GENERALIZED DIFFERENTIATION*

BY

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Introduction

The history of the subject of generalized differentiation can be traced back to Leibnitz.† In the earlier literature the term fractional differentiation is used as an alternative, and the aim was to generalize the concept of the nth derivative of a function to non-integral values of n. Four different attacks on this problem may be noted. The earliest was that of Liouville, who expanded the functions operated upon in series of exponentials, and assumed, as a basis, $D^n e^{ax} = a^n e^{ax}$, where D symbolizes differentiation. Riemann considered power series with non-integral exponents as analogues of Taylor's series, and through their coefficients was led to the expression of generalized derivatives in terms of a definite integral plus an infinite series with arbitrary constant coefficients.‡ Liouville's and Riemann's results proved to be in disagreement. Prior to the publication of Riemann's work, Grünwald was led by the restrictions of Liouville's method to generalize directly the definition of a derivative as the limit of a finite difference quotient, and, by rigorous methods, also arrived at definite integral formulas.§ Grünwald's definition involved the idea of differentiation between limits which later resulted in the coördination of Liouville's and Riemann's results. This, for example, was effected by Krug,¶ who introduced a new development, based on Cauchy's contour integral for ordinary derivatives, which also involved limits of differentiation, in terms of which he showed that Riemann's definite integral corresponded to finite lower limit, Liouville's development to lower limit $-\infty$. The Riemann-Grünwald definite integral form has become standard in the literature, and has been intensively studied.||

* Presented to the Society, October 27, 1923; received by the editors June 29, 1929.

† For references, see S. Pincherle, *Equations et opérations fonctionnelles*, Encyclopédie des Sciences Mathématiques, Paris, 1912, tome 2, vol. 5, fasc. 1, pp. 1–81; also Eugene Stephens, *Symbolic calculus. Bibliography on general* (*or fractional*) *differentiation*, Washington University Studies, vol. 12 (1925), No. 2, pp. 137–152.

‡ B. Riemann, Gesammelte Mathematische Werke, Leipzig, 1876, pp. 331–344.

§ K. A. Grünwald, Zeitschrift für Mathematik und Physik, vol. 12 (1867), pp. 441–480.

¶ A. Krug, *$d^n f(x)/dx^n$ regarded as a function of n*, Akademie der Wissenschaften, Wien, Denkschriften, Mathematisch-Naturwissenschaftliche Klasse, vol. 57 (1890), pp. 151–228.

|| E.g. by A. Marchaud, *Sur les dérivées et sur les différences des fonctions de variables réelles*, Journal de Mathématiques, (9), vol. 6 (1927), pp. 337–425.

Another trend was introduced by the symbolic treatment of linear differential equations with constant coefficients as exemplified by the work of Boole. The polynomial operators thus occurring led naturally to the concept of an arbitrary operator $f(D)$, which was to be formally expanded in powers of D, and thus applied to the operand. This treatment has since been established on a rigorous basis for operators $f(D)$ corresponding to entire transcendental functions $f(z)$ of genus zero, operating upon functions which are analytic in a given region.* Boole's methods for linear differential equations with variable coefficients were extended by other writers to yield formal solutions in terms of operators algebraic in D; but in many cases no further attempt was made to assign a meaning to such expressions.

We may think of the more recent work of Heaviside as the next step in this development.† Of course Heaviside's contribution assumes an importance far beyond this formal juggling of symbols, through its application to important physical problems, and its skillful methods for evaluating the operations that are used. This "operational calculus," however, was developed with physical intuition, rather than mathematical rigor, as guide. A more rigorous mathematical basis has since been supplied by Carson in terms of solutions of Laplace integral equations, and their use in a definite integral formula.‡ Carson's formulas include the Riemann-Grünwald definite integral for D^n as a special case when the real part of n is less than one; but his treatment is quite unrelated to the theory of entire operators of genus zero.§

In the present paper Grünwald's method of arriving at a definition for operators D^n is carried forward by means of an artifice of Arbogast to yield a definition of generalized differentiation for operators $f(D)$.¶ This definition is shown to include Carson's operators, and entire operators of genus zero, as special cases.|| The major part of the paper is devoted to operators $f(D)$,

* C. Bourlet, *Sur les opérations en général et les équations linéaires différentielles d'ordre infini*, Annales de l'Ecole Normale, (3), vol. 33 (1897), pp. 133–190.

J. F. Ritt, *On a general class of linear homogeneous differential equations of infinite order with constant coefficients*, these Transactions, vol. 18 (1917), pp. 27–49.

† Oliver Heaviside, *Electromagnetic Theory*, London, 1922, vol. 2, chapters 7, 8.

‡ J. R. Carson, *The Heaviside operational calculus*, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 43–68; also numerous papers in the Bell System Technical Journal.

§ Another theory is developed by Norbert Wiener: *The operational calculus*, Mathematische Annalen, vol. 95 (1926), pp. 557–584.

¶ Our definition may therefore be called the extended Grünwald definition.

|| This is not strictly correct as far as Carson's development is concerned, since we derive his formulas only under certain hypotheses on the functions involved. It should be noted, however, that Carson does not explicitly state the domain of applicability of these formulas, and that the hypotheses in question are of considerable generality.

termed of type zero, which correspond to functions of a complex variable, $f(z)$, which are analytic in a certain sector of the z-plane of angle greater than π, and whose moduli satisfy within this sector the Poincaré inequality for entire functions of genus zero.* Existence theorems are established for such operators, and certain formal properties are investigated, such as the law of successive operations, and the generalized Leibnitz theorem for differentiation of a product.† The last three sections are chiefly devoted to operators given by a Laplace integral. For these we have only established the existence theorem, and investigated the application to Carson's development where not $f(z)$, but $f(z)/z$, is given by a Laplace integral.

Next to the definition of generalized differentiation itself, the writer wishes to call attention to the associated operator $A[f](t)$, defined by

$$A[f](t) = \lim_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} \frac{(-1)^r f^{(r)}(1/\Delta x)}{r!\Delta x^{r+1}} \cdot$$

The existence of this limit, for the cases considered, serves as the basis of our theory. It possesses the notable property of inverting the Laplace transformation, and enjoys numerous formal relations which flow directly from its definition. For $f(D)$ of type zero, it can be expressed by a contour integral, which is the Fourier integral with the vertical line contour replaced by two half-lines running into the negative half of the z-plane. The exponential factor of the integrand thus acquires a potency for convergence which should be of considerable use in most practical applications, where the loss of generality incurred by requiring analyticity in a sector of angle greater than π, as opposed to analyticity in a half-plane, is irrelevant, due to the presence of but a finite number of singularities.

