we may obtain the equivalent of Lane's canonical form. Again by assuming the parametric curves to be non-conjugate we obtain a canonical form for Green's differential equations§ defining a non-conjugate net. These canonical forms may be written with covariant derivatives.

2. An intermediate form of Green's equations

Let the homogeneous coördinates $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$ of a point y of a non-degenerate surface S_y be given as analytic functions of two independent variables u, v. Let also $z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}$ be the homogeneous coördinates of a point not in the tangent plane to S_y at y. The four pairs of functions (y, z) are solutions of a system of differential equations of the form¶

$$
\begin{aligned}
y_{uu} &= \alpha y_u + \beta y_v + py + Lz,\\
y_{uv} &= ay_u + by_v + cy + Mz,\\
y_{vv} &= \gamma y_u + \delta y_v + qy + Nz,\\
z_u &= my_u + sy_v + fy + Az,\\
z_v &= ty_u + ny_v + gy + Bz.
\end{aligned}
\tag{1}
$$

Since system (1) is a symmetrical system, the formulas appear in pairs.

* Presented to the Society, December 31, 1928; received by the editors January 25, 1930.

† G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), p. 148. Hereafter referred to as Green, *Surfaces*.

‡ E. P. Lane, *Contributions to the theory of conjugate nets*, American Journal of Mathematics, vol. 49 (1927), pp. 565–576. Hereafter referred to as Lane, *Nets*.

§ G. M. Green, *Nets of space curves*, these Transactions, vol. 21 (1920), p. 207.

¶ Green, *Surfaces*, p. 149.

We shall give one of each pair; the other is obtained from the one given by the transformation scheme

$$S\begin{pmatrix} u, & \alpha, & \beta, & p, & L, & a, & c, & M, & m, & s, & f, & A \\ v, & \delta, & \gamma, & q, & N, & b, & c, & M, & n, & t, & g, & B \end{pmatrix},$$

any symbol being replaced by the one immediately above or below it.

The integrability conditions of system (1) are given by the following equations* and by the scheme S:

$$\begin{aligned} a_u - \alpha_v + ab + \beta\gamma + c &= Lt - Mm, \\ b_u - \beta_v + a\beta + b(b-\alpha) - \beta\delta - p &= Ln - Ms, \\ c_u - p_v + ap + c(b-\alpha) - \beta q &= Lg - Mf, \\ M_u - L_v + aL + M(b-\alpha) - \beta N &= LB - MA; \end{aligned} \tag{2a}$$

$$\begin{aligned} t_u - m_v + \alpha t + a(n-m) - s\gamma + g &= At - Bm, \\ g_u - f_v + pt + c(n-m) - sq &= Ag - Bf, \\ B_u - A_v + Lt + M(n-m) - sN &= 0. \end{aligned} \tag{2b}$$

Green has shown† from these conditions that

$$(a + \delta + A)_u = (b + \alpha + B)_v. \tag{3}$$

If we transform the independent variables by the transformation

$$\bar{u} = \phi(u,v), \quad \bar{v} = \psi(u,v), \tag{4}$$

where

$$\phi_u = -\theta\phi_v, \quad \psi_v = -\omega\psi_u, \quad \theta\omega - 1 \neq 0, \tag{5}$$

we make the integral curves of

$$(\theta du - dv)(\omega dv - du) = 0$$

parametric. Transformation (4) changes system (1) into a system of like form, six of whose coefficients are given by S and the formulas

$$\begin{aligned} \bar{\alpha} &= \frac{1}{\phi_v(\theta\omega - 1)^2}[\beta\omega^2 + 2b\omega + \delta - \theta(\alpha\omega^2 + 2a\omega + \gamma) \\ &\quad + \omega^2\theta_u + (\theta\omega - 2)\omega\theta_v] - \frac{\phi_{vv}}{\phi_v{}^2}, \\ \bar{\beta} &= \frac{\psi_u}{\phi_v{}^2(\theta\omega - 1)^2}[\alpha\omega^2 + 2a\omega + \gamma - \omega(\beta\omega^2 + 2b\omega + \delta) + \omega\omega_u + \omega_v], \end{aligned} \tag{6}$$

* Green, *Surfaces*, p. 150.