In order to bring this paper to a conclusion, many topics have been excluded either because they involve an extension of our definition, or because they depend on the formulas flowing from our definition, rather than on the definition itself. Among these omitted topics are an extension of our definition to complex limits of differentiation, the application to Volterra's functions of composition of the closed cycle group, and the derivation of the Heaviside expansions.

* By thus breaking away from the classic assumption of analyticity in a half-plane, the restriction on the modulus of $f(z)$ is so greatly lightened (see footnote following proof of Theorem I) that we are able to include all operators $f(D)$ for which $f(z)$ is an entire function of genus zero, as well as all operators for which $f(z)$ is algebraic. The half-plane assumption would exclude these classes, as such, and would also hinder the theory by not permitting indiscriminate differentiation with respect to the upper limit.

† This treatment of the extended Grünwald definition follows very closely Grünwald's development of the original definition.

1. The definition. Our definition of generalized differentiation grows out of the expression of the nth derivative of a function as the limit of its nth difference quotient. For our purpose, this expression is best written in the form

$$D^n\phi(x) = \lim_{\Delta x\to 0} \frac{\phi(x) - n\phi(x-\Delta x) + [n(n-1)/2!]\phi(x-2\Delta x) - \cdots + (-1)^n\phi(x-n\Delta x)}{\Delta x^n},$$

where Δx is the negative of the increment of x as ordinarily defined. Since the right hand member of this equation continues to have meaning when n is not a positive integer, it can be used to define $D^n\phi(x)$ for arbitrary n.

When $n=-1$, the suggested definition becomes

$$D^{-1}\phi(x) = \lim_{\Delta x\to +0} [\phi(x)\Delta x + \phi(x-\Delta x)\Delta x + \phi(x-2\Delta x)\Delta x + \cdots],$$

i.e., the limit of an infinite series. If, however, we arbitrarily terminate the series at the $(p+1)$st term, where $x_0 < x - p\Delta x \leqq x_0 + \Delta x$, the result, at least for $\phi(x)$ continuous, will be the definite integral of $\phi(x)$ with finite lower limit x_0. Replacing the upper limit by X, we shall use the notation $\{D^{-1}\}_{x_0}^{X}\phi(x)$ for the limit of the finite sum, $\{D^{-1}\}[_{-\infty}^{X}\phi(x)$ for the limit of the infinite series. Similarly for D^n, n arbitrary.*

To extend this definition still further, let us momentarily introduce the operator E^m, defined by the relation $E^m\phi(x) = \phi(x+m)$. The reader will have seen the analogy between the above expression for $D^n\phi(x)$ and the binomial series expansion. Through the operator E^m, this analogy becomes formal, and we can write, apart from refinements,

$$D^n\phi(x) = \lim_{\Delta x\to 0}\left(\frac{1-E^{-\Delta x}}{\Delta x}\right)^n\phi(x) = \lim_{\Delta x\to 0}\left(\frac{\Delta}{\Delta x}\right)^n\phi(x).$$

This immediately suggests the desired definition of $f(D)\phi(x)$, i.e.,

$$f(D)\phi(x) = \lim_{\Delta x\to 0} f\left(\frac{\Delta}{\Delta x}\right)\phi(x) = \lim_{\Delta x\to 0} f\left(\frac{1-E^{-\Delta x}}{\Delta x}\right)\phi(x),$$

where $f(1/\Delta x - E^{-\Delta x}/\Delta x)$ is to be expanded by Taylor's series, and formally applied to $\phi(x)$. As in the case of D^{-1} the limit of the infinite series will be written $\{f(D)\}_{-\infty}^{X}\phi(x)$. In the case of finite lower limit x_0, the choice of p through the inequalities $x_0 < X - p\Delta x \leqq x_0 + \Delta x$ allows Δx to approach zero independently of $X - x_0$, except for sign. p is then the largest integer less

* Up to this point the writer essentially retraces Grünwald's argument.

than $(X-x_0)/\Delta x$. We shall use the notation $p=\{(X-x_0)/\Delta x\}$. Though otherwise arbitrary, Δx must have the same sign as $X-x_0$. In this paper Δx will approach zero positively, that is, X will be greater than x_0. Our completed definitions thus become

$$(1)\quad \{f(D)\}_{x_0}^{X}\phi(x) = \lim_{\Delta x\to+0}\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x) = \lim_{\Delta x\to+0}\left\{f\left(\frac{1-E^{-\Delta x}}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x)$$

$$= \lim_{\substack{\Delta x\to+0\\ p=\{(X-x_0)/\Delta x\}}}\left[f\left(\frac{1}{\Delta x}\right)\phi(X) - \frac{f'\left(\frac{1}{\Delta x}\right)}{1!\Delta x}\phi(X-\Delta x) + \cdots + \frac{(-1)^p f^{(p)}\left(\frac{1}{\Delta x}\right)}{p!\Delta x^p}\phi(X-p\Delta x)\right],$$

$$(2)\quad \{f(D)\}_{-\infty}^{X}\phi(x) = \lim_{\Delta x\to+0}\left[f\left(\frac{1}{\Delta x}\right)\phi(X) - \frac{f'\left(\frac{1}{\Delta x}\right)}{1!\Delta x}\phi(X-\Delta x) + \cdots\right].$$

The existence of the limits involved in these definitions is proved for various classes of operators in subsequent sections of the paper. In certain simple cases they can be evaluated directly. For example, consider $f(D) = \log D$, $\phi(x)\equiv 1$, and x_0 finite. Then by (1)

$$\left\{\log\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}1 = \log\left(\frac{1}{\Delta x}\right) - 1 - \frac{1}{2} - \cdots - \frac{1}{p}$$

$$= -\left[1+\frac{1}{2}+\cdots\frac{1}{p} - \log p\right] - \log(p\Delta x).$$

As $\Delta x\to+0$, p increases indefinitely, while $p\Delta x\to X-x_0$. Furthermore, as $p\to\infty$, the bracket has for limit γ, the Eulerian constant. We thus find

$$(3)\qquad \{\log(D)\}_{x_0}^{X}1 = -\gamma - \log(X-x_0).$$

Besides proving existence theorems, with their associated formulas, we shall be concerned with the verification of the formal laws of generalized differentiation for our definitions. Of these the most important is the law of successive operations. This can be immediately verified for the finite difference operators on which our generalized derivatives are based. Assuming the necessary number of derivatives of f_1 and f_2, we have

$$\left\{f_1\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\left\{f_2\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{x}\phi(\xi) = f_1\left(\frac{1}{\Delta x}\right)\left[f_2\left(\frac{1}{\Delta x}\right)\phi(X)\right.$$

$$\left. - \frac{f_2'\left(\frac{1}{\Delta x}\right)}{1!\Delta x}\phi(X-\Delta x) + \cdots + \frac{(-1)^p f_2^{(p)}\left(\frac{1}{\Delta x}\right)}{p!\Delta x^p}\phi(X-p\Delta x)\right]$$

$$-\frac{f_1'\left(\frac{1}{\Delta x}\right)}{1!\Delta x}\left[f_2\left(\frac{1}{\Delta x}\right)\phi(X-\Delta x) - \cdots + \frac{(-1)^{p-1} f_2^{(p-1)}\left(\frac{1}{\Delta x}\right)}{(p-1)!\Delta x^{p-1}}\phi(X-p\Delta x)\right]\cdot$$

$$+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$+ \frac{(-1)^p f_1^{(p)}\left(\frac{1}{\Delta x}\right)}{p!\Delta x^p}\left[f_2\left(\frac{1}{\Delta x}\right)\phi(X-p\Delta x)\right]$$

$$= \left[f_1\left(\frac{1}{\Delta x}\right)f_2\left(\frac{1}{\Delta x}\right)\right]\phi(X)$$

$$- \frac{\left[f_1\left(\frac{1}{\Delta x}\right)f_2'\left(\frac{1}{\Delta x}\right) + f_1'\left(\frac{1}{\Delta x}\right)f_2\left(\frac{1}{\Delta x}\right)\right]\cdot}{1!\Delta x}\phi(X-\Delta x)$$

$$+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$+ (-1)^p\frac{\left[f_1\left(\frac{1}{\Delta x}\right)f_2^{(p)}\left(\frac{1}{\Delta x}\right) + \frac{p}{1!}f_1'\left(\frac{1}{\Delta x}\right)f_2^{(p-1)}\left(\frac{1}{\Delta x}\right) + \cdots + f_1^{(p)}\left(\frac{1}{\Delta x}\right)f_2\left(\frac{1}{\Delta x}\right)\right]}{.p!\Delta x^p}$$

$$\cdot\ \phi(X-p\Delta x),$$

so that, by Leibnitz's theorem and our definition, we obtain

$$\left\{f_1\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\left\{f_2\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{x}\phi(\xi) = \left\{f_1\left(\frac{\Delta}{\Delta x}\right)f_2\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x). \tag{4}$$

The character of definition (1) is more apparent in the following form:

$$\begin{aligned} &\{f(D)\}_{x_0}^{X}\phi(x) = \lim_{\substack{\Delta x\to+0 \\ p=\{(X-x_0)/\Delta x\}}} \sum_{r=0}^{p} A[f](r,\Delta x)\phi(X-r\Delta x)\Delta x; \\ &A[f](r,\Delta x) = \frac{(-1)^r f^{(r)}\left(\frac{1}{\Delta x}\right)}{r!\Delta x^{r+1}}\cdot \end{aligned} \tag{5}$$

(5) would lead directly to a definite integral if in place of $A[f](r, \Delta x)$ we had a function of $r\Delta x$. The way in which this difficulty can be partly overcome may be indicated by means of the operator D^n, n arbitrary. Using the notation $A[f(u)](r, \Delta x)$ when the form of f is specified, we have, in this case,

$$A[u^n](r, \Delta x) = \frac{(-1)^r n(n-1)\cdots(n-r+1)}{r!\Delta x^{n+1}}$$
$$= \frac{(-n)(-n+1)\cdots(-n+r-1)}{r!r^{-n-1}}(r\Delta x)^{-n-1}.$$

The Gaussian form of the gamma function gives

$$\lim_{r\to\infty}\frac{(-n)(-n+1)\cdots(-n+r-1)}{r!r^{-n-1}} = \frac{1}{\Gamma(-n)}\cdot$$

If at the same time $\Delta x\to +0$ in such a way that $r\Delta x\to t$, $t>0$, it will follow that $A[u^n]$ $(r, \Delta x)$ will approach a function of t as limit. Symbolizing this function by $A[u^n](t)$, we thus find

(6) $$A[u^n](t) = \lim_{\substack{\Delta x\to +0\\ r\Delta x\to t}} A[u^n](r, \Delta x) = \frac{t^{-n-1}}{\Gamma(-n)}\cdot$$

More generally we shall look for the existence of a limit function $A[f](t)$ corresponding to $A[f](r, \Delta x)$. When such a limit exists, then for Δx sufficiently small, and $r\Delta x$ greater than some positive ϵ, $A[f](r, \Delta x)$ will be approximated by $A[f](r\Delta x)$; and so, (5) will lead in part to a definite integral. Nevertheless, the restriction $r\Delta x>\epsilon$ leaves terms of (5) with small r unaccounted for. These will usually require separate treatment, where, however, the $A[f](t)$'s of operators connected with $f(D)$ will be used. The first step in our theory is therefore the establishing of the existence of $A[f](t)$ for a sufficiently wide class of operators $f(D)$. We pause a moment to consider a certain limit criterion which will unify the last stages of many of our proofs.