† Green, *Surfaces*, p. 152.

$$\bar{L} = \frac{1}{\phi_v^2(\theta\omega - 1)^2}[L\omega^2 + 2M\omega + N].$$

Since the asymptotic curves are not parametric, we may make $\bar{L}=0$ by taking

$$\omega = (d^{1/2} - M)/L, \quad d = M^2 - LN. \tag{7}$$

For this choice of ω, the coefficient $\bar{\beta}$ assumes the form

$$\bar{\beta} = \frac{\psi_u \omega}{2\phi_v^2(\theta\omega - 1)d^{1/2}}(\omega L Q^{(u)} + N Q^{(v)}) \tag{8}$$

wherein $Q^{(u)}$ and $Q^{(v)}$ are defined by S and the formula

$$\begin{aligned} Q^{(u)} = 2\left[\frac{2(aN - \gamma M)}{N} - \frac{\delta L - \beta N}{L}\right] + 2\frac{M}{L}\left[\frac{2(bL - \beta M)}{L} - \frac{\alpha N - \gamma L}{N}\right] \\ - 2\left(\frac{M}{L}\right)_u - \frac{\partial}{\partial v}\log\frac{L}{N}\cdot \end{aligned} \tag{9}$$

From (8) and S we find readily that

$$\bar{\beta}\bar{\gamma} = \frac{LNR}{16d(d^{1/2} - M)\phi_v\psi_u}, \tag{10}$$

wherein

$$R = LN(LQ^{(u)^2} - 2MQ^{(u)}Q^{(v)} + NQ^{(v)^2})/d. \tag{11}$$

Hence *the surface S_y is ruled if and only if* $R=0$.

If we make the transformation

$$y = \lambda\bar{y} \tag{12}$$

on system (1), we obtain a new system of the same form with the following coefficients:

$$\begin{aligned} &\bar{\alpha} = \alpha - 2\lambda_u/\lambda,\ \bar{\beta} = \beta,\ \bar{p} = p + \alpha\lambda_u/\lambda + \beta\lambda_v/\lambda - \lambda_{uu}/\lambda,\ \bar{L} = L/\lambda, \\ &\bar{a} = a - \lambda_v/\lambda, \quad \bar{c} = c + a\lambda_u/\lambda + b\lambda_v/\lambda - \lambda_{uv}/\lambda, \qquad \bar{M} = M/\lambda. \end{aligned} \tag{13}$$

The remaining coefficients are either obtainable by S or unnecessary for our purposes. The functions d, $Q^{(u)}$, $Q^{(v)}$, and R are transformed by (12) into $\bar{d}$, etc. defined by the formulas

$$\bar{d} = d/\lambda^2, \quad \bar{Q}^{(u)} = Q^{(u)}, \quad \bar{Q}^{(v)} = Q^{(v)}, \quad \bar{R} = R/\lambda. \tag{14}$$

It follows therefore from (3) and (14) that there exists a function $F(u, v)$ such that

$$
(15) \qquad \begin{aligned} F_u &= \alpha + b + A - 2R_u/R + (3/2)d_u/d, \\ F_v &= \delta + a + B - 2R_v/R + (3/2)d_v/d. \end{aligned}
$$

Transformation (12) transforms the functions F_u, F_v into

$$
(16) \qquad \overline{F}_u = F_u - 4\lambda_u/\lambda, \qquad \overline{F}_v = F_v - 4\lambda_v/\lambda.
$$

Hence if we choose $\lambda = e^{(1/4)F}$, we make $\overline{F}_u = \overline{F}_v = 0$. The resulting differential equations with coefficients (13) for this choice of λ will be called the *intermediate form* of (1).