2. Limit criterion. *If $F(\Delta x)$ can be expressed as a sum $G(\Delta x, \nu)+H(\Delta x, \nu)$, ν independent of Δx, such that*

$$\lim_{\Delta x\to +0} G(\Delta x, \nu) = g(\nu),\ \limsup_{\Delta x\to +0} |H(\Delta x, \nu)| \leqq h(\nu),\ \lim_{\nu\to\nu_0} h(\nu) = 0,$$

then $\lim_{\Delta x\to +0}F(\Delta x)$ *exists, and is given by*

(7) $$\lim_{\Delta x\to +0} F(\Delta x) = \lim_{\nu\to\nu_0} g(\nu).$$

In fact the stated conditions show that

$$\limsup_{\Delta x \to +0} F(\Delta x) \leqq g(\nu) + h(\nu), \qquad \liminf_{\Delta x \to +0} F(\Delta x) \geqq g(\nu) - h(\nu).$$

As these upper and lower limits are independent of ν, we find, by letting $\nu \to \nu_0$, and observing that $h(\nu) \to 0$,

$$\limsup_{\Delta x \to +0} F(\Delta x) = \liminf_{\Delta x \to +0} F(\Delta x) = \lim_{\nu \to \nu_0} g(\nu).$$

Since for any one ν the values of $g(\nu)+h(\nu)$ and $g(\nu)-h(\nu)$ are finite, the above inequalities yield the

COROLLARY. *Under the given hypothesis* $\lim_{\Delta x \to +0} F(\Delta x)$, *and hence also* $\lim_{\nu \to \nu_0} g(\nu)$, *is finite.**

3. **Existence of $A[f](t)$ for $f(D)$ of type zero.** An operator $f(D)$, corresponding to a function of a complex variable $f(z)$, will be said to be of *type zero* when $f(z)$ satisfies the following two conditions:

(a) $f(z)$ *is analytic in a sector of the z-plane of angle greater than* π *bisected by the positive direction of the axis of reals,*

(b) *for each real and positive* κ, *however small, there corresponds a real and positive* K, *such that*

$$| f(z) | \leqq K e^{\kappa |z|},$$

for every z in the analytic sector.

We shall designate the positive acute angle made by the sides of the sector with the negative direction of the axis of reals by α.

Operators of finite order, defined later, which include all operators whose $f(z)$ is algebraic, are also of type zero. Other examples are e^{aD^m}, $m<1$, and operators for which $f(z)$ is an entire function of genus zero.

We shall now prove the fundamental

THEOREM I. *If $f(D)$ is of type zero, $A[f](r, \Delta x)$ approaches a finite limit as $\Delta x \to +0$, $r\Delta x \to t$, for every real and positive t. The resulting function of t, $A[f](t)$, is given by the formula*

$$A[f](t) = \frac{1}{2\pi i}\int_C e^{tz} f(z)\,dz, \tag{8}$$

where C is formed by two rays within the analytic sector and parallel to its edges, with common end point on the axis of reals, and traversed so that, along it, the imaginary part of z increases.

The proof is based on Cauchy's second integral formula. For Δx sufficiently small, $1/\Delta x$ will be in the analytic sector posited by condition (a).

* Henceforth, existence will include finiteness.

If C''' is a simple contour about $1/\Delta x$, traversed counterclockwise, and also contained in this sector, then, by the formula in question, we shall have

$$f^{(r)}\left(\frac{1}{\Delta x}\right) = \frac{r!}{2\pi i}\int_{C'''}\frac{f(z)dz}{\left(z - \dfrac{1}{\Delta x}\right)^{r+1}};$$

and so, by the definition of $A[f](r, \Delta x)$ given in (5),

$$A[f](r, \Delta x) = -\frac{1}{2\pi i}\int_{C'''}\frac{f(z)dz}{(1 - z\Delta x)^{r+1}}.$$

Assuming Δx sufficiently small, we can choose for C''' the contour formed by C'', an arc of a circle, center the origin, radius $k/\Delta x$, $k>1$, joined to C', the finite portion of C cut off by this circle. If C'' is traversed in the same direction as C''', but C' in the opposite direction, i.e., in the same direction as C, we can write schematically

$$A[f](r, \Delta x) = -\frac{1}{2\pi i}\int_{C''} + \frac{1}{2\pi i}\int_{C'}.$$

Along C'' we have $|z| = k/\Delta x$. Hence $|1 - z\Delta x| \geqq k - 1$, so that, by inequality (b), we find

$$\left|\int_{C''}\frac{f(z)dz}{(1 - z\Delta x)^{r+1}}\right| < \frac{K\cdot 2\pi k}{\Delta x(k - 1)}\left[\frac{e^{k\kappa/(r\Delta x)}}{k - 1}\right]^{r}.$$

Choose $k>2$, and then κ so that $e^{k\kappa/t}/(k-1)<1$. The right hand member of the inequality will then approach zero as limit as $\Delta x \to +0, r\Delta x \to t$. The C'' contribution can therefore be neglected.

Now break up C' into $C_{l,m}$ and $C'_{l,m}$, where $C_{l,m}$ extends distances l and m respectively along the two segments of C' from their meeting point, while $C'_{l,m}$ consists of the rest of C'. For l and m sufficiently large, $R(z)$, the real part of z, will be negative along $C'_{l,m}$, and we shall have

$$\left|\int_{C'_{l,m}}\frac{f(z)dz}{(1 - z\Delta x)^{r+1}}\right| < K\int_{C'_{l,m}}\frac{e^{\kappa|z|}\,|dz|}{[1 - R(z)\Delta x]^{r+1}}.$$

If we set

$$1 - R(z)\Delta x = e^{-\lambda R(z)\Delta x},$$

we observe that λ stays positive and decreases monotonically as $-R(z)\Delta x$ increases. Now the largest value of $-R(z)\Delta x$ along any one $C'_{l,m}$ corresponds to $|z| = k/\Delta x$. As $\Delta x \to +0$ this largest value approaches $k \cos \alpha$ as limit. Hence for all Δx's sufficiently small, and all corresponding $C'_{l,m}$'s, $-R(z)\Delta x$

remains less than some fixed positive quantity. The corresponding λ's therefore have a positive lower bound λ_0. Since $\lambda \geqq \lambda_0 > 0$, we shall thereby have

$$1 - R(z)\Delta x > e^{-\lambda_0 R(z)\Delta x},$$

and so

$$\left| \int_{C'_{l,m}} \frac{f(z)dz}{(1 - z\Delta x)^{r+1}} \right| < K \int_{C'_{l,m}} e^{\kappa|z| + \lambda_0 (r+1)\Delta x \cdot R(z)} \, | dz | .$$

Since $R(z)/|z| \to -\cos\alpha$, as $|z| \to \infty$, if we choose κ less than $\lambda_0 t \cos\alpha$ the last integral will be less than

$$\int_{C'_{l,m}} e^{-\mu|z|} \, | dz | ,$$

with fixed positive μ, for l and m sufficiently large, and $(r+1)\Delta x$ sufficiently near t. The contour $C'_{l,m}$ depends on Δx. If we replace it by $\overline{C}'_{l,m}$, which consists of C with $C_{l,m}$ removed, and of which $C'_{l,m}$ is a part, we observe that the resulting integral converges, and so approaches zero as limit as l and m increase indefinitely. We therefore have

$$\limsup_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} \left| \int_{C'_{l,m}} \frac{f(z)dz}{(1 - z\Delta x)^{r+1}} \right| \leqq K \int_{\overline{C}'_{l,m}} e^{-\mu|z|} \, | dz | , \quad \lim_{\substack{l \to \infty \\ m \to \infty}} \int_{\overline{C}'_{l,m}} e^{-\mu|z|} \, | dz | = 0.$$

Finally $f(z)/(1-z\Delta x)^{r+1}$ uniformly approaches $e^{tz}f(z)$ along $C_{l,m}$. Hence

$$\lim_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} \int_{C_{l,m}} \frac{f(z)dz}{(1 - z\Delta x)^{r+1}} = \int_{C_{l,m}} e^{tz} f(z) dz.$$

We can therefore apply our limit criterion,* and obtain

$$\lim_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} \int_{C'} \frac{f(z)dz}{(1 - z\Delta x)^{r+1}} = \lim_{\substack{l \to \infty \\ m \to \infty}} \int_{C_{l,m}} e^{tz} f(z) dz = \int_{C} e^{tz} f(z) dz.$$

Since we saw that the integral along C'' could be neglected, this establishes both the existence of $A[f](t)$, and the formula given for it.†

* This was stated for one independent variable Δx. It can obviously be extended to any number of independent variables, in this case two.

† If the analyticity condition imposed on $f(z)$ be weakened to analyticity in the half-plane to the right of some line $R(z) = a$, while within that half-plane the modulus of $f(z)$ satisfies the far stronger inequality

$$| f(x) | \leqq K / | z |^{1+\epsilon}, \; \epsilon > 0,$$

then essentially the same proof will yield the existence of $A[f](t)$, and its expression by means of a Fourier integral. Furthermore, to anticipate the later developments, the argument leading to (48) in §11 can be duplicated in this case, so that our methods would yield the classic solution of the Laplace integral equation.

By real-variable methods, it can be proved that if $A[f](t)$ exists for each t in a closed interval (t_1, t_2), then, without any further hypothesis on the function f, or even on the form of $A[f](r, \Delta x)$, the following consequences hold:

(a) $A[f](t)$ is continuous in the closed interval (t_1, t_2);

(b) for each positive ϵ however small, there corresponds a positive η, such that, for $\Delta x < \eta$,

$$| A[f](r, \Delta x) - A[f](r\Delta x) | < \epsilon$$

for all r's for which $r\Delta x$ is in the interval (t_1, t_2).*

From our definition we have, for $x_0 < x_1 < X$,

$$\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x) = \left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_1}^{X}\phi(x) + \sum_{r=q+1}^{p} A[f](r, \Delta x)\phi(X - r\Delta x)\Delta x,$$

where $p = \{(X - x_0)/\Delta x\}$, $q = \{(X - x_1)/\Delta x\}$. If we set $X - r\Delta x = x$, then, as r varies from $q+1$ to p, x stays in the closed interval (x_0, x_1), while $r\Delta x$ stays in the interval $(X - x_1, X - x_0)$. Suppose then that $\phi(x)$ is continuous in (x_0, x_1), while $A[f](t)$ exists in $(X - x_1, X - x_0)$. Then by (b), $A[f](r, \Delta x)$ can be replaced by $A[f](r\Delta x)$, i.e., by $A[f](X - x)$, in the above sum. But by (a), $A[f](X - x)$ will be continuous in x in the interval (x_0, x_1). The new sum will therefore lead to a definite integral as $\Delta x \to +0$. Hence

THEOREM II. *If $A[f](t)$ exists in the interval $(X - x_1, X - x_0)$, where $x_0 < x_1 < X$ and if $\phi(x)$ is continuous in the interval (x_0, x_1), then*

$$\{f(D)\}_{x_0}^{X}\phi(x) = \{f(D)\}_{x_1}^{X}\phi(x) + \int_{x_0}^{x_1} A[f](X - x)\phi(x)dx, \tag{9}$$

provided either of the indicated operations exist.

When $f(D)$ is of type zero $A[f](t)$ exists for all positive t's. If then $\phi(x)$ is continuous in (x_0, X), (9) will be valid with x_1 anywhere in this interval.

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* The proof runs as follows: Choose any positive ϵ. For each point t' of the interval (t_1, t_2), $A[f](t')$ exists, and is defined as $\lim A[f](r, \Delta x)$ as $\Delta x \to +0$, $r\Delta x \to t'$. Hence for each t' of (t_1, t_2) there is a positive η' such that $|A[f](r, \Delta x) - A[f](t')| < \epsilon/2$ provided $\Delta x < \eta'$ and $|r\Delta x - t'| < \eta'$. By letting $\Delta x \to +0$ and $r\Delta x \to t$ we obtain $|A[f](t) - A[f](t')| \leqq \epsilon/2$ provided $|t - t'| < \eta'$ and t is in (t_1, t_2). Hence $A[f](t)$ is continuous at every point t' of (t_1, t_2).

Consider now the open intervals $(t' - \eta', t' + \eta')$ thus associated with the above pairs (t', η'). Every point of (t_1, t_2) is in fact the midpoint of such an interval. Hence by the Heine-Borel theorem a finite number of these intervals suffice to cover (t_1, t_2). Each interval uniquely determines the corresponding t' and η'. Let η be the smallest of the η''s of this finite set of intervals, and let $\Delta x < \eta$. Any $r\Delta x$ in (t_1, t_2) will be in one of these intervals. Since, for the (t', η') of this interval, $\Delta x < \eta \leqq \eta'$ and $|r\Delta x - t'| < \eta'$, we will have $|A[f](r, \Delta x) - A[f](t')| < \epsilon/2$ and $|A[f](r\Delta x) - A[f](t')| \leqq \epsilon/2$. Hence $|A[f](r, \Delta x) - A[f](r\Delta x)| < \epsilon$. That is, this inequality holds for every r and Δx for which $\Delta x < \eta$ and $r\Delta x$ is in (t_1, t_2).