The transformation

$$
(17) \qquad \bar{u} = \phi(u), \ \bar{v} = \psi(v)
$$

changes system (1) into another system whose coefficients are defined by S and by

$$
(18) \qquad \begin{aligned} &\bar{\alpha} = (\alpha - \phi_{uu}/\phi_u)/\phi_u, \quad \bar{\beta} = \psi_v\beta/\phi_u, \quad \bar{p} = p/\phi_u^2, \quad \bar{L} = L/\phi_u^2, \\ &\bar{a} = a/\phi_v, \quad \bar{c} = c/(\phi_u\psi_v), \quad \overline{M} = M/(\phi_u\psi_v), \\ &\bar{m} = m, \quad \bar{s} = \psi_v s/\phi_u, \quad \bar{f} = f/\phi_u, \quad \overline{A} = A/\phi_u. \end{aligned}
$$

The invariants d, $Q^{(u)}$, $Q^{(v)}$, and R are transformed according to the formulas

$$
d = d/(\phi_u\psi_v), \quad \overline{Q}^{(u)} = Q^{(u)}/\psi_v, \quad \overline{Q}^{(v)} = Q^{(v)}/\phi_u, \quad \overline{R} = R/(\phi_u\psi_v)^2.
$$

Hence the conditions $F_u = F_v = 0$, characterizing the intermediate form, are unchanged by the transformation (17).

The polar reciprocal of the projective normal with respect to the quadric of Lie of the surface S_y at y intersects the asymptotic tangents in the points*

$$
(19) \qquad \bar{r} = \frac{\partial y}{\partial \bar{u}} - \frac{1}{2}\left[\bar{\alpha} + \frac{\partial}{\partial \bar{u}} \log (\bar{\beta}\bar{\gamma})\right] y, \qquad \bar{s} = \frac{\partial y}{\partial \bar{v}} - \frac{1}{2}\left[\bar{\delta} + \frac{\partial}{\partial \bar{v}} \log (\bar{\beta}\bar{\gamma})\right] y,
$$

wherein $\bar{u}$, $\bar{v}$ are the asymptotic parameters, and $\bar{\alpha}$, $\bar{\delta}$, $\bar{\beta}$, $\bar{\gamma}$ are defined by (6) and S. By means of (6) we may show that the reciprocal of the projective normal intersects the tangents to the parametric curves in the points

$$
\begin{aligned} r &= y_u - \tfrac{1}{4}F_u y, \\ s &= y_v - \tfrac{1}{4}F_v y. \end{aligned}
$$

The points y_u and y_v of the intermediate form are therefore characterized geometrically as the intersections of the reciprocal of the projective normal with the parametric tangents.

* Green, *Surfaces*, p. 128.

3. The projective normal. The canonical form

The Darboux curves of the surface S_y are defined by the differential equation

$$\bar{\beta} d\bar{u}^3 + \bar{\gamma} d\bar{v}^3 = 0, \tag{20}$$

wherein $\bar{\beta}$ and $\bar{\gamma}$ are defined by means of (6) and by S, and $\bar{u}$ and $\bar{v}$ are asymptotic parameters. In terms of the coefficients and variables of system (1) equation (20) may be written in the form

$$L^2(NQ^{(v)} - 2MQ^{(u)})du^3 - 3L^2NQ^{(u)}du^2dv \\ - 3LN^2Q^{(v)}dudv^2 + N^2(LQ^{(u)} - 2MQ^{(v)})dv^3 = 0.$$

The asymptotic curves on S_y are defined by the differential equation

$$Ldu^2 + 2Mdudv + Ndv^2 = 0.$$

Consider now the two forms

$$\begin{aligned} \phi_2 &= R(Ldu^2 + 2Mdudv + Ndv^2)/d, \\ \phi_3 &= R[L^2(NQ^{(v)} - 2MQ^{(u)})du^3 - 2L^2NQ^{(u)}du^2dv \\ &\qquad - 3LN^2Q^{(v)}dudv^2 + N^2(LQ^{(u)} - 2MQ^{(v)})dv^3]/d^2. \end{aligned} \tag{21}$$