The suggestion made at the end of §1 is thus verified. Before we see how to deal with $\{f(D)\}_{x_0}^{X}\phi(x)$, we shall consider $\phi(x)$ a polynomial. In this case, $\{f(D)\}_{x_0}^{X}\phi(x)$ can be immediately evaluated.

4. **Finite difference reduction formula; $\phi(x)$ a polynomial, unity.** For the formula about to be derived it is essential to distinguish between $\Delta\phi(x)/\Delta x$, which in this paper is written $[\phi(x)-\phi(x-\Delta x)]/\Delta x$, and $\{\Delta/\Delta x\}_{x_0}^{x}\phi(\xi)$ which, while the same as $\Delta\phi(x)/\Delta x$ for $x-x_0>\Delta x$, is but $\phi(x)/\Delta x$ for $x-x_0\leqq\Delta x$. While (4) would hold with $\{\Delta/\Delta x\}_{x_0}^{x}\phi(\xi)$ for $\{f_2(\Delta/\Delta x)\}_{x_0}^{x}\phi(\xi)$, it does not hold with $\Delta\phi(x)/\Delta x$ in place of $\{\Delta/\Delta x\}_{x_0}^{x}\phi(\xi)$. Instead, we obtain by the same method

$$\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\frac{\Delta\phi(x)}{\Delta x}=\left\{\frac{\Delta}{\Delta x}f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x)-A\,[f(u)](p,\Delta x)\phi(x_0');$$

$$p=\left\{\frac{X-x_0}{\Delta x}\right\},\quad x_0'=X-(p+1)\Delta x.^*$$

It will be noticed that x_0' does not vary with X as long as X changes by multiples of Δx. Also $x_0-\Delta x<x_0'\leqq x_0$. If we replace $f(u)$ by $u^{-1}f(u)$ and rearrange its terms, this formula becomes

$$\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x)=\left\{\left(\frac{\Delta}{\Delta x}\right)^{-1}f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\frac{\Delta\phi(x)}{\Delta x}+A\,[u^{-1}f(u)](p,\Delta x)\phi(x_0').$$

By induction, we are thus led to the fundamental formula

$$\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x)=\left\{\left(\frac{\Delta}{\Delta x}\right)^{-m}f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\frac{\Delta^m\phi(x)}{\Delta x^m}+\sum_{\mu=0}^{m-1}A\,[u^{-\mu-1}f(u)](p,\Delta x)\frac{\Delta^\mu\phi(x_0')}{\Delta x^\mu}.\tag{10}$$

It will be referred to as the finite difference reduction formula.

If $P(x)$ is a polynomial of degree n, we have

$$\frac{\Delta^{n+1}P(x)}{\Delta x^{n+1}}\equiv 0.$$

Let then $\phi(x)=P(x)$, $m=n+1$, in (10). It becomes

$$\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}P(x)=\sum_{\mu=0}^{n}A\,[u^{-\mu-1}f(u)](p,\Delta x)\frac{\Delta^\mu P(x_0')}{\Delta x^\mu}.$$

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* The definition of $\{f(\Delta/\Delta x)\}_{x_0}^{X}\phi(x)$ only requires $\phi(x)$ to be defined in the interval (x_0, X), whereas both members of this formula use values of x less than x_0, when $(X-x_0)/\Delta x$ is not an integer. We must therefore arbitrarily define $\phi(x)$ for a suitable interval beyond x_0 to render the formula applicable. The validity of the formula, however, does not depend on the particular way in which this prolongation is effected.

If we now let $\Delta x \to +0$, it will follow that $p\Delta x \to X - x_0$, and $x_0' \to x_0$. Hence *if* $A[u^{-\mu-1}f(u)](X-x_0)$ *exists for* $\mu = 0, 1, \cdots, n$, $\{f(D)\}_{x_0}^{X}P(x)$ *will also exist, and will be given by*

$$(11) \qquad \{f(D)\}_{x_0}^{X}P(x) = \sum_{\mu=0}^{n} A[u^{-\mu-1}f(u)](X - x_0)P^{(\mu)}(x_0).$$

For $P(x) \equiv 1$, we can use $n = 0$, and so obtain

$$\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X} 1 = A[u^{-1}f(u)](p, \Delta x).$$

Hence, *if* $A[u^{-1}f(u)](X-x_0)$ *exists*, $\{f(D)\}_{x_0}^{X}1$ *exists, and is given by*

$$(12) \qquad \{f(D)\}_{x_0}^{X}1 = A[u^{-1}f(u)](X - x_0).$$

It is easily verified that if $f(D)$ is of type zero then $D^{-\mu-1}f(D)$ also is of type zero. Hence the existence theorem for $\phi(x)$ a polynomial is completely established for operators of type zero.

The relation (12) is of special interest and importance. Note that in spite of the finite difference relation preceding it, we cannot conclude that if $\{f(D)\}_{x_0}^{X}1$ exists, $A[u^{-1}f(u)](X-x_0)$ also will exist; for in the former p depends on Δx, while for the latter $p\Delta x$ should vary independently of Δx.

5. **Operators of finite order; existence theorems.** In establishing existence theorems for $\{f(D)\}_{x_0}^{X}\phi(x)$, we shall find that the wider the class of operators $f(D)$ we consider, the greater the restrictions we have to impose on $\phi(x)$. Hence the following specialization of operators of type zero.

An operator $f(D)$ will be said to be of *finite order* ρ, ρ zero or a positive integer, when

(a) $f(z)$ *satisfies the analyticity condition for operators of type zero,*

(b) *there exists a positive* ϵ, *and corresponding* K, *such that*

$$|f(z)| \leqq K|z|^{\rho-\epsilon}$$

for every z *in the analytic sector for which* $|z| > \delta$, $\delta > 0$.

According to this definition, if an operator is of order ρ, it is also of any order greater than ρ.

D^n, n arbitrary, is a typical example. All algebraic operators are of finite order. It is evident that operators of finite order are also of type zero. Hence §3 is immediately applicable.