The ratio of the discriminant of ϕ_3 to the cube of the discriminant of ϕ_2 is a constant. Hence these forms are the forms ϕ_2 and ϕ_3 of Fubini.*

Since, for the intermediate form, y_u and y_v are the points in which the reciprocal of the projective normal intersects the parametric tangents, it follows that the projective normal joins y to the point†

$$\zeta = Ny_{11} - 2My_{12} + Ly_{22},$$

wherein the y_{ik} are the second covariant derivatives of the scalar y with respect to the form ϕ_2.

The Christoffel symbols of the second kind for the quadratic form ϕ_2 are readily calculated to be defined by the formulas

$$\begin{aligned} 2d\begin{Bmatrix} 1 & 1 \\ & 1 \end{Bmatrix} &= 2LM(M/L)_u + (2M^2 - LN)[\log(LR/d)]_u - LM[\log(LR/d)]_v, \\ 2d\begin{Bmatrix} 2 & 2 \\ & 1 \end{Bmatrix} &= MN[\log(NR/d)]_v - LN[\log(LR/d)]_u, \\ 2d\begin{Bmatrix} 1 & 1 \\ & 2 \end{Bmatrix} &= L^2[\log(LR/d)]_v - LM[\log(LR/d)]_u - 2L^2(M/L)_u, \end{aligned} \tag{22}$$

* Fubini and Čech, *Geometria Proiettiva Differenziale*, Bologna, Zanichelli, 1926, pp. 84–87.

† Ibid., p. 87.

and by the substitution S.

In terms of the ordinary derivatives and the coefficients and variables of the intermediate form, the expression for ζ may be written

$$\zeta = \lambda y_u + \mu y_v + (\ )y - 2dz$$

wherein λ and μ are defined by S and the formula

$$2d\lambda = 2d(\alpha N - 2aM + \gamma L) + N(LN - 2M^2)[\log L/N]_u - LMN[\log L/N]_v - 2LMN(M/L)_u + 2LN^2(M/N)_v.$$

The point Z defined by the expression

$$2dZ = -\lambda y_u - \mu y_v + 2dz$$

is evidently on the projective normal. If we make the transformation

$$z = Z + \tfrac{1}{2}\lambda y_u/d + \tfrac{1}{2}\mu y_v/d \tag{23}$$

on the intermediate form, we find that (y, Z) are the solutions of differential equations of the form

$$\begin{aligned} y_{uu} &= \alpha y_u + \beta y_v + py + LZ, \\ y_{uv} &= ay_u + by_v + cy + MZ, \\ y_{vv} &= \gamma y_u + \delta y_v + qy + NZ, \\ Z_u &= \bar{m} y_u + \bar{s} y_v + \bar{f} y + \bar{A} Z, \\ Z_v &= \bar{l} y_u + \bar{n} y_v + \bar{g} y + \bar{B} Z, \end{aligned} \tag{24}$$

wherein the new coefficients α, β, γ, δ, a, b are defined by the formulas

$$\begin{aligned} 2d^2\alpha &= \tfrac{1}{4}L^2MNQ^{(u)} - \tfrac{1}{4}L^2N^2Q^{(v)} + d(2M^2 - LN)[\log (R/N)]_u \\ &\quad - dLM[\log (R/L)]_v - M^2(2M^2 - LN)[\log (L/N)]_u \\ &\quad - LM(2M^2 - LN)[\log (L/N)]_v + 2LM^2N(M/N)_v - 2LM^3(M/L)_u, \\ 2d\beta &= \tfrac{1}{4}L^2(LN - 2M^2)Q^{(u)} + \tfrac{1}{4}L^2MNQ^{(v)} - dLM[\log (R/N)]_u \\ &\quad - dL^2[\log (R/L)]_v + LM^2[\log (L/N)]_u \\ &\quad - L^2(LN - 2M^2)[\log (L/N)]_v + 2L^3N(M/L)_u - 2L^2MN(M/N)_v, \\ 2da &= \tfrac{1}{4}L^2NQ^{(u)} - \tfrac{1}{4}LMN^2Q^{(v)} + dMN[\log(R/N)]_u - dLN[\log (R/L)]_v \\ &\quad + MN(LN - 2M^2)[\log (L/N)]_u + LN(LN - 2M^2)[\log (L/N)]_v \\ &\quad - 2LM^2N(M/L)_u + 2LMN^2(M/N)_v, \end{aligned} \tag{25}$$

and by S. The coefficients p, c, q, L, M, N are unchanged by (23). The remaining eight coefficients may be found in terms of the twelve thus defined by solving the proper equations of (2a) for the system (24).

If, as the point y describes the curve $v=v(u)$, the line yZ generates a developable, the function $v(u)$ must satisfy the following differential equation:

$$\bar{s}du^2 + (\bar{n} - \bar{m})dudv - \bar{t}dv^2 = 0. \tag{26}$$

If $Z+\lambda y$ is the point of contact of yZ with the edge of regression of the developable, the function λ must satisfy the quadratic equation

$$\lambda^2 + (\bar{m} + \bar{n})\lambda + \bar{m}\bar{n} - \bar{s}\bar{t} = 0. \tag{27}$$

From (27) we see that the harmonic conjugate of y with respect to the focal points of yZ is the point z defined by

$$z = Z - \tfrac{1}{2}(\bar{m} + \bar{n})y.$$

If on system (24) we make the transformation

$$Z = z + \tfrac{1}{2}(\bar{m} + \bar{n})y \tag{28}$$

the coefficients $\bar{p}$, $\bar{c}$, $\bar{m}$, $\bar{f}$ are transformed according to the formulas

$$\begin{aligned} \mathbf{p} &= \bar{p} + \tfrac{1}{2}(\bar{m} + \bar{n}), \quad \mathbf{c} = \bar{c} + \tfrac{1}{2}(\bar{m} + \bar{n}), \\ \mathbf{m} &= \tfrac{1}{2}(\bar{m} - \bar{n}), \quad \mathbf{f} = \bar{f} + \tfrac{1}{2}\bar{A}(\bar{m} + \bar{n}) - \tfrac{1}{2}(\bar{m} + \bar{n})_u. \end{aligned} \tag{29}$$

The coefficients $\mathbf{q}$, $\mathbf{n}$ are obtained from (29) by S. The coefficients L, M, N, $\bar{A}$, $\bar{B}$ are unchanged by (28).

Since the projective lines of curvature defined by (26) form a conjugate system, it follows that

$$L\mathbf{t} + (\mathbf{n} - \mathbf{m})M - \mathbf{s}N = 0.$$

Therefore from the last of the integrability conditions (2b), we find that

$$\bar{A}_v - \bar{B}_u = 0.$$

If we now make the transformation

$$\mathbf{z} = e^{\mu}z,$$

wherein

$$\mu_u = \bar{A}, \quad \mu_v = \bar{B}, \tag{30}$$

the new system of equations will have $A=B=0$. Hence *system* (1) *may be reduced to the following canonical form*:

$$\begin{aligned} y_{uu} &= \alpha y_u + \beta y_v + py + Lz, \\ y_{uv} &= ay_u + by_v + cy + Mz, \\ y_{vv} &= \gamma y_u + \delta y_v + qy + Nz, \\ z_u &= my_u + sy_v + fy, \\ z_v &= ty_n + ny_v + gy. \end{aligned} \tag{31}$$

The coefficients of this system are expressible in terms of the coefficients and variables of system (1) by equations (13), (16), (25), (29) and (30), and by solving the proper equations of (2a). *The system* (31) *is characterized analytically by the following conditions*:

$$
(32)\quad
\begin{aligned}
&\text{(a)}\quad F_u = F_v = 0,\\
&\text{(b)}\quad 2d(\alpha N - 2aM + \gamma L) + N(LN - 2M^2)[\log (L/N)]_u\\
&\qquad\qquad - LMN[\log (L/N)]_v - 2LMN(M/L)_u + 2LN^2(M/N)_v,\\
&\text{(c)}\quad m + n = 0,\\
&\text{(d)}\quad A = B = 0,
\end{aligned}
$$

and by the counterpart of (b) *in* S.