We shall first derive an inequality for $A[f](r, \Delta x)$ which is essential for our existence proofs. With the notation of §3 we obtain by the new condition (b)

$$\left|\int_{C''}\frac{f(z)dz}{(1-z\Delta x)^{r+1}}\right| < 2\pi K\left(\frac{k}{\Delta x}\right)^{\rho+1-\epsilon}/(k-1)^{r+1},\ k/\Delta x > \delta.$$

For fixed Δx, and $r\geqq\rho$, this expression approaches zero as limit as k increases indefinitely, i.e., as the radius of C'' is made to increase indefinitely. We can therefore replace C''' by C, the infinite contour of §3, and write, for $r\geqq\rho$,

$$A[f](r,\Delta x) = \frac{1}{2\pi i}\int_C\frac{f(z)dz}{(1-z\Delta x)^{r+1}}\cdot$$

Choose C so that its vertex is at $z=1/[2(r+1)\Delta x]$, and use condition (b) along it. This will be possible for $(r+1)\Delta x$ not too large. Through the change of variable $\zeta=z(r+1)\Delta x$, we then obtain

$$|A[f](r,\Delta x)| \leqq \frac{K}{2\pi}[(r+1)\Delta x]^{-\rho-1+\epsilon}\int_{C_{1/2}}\frac{|\zeta|^{\rho-\epsilon}|d\zeta|}{\left|1-\dfrac{\zeta}{r+1}\right|^{r+1}},$$

where $C_{1/2}$ has its vertex at $\zeta=1/2$. Since $R(\zeta)\leqq 1/2$ along $C_{1/2}$, we have

$$\frac{1}{\left|1-\dfrac{\zeta}{r+1}\right|^{r+1}} \leqq \frac{1}{\left(1-\dfrac{R(\zeta)}{r+1}\right)^{r+1}}\cdot$$

Now we know that $[1+x/(r+1)]^{r+1}$ is an increasing function of r, $r\geqq 0$, both for positive x, and for negative x with $|x|<1$. We therefore have along $C_{1/2}$

$$\frac{1}{\left(1-\dfrac{R(\zeta)}{r+1}\right)^{r+1}} \leqq \frac{1}{\left(1-\dfrac{R(\zeta)}{\rho+1}\right)^{\rho+1}}\cdot$$

The integral along $C_{1/2}$, which depends only on r, is thus seen to be bounded for $r\geqq\rho$, so that we obtain

$$|A[f](r,\Delta x)| < L'[(r+1)\Delta x]^{-\rho-1+\epsilon}.$$

A like inequality is obtained for $r<\rho$ by using for C''' a circle center $1/\Delta x$, radius $\theta/\Delta x$, $0<\theta<1$. We thus arrive at the following result:

For all Δx's and r's with $(r+1)\Delta x<a$, where a is some fixed positive quantity, we have the inequality

$$|A[f](r,\Delta x)| < L[(r+1)\Delta x]^{-\rho-1+\epsilon}. \tag{13}$$

Hence also, for $t<a$, we have

$$|A[f](t)| \leqq Lt^{-\rho-1+\epsilon}. \tag{14}$$

The simplest formula for $\{f(D)\}_{x_0}^{X}\phi(x)$ results when $f(D)$ is of order zero. As at the end of §3, we shall write

$$\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x) = \left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{X-h}^{X}\phi(x) + \sum_{r=q+1}^{p} A[f](r,\Delta x)\phi(X - r\Delta x)\Delta x;$$
$$q = \{h/\Delta x\}.$$

In the same manner also, we find for fixed positive h,

$$\lim_{\Delta x\to+0}\sum_{r=q+1}^{p} A[f](r,\Delta x)\phi(X - r\Delta x)\Delta x = \int_{x_0}^{X-h} A[f](X-x)\phi(x)dx.$$

Now $\phi(x)$ is to be assumed continuous in (x_0, X). Let M be the upper bound of $|\phi(x)|$ in this interval. For $h<a$ we can apply (13), with $\rho=0$, and obtain

$$\left|\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{X-h}^{X}\phi(x)\right| < LM\sum_{r=0}^{\{h/\Delta x\}}[(r+1)\Delta x]^{-1+\epsilon}\Delta x.$$

By comparing the indicated sum with the integral of $\xi^{-1+\epsilon}d\xi$ between limits 0 and $h+\Delta x$ or Δx and $h+2\Delta x$, according as $0<\epsilon<1$, or $\epsilon\geqq 1$, we see that

$$\limsup_{\Delta x\to+0}\left|\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{X-h}^{X}\phi(x)\right| \leqq LM\frac{h^{\epsilon}}{\epsilon}\cdot$$

Since $h^{\epsilon}/\epsilon\to 0$, as $h\to+0$, we can apply the limit criterion, and obtain

THEOREM III. *If $f(D)$ is of order zero, and $\phi(x)$ is continuous in (x_0, X), then $\{f(D)\}_{x_0}^{X}\phi(x)$ exists, and is given by*

(15)
$$\{f(D)\}_{x_0}^{X}\phi(x) = \lim_{h\to+0}\int_{x_0}^{X-h} A[f](X-x)\phi(x)dx = \int_{x_0}^{X} A[f](X-x)\phi(x)dx.*$$

* If (8) is introduced in (15), we obtain

$$\{f(D)\}_{x_0}^{X}\phi(x) = \frac{1}{2\pi i}\int_{x_0}^{X}\left[\int_C e^{(X-x)z}f(z)dz\right]\phi(x)dx.$$

It may interest the reader to note that this formula might have suggested itself in the following formal manner. Symbolic use of Cauchy's second integral formula gives

$$f(D) = \frac{1}{2\pi i}\int_K \frac{f(z)dz}{z-D}\cdot$$

On the other hand, the linear differential equation would suggest

$$\left\{\frac{1}{Z-D}\right\}_{x_0}^{X}\phi(x) = -\int_{x_0}^{X} e^{(X-x)z}\phi(x)dx,$$

so that we would be led to

$$\{f(D)\}_{x_0}^{X}\phi(x) = -\frac{1}{2\pi i}\int_K\left[\int_{x_0}^{X} e^{(X-x)z}\phi(x)dx\right]f(z)dz.$$

Reversing the sense in which "contour" K is traversed, and changing the order of integration leads to the actual formula.