Conditions (a) imply that the line $y_u\, y_v$ is the reciprocal of the projective normal; the conditions (b) and (a) imply that yz is the projective normal; condition (c) implies that z is the harmonic conjugate of y with respect to the focal points on the projective normal; and the conditions (d) imply that the line of intersection of the tangent planes at y and z to S_y and S_z respectively is the line $z_u z_v$. *It is therefore evident geometrically that the system* (31) *characterized by* (32) *is unchanged in form by the transformation*

$$y = c\bar{y},\quad z = c'\bar{z},\quad \bar{u} = \phi(u,v),\quad \bar{v} = \psi(u,v),$$

provided the transforms of **L** *and* **N** *are not zero.*

We may write system (31) with covariant derivatives as follows:

$$
(33)\quad
\begin{aligned}
y_{11} &= \mathfrak{A} y_1 + \mathfrak{B} y_2 + py + Lz,\\
y_{12} &= \mathfrak{a} y_1 + \mathfrak{b} y_2 + cy + Mz,\\
y_{22} &= \mathfrak{C} y_1 + \mathfrak{D} y_2 + qy + Nz,\\
z_u &= my_1 + sy_2 + fy,\\
z_v &= ty_1 - my_2 + gy,
\end{aligned}
$$

wherein the new coefficients $\mathfrak{A}$, $\mathfrak{B}$ etc. are defined by the following equations and by S:

$$
(34)\quad
\begin{aligned}
8d^2\mathfrak{A} &= L^2N(MQ^{(u)} - NQ^{(v)}),\\
8d^2\mathfrak{B} &= L^2MNQ^{(v)} + L^2(LN - 2M^2)Q^{(u)},\\
8d^2\mathfrak{a} &= L^2NQ^{(u)} - LMNQ^{(v)}.
\end{aligned}
$$

4. Geometrical interpretation of the invariant coefficients of the canonical form

We shall now give geometrical interpretations of the invariant coefficients of system (31). We see readily that if $\beta=0(\gamma=0)$ the curve v = const.

(u=const.) is a union curve of the congruence of projective normals. If $\beta=\gamma=0$ the axis congruence of the given parametric net coincides with the projective normal congruence. If $p=0$ ($q=0$) the tangent to v=const. (u=const.) on the surface $S_{y_u}(S_{y_v})$ generated by y_u (y_v) intersects the line joining y_v (y_u) to z.

The point y_{uv} is the point of intersection of the tangents to u=const. on S_{y_u} and the tangent to v=const. on S_{y_v}. Hence if $a=0$ ($b=0$) the point y_{uv} lies in the plane determined by the tangent to u=const. (v=const.) and the projective normal. If the parametric net is non-conjugate, the projective normal and its reciprocal with respect to the quadric of Lie will be in Green's relation R with respect to the given parametric net if and only if $a=b=0$. In case $M=0$, the parametric net is conjugate and the vanishing of a and b imply that the ray of the point y with respect to the parametric net is the reciprocal of the projective normal.*

If $m=0$ the tangents to u=const. and v=const. on S_z intersect respectively the tangents to v=const. and u=const. on S_y. If $s=0$ ($t=0$) the tangent to v=const. (u=const.) on S_z intersects the tangent to v=const. (u=const.) on S_y. If $s=t=M=0$, the developables of the projective normal congruence intersect S_y in the parametric curves. If $s=t=0$, $M\neq 0$, the projective normals pass through a point. If $f=0$ ($g=0$), the tangent to v= const. (u=const.) on S_z intersects the reciprocal of the projective normal.

We may easily show that the developables of the reciprocal of the projective normal congruence correspond to the curves defined by the differential equation

$$(35)\qquad (cL-pM)du^2+(qL-pN)dudv+(qM-cN)dv^2=0.$$

These curves have been called the *reciprocal projective lines of curvature.*† The focal points of the reciprocal of the projective normal are defined by the expression $y_u+\lambda y_v$ where λ satisfies the equation

$$(36)\qquad (cN-pM)\lambda^2+(pN-qL)\lambda+pM-cL=0.$$

The associate conjugate net of the parametric net is defined by the differential equation

$$(37)\qquad Ldu^2-Ndv^2=0.$$

The reciprocal projective lines of curvature therefore coincide with the associate conjugate curves if and only if

* Lane, *Nets*, p. 570.
† Lane, *Nets*, p. 570.

$$pN - qL = 0, \tag{38}$$

that is, if and only if the focal points of the reciprocal of the projective normal separate the points y_u and y_v harmonically.

Necessary and sufficient conditions that the ray of the point y with respect to the parametric net coincide with the reciprocal of the projective normal are*

$$\mu = a - \gamma M/N = 0, \quad \lambda = b - \beta M/L = 0. \tag{39}$$

The condition that the ray tangents separate the parametric tangents harmonically is the vanishing of a certain invariant† $\mathfrak{D}$ defined by the formula

$$\begin{aligned} N\mathfrak{D} = Np - Lq + L\mu^2 - N\lambda^2 + L\mu_v - N\lambda_u \\ + (\beta N - L\delta)\mu - (\gamma L - \alpha N)\lambda + M(\mu_u - \lambda_v). \end{aligned} \tag{40}$$

Hence if conditions (38) and (39) are satisfied, the invariant $\mathfrak{D}$ vanishes. Therefore *if the associate conjugate net of the given net is the reciprocal projective lines of curvature, and if the ray of the given net is the reciprocal of the projective normal, the given net has equal point invariants of the first kind*‡ *and the ray tangents separate the tangents to the curves of the given net harmonically*. In case the given net is conjugate we have the theorem of Lane:§ If the associate of the given (conjugate) net is the reciprocal projective lines of curvature, and if the ray of the given net is the reciprocal of the projective normal, the given net has equal Laplace-Darboux invariants and is moreover harmonic.

5. A canonical form for the defining differential equations of a non-conjugate net

Let us consider a non-conjugate net. Since $M \neq 0$, we may eliminate z from the first three of (31), obtaining the following system:

$$\begin{aligned} y_{uu} &= \mathfrak{A} y_{uv} + B y_u + \mathfrak{C} y_v + \mathfrak{D} y, \\ y_{vv} &= \mathfrak{A}' y_{uv} + \mathfrak{B}' y_u + C' y_v + \mathfrak{D}' y, \end{aligned} \tag{41}$$

wherein the coefficients $\mathfrak{A}$, B, $\cdots$ are defined by the formulas

$$\begin{aligned} &\mathfrak{A} = L/M, \quad B = \alpha - a\mathfrak{A}, \quad \mathfrak{C} = \beta - b\mathfrak{A}, \quad \mathfrak{D} = p - c\mathfrak{A}, \\ &\mathfrak{A}' = N/M, \ \mathfrak{B}' = \gamma - a\mathfrak{A}', \ C' = \delta - b\mathfrak{A}', \ \mathfrak{D}' = q - c\mathfrak{A}'. \end{aligned} \tag{42}$$

* V. G. Grove, *Transformations of nets*, these Transactions, vol. 30 (1928), p. 489.

† V. G. Grove, *Nets with equal W invariants*, these Transactions, vol. 31 (1929), pp. 846–847.

‡ Ibid., p. 847.

§ Lane, *Nets*, pp. 570–571.

The coefficients written with German letters are invariants of the net.

The functions (15) in terms of the coefficients (42) are

$$F_u = f_u - 2\frac{R_u'}{R'} + \frac{3}{2}\frac{\partial}{\partial u}\log(1 - \mathfrak{A}\mathfrak{A}'),$$

$$F_v = f_v - 2\frac{R_v'}{R'} + \frac{3}{2}\frac{\partial}{\partial v}\log(1 - \mathfrak{A}\mathfrak{A}'),$$

wherein

$$R' = R/M$$

and the functions f_u and f_v are the same as the like named functions used by Green.*

The conditions (32b) may be written in terms of the coefficients (42) as follows:

$$(43)\quad \begin{aligned} 2(1 - \mathfrak{A}\mathfrak{A}')a &= \mathfrak{A}'B + \mathfrak{A}\mathfrak{B}' + \mathfrak{A}_u' \\ &\quad + \frac{1}{2}\mathfrak{A}\mathfrak{A}'\left[\frac{\partial}{\partial v}\log(1 - \mathfrak{A}\mathfrak{A}') - \mathfrak{A}'\frac{\partial}{\partial u}\log(1 - \mathfrak{A}\mathfrak{A}')\right], \\ 2(1 - \mathfrak{A}\mathfrak{A}')b &= \mathfrak{A}C' + \mathfrak{A}'\mathfrak{C} + \mathfrak{A}_v \\ &\quad + \frac{1}{2}\mathfrak{A}\mathfrak{A}'\left[\frac{\partial}{\partial u}\log(1 - \mathfrak{A}\mathfrak{A}') - \mathfrak{A}\frac{\partial}{\partial v}\log(1 - \mathfrak{A}\mathfrak{A}')\right]. \end{aligned}$$

If therefore the points y_u, y_v are the points in which the reciprocal of the projective normal intersects the parametric tangents, the projective normal joins the point y to the point ζ defined by

$$(44)\quad \zeta = y_{uv} - ay_u - by_v.$$

In terms of covariant derivatives, system (41) assumes the form

$$(45)\quad \begin{aligned} y_{11} &= \mathfrak{A}y_{12} + \frac{\mathfrak{A}^2\mathfrak{A}'Q^{(u)}}{8(1 - \mathfrak{A}\mathfrak{A}')}y_1 + \frac{\mathfrak{A}^2(\mathfrak{A}'Q^{(v)} - 2Q^{(u)})}{8(1 - \mathfrak{A}\mathfrak{A}')}y_2 + \mathfrak{D}y, \\ y_{22} &= \mathfrak{A}'y_{12} + \frac{\mathfrak{A}'^2(\mathfrak{A}Q^{(u)} - 2Q^{(v)})}{8(1 - \mathfrak{A}\mathfrak{A}')}y_1 + \frac{\mathfrak{A}\mathfrak{A}'^2Q^{(v)}}{8(1 - \mathfrak{A}\mathfrak{A}')}y_2 + \mathfrak{D}y. \end{aligned}$$

The invariant coefficients of system (41) are readily interpreted. The line joining y to y_{uv} is the line in Green's relation R to the reciprocal of the projective normal. Hence if $\mathfrak{C} = 0$ ($\mathfrak{B}' = 0$) the osculating plane of $v = \text{const.}$ ($u = \text{const.}$) passes through the line yy_{uv}. If $\mathfrak{B}' = \mathfrak{C} = 0$ the projective normal and its reciprocal are the axis and ray of the point y with respect to the given

* G. M. Green, *Nets of space curves*, these Transactions, vol. 21 (1920), p. 212.

net and are therefore in Green's relation R to that net. If $\mathfrak{D}=0$ ($\mathfrak{D}'=0$) the tangent to $v=\text{const.}$ ($u=\text{const.}$) on the surface generated by y_u (y_v) intersects the the tangent to $v=\text{const.}$ ($u=\text{const.}$) on the surface generated by y_v (y_u). If $\mathfrak{D}=\mathfrak{D}'=0$, the reciprocals of the projective normals lie in a fixed plane.

MICHIGAN STATE COLLEGE,
EAST LANSING, MICH.