It will be noticed that the last integral may be improper for $x=X$; but of its convergence we are assured both by the limit criterion, and by the direct application of (14). To illustrate (15), we may take $f(D)=D^n$, $R(n)<0$. The operator is then of order zero. By (6) and (15), we thus get the standard Riemann-Grünwald form

$$(16)\qquad \{D^n\}_{x_0}^{X}\phi(x) = \frac{1}{\Gamma(-n)}\int_{x_0}^{X}(X-x)^{-n-1}\phi(x)dx,\ R(n)<0.$$

We shall give two existence proofs for operators of arbitrary finite order. The first proof depends on the observation that if $f(D)$ is of order ρ, then $D^{-\rho}f(D)$ is of order zero, and so comes under Theorem III. Inasmuch as additional assumptions on $\phi(x)$ in a left neighborhood of $x=X$ will be required, we shall use Theorem II with x_1 in this neighborhood.

Formula (10), with x_1 in place of x_0, and $m=\rho$, becomes

$$\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_1}^{X}\phi(x) = \left\{\left(\frac{\Delta}{\Delta x}\right)^{-\rho} f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_1}^{X}\frac{\Delta^\rho\phi(x)}{\Delta x^\rho}$$
$$+\sum_{\mu=0}^{\rho-1}A\,[u^{-\mu-1}f(u)](q,\Delta x)\frac{\Delta^\mu\phi(x_1')}{\Delta x^\mu},\quad q=\{(X-x_1)/\Delta x\}.$$

Let $\phi(x)$ possess a continuous ρth derivative in an interval (k,X), $X>k$,* and assign to x_1 a value within this interval. By the law of the mean, and the continuity of $\phi^{(\rho)}(x)$ in (k,X), it follows that, for sufficiently small Δx,

$$\left|\frac{\Delta^\rho\phi(x)}{\Delta x^\rho}-\phi^{(\rho)}(x)\right|<\delta_{\Delta x},\quad \lim_{\Delta x\to+0}\delta_{\Delta x}=0,$$

for all x's in (x_1,X).† Hence if $\phi^{(\rho)}(x)$ be substituted for $\Delta^\rho\phi(x)/\Delta x^\rho$ in the above equation, the result will be changed in absolute value by no more than

$$\delta_{\Delta x}\sum_{r=0}^{q}\left|A\,[u^{-\rho}f(u)](r,\Delta x)\right|\Delta x.$$

Since $D^{-\rho}f(D)$ is of order zero, the proof of Theorem III shows this sum to be bounded, and hence the change in question to approach zero as limit as $\Delta x\to+0$. We then easily derive

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* Here, as later, when a derivative is assumed to exist in a left neighborhood of X, the derivative at X is to be but a left derivative.

† See de la Vallée Poussin, *Cours d'Analyse*, vol. 1, 1914, §109, for the case $\rho=1$. By (1), §118, this is directly extended to arbitrary ρ.

THEOREM IV. *If $f(D)$ is of order ρ, and if $\phi(x)$ is continuous in (x_0, X), and possesses a continuous ρth derivative in a left neighborhood of $x=X$, then $\{f(D)\}_{x_0}^{X}\phi(x)$ exists; and, if x_1 is within this neighborhood, is given by*

$$\tag{17} \begin{aligned} \{f(D)\}_{x_0}^{X}\phi(x) &= \int_{x_1}^{X} A[u^{-\rho}f(u)](X-x)\phi^{(\rho)}(x)dx \\ &+ \sum_{\mu=0}^{\rho-1} A[u^{-\mu-1}f(u)](X-x_1)\phi^{(\mu)}(x_1) + \int_{x_0}^{x_1} A[f](X-x)\phi(x)dx. \end{aligned}$$

For illustration, again take D^n, with $\rho-1 \leqq R(n) < \rho$. We obtain

$$\tag{18} \begin{aligned} \{D^n\}_{x_0}^{X}\phi(x) &= \frac{1}{\Gamma(\rho-n)}\int_{x_1}^{X}(X-x)^{\rho-n-1}\phi^{(\rho)}(x)dx \\ &+ \sum_{\mu=0}^{\rho-1}\frac{(X-x_1)^{\mu-n}}{\Gamma(\mu-n+1)}\phi^{(\mu)}(x_1) + \frac{1}{\Gamma(-n)}\int_{x_0}^{x_1}(X-x)^{-n-1}\phi(x)dx.* \end{aligned}$$

For the second proof note that, except for successive operations, the upper limit X plays the part of a constant. It can therefore appear as such in the operand $\phi(x)$. Now suppose that a continuous $\phi(x)$ satisfies the inequality

$$|\phi(x)| \leqq N(X-x)^{\rho}$$

in a left neighborhood of $x=X$. This, with (13), leads to the identical inequalities that gave us Theorem III. Hence (15) holds for such a $\phi(x)$.

Let then a continuous $\phi(x)$ possess a finite ρth left derivative at $x=X$, and hence also left derivatives of all lower orders. Then, as a result of a first theorem in Taylor's expansion, we have with finite N, in a left neighborhood of $x=X$,

$$|\phi(x) - P(x)| < N(X-x)^{\rho}, \quad P(x) = \sum_{\mu=0}^{\rho-1}\frac{\phi^{(\mu)}(X)}{\mu!}(x-X)^{\mu}.$$

If $\phi(x)$ is continuous in (x_0, X), $\{f(D)\}_{x_0}^{X}[\phi(x)-P(x)]$ will exist, as seen above, and be given by (15). On the other hand, $P(x)$ is but a polynomial in x, so that $\{f(D)\}_{x_0}^{X}P(x)$ exists, and could be evaluated by (11). Hence

THEOREM V. *If $f(D)$ is of order ρ, and if $\phi(x)$ is continuous in (x_0, X), and possesses a finite ρth left derivative at $x=X$, then $\{f(D)\}_{x_0}^{X}\phi(x)$ exists, and is given by*

$$\tag{19} \begin{aligned} \{f(D)\}_{x_0}^{X}\phi(x) &= \int_{x_0}^{X} A[f](X-x)\left[\phi(x) - \sum_{\mu=0}^{\rho-1}\frac{\phi^{(\mu)}(X)}{\mu!}(x-X)^{\mu}\right]dx \\ &+ \{f(D)\}_{x_0}^{X}\sum_{\mu=0}^{\rho-1}\frac{\phi^{(\mu)}(X)}{\mu!}(x-X)^{\mu}. \end{aligned}$$

* This reduces to Grünwald's result if $x_1 = x_0$.

Though this theorem requires less of $\phi(x)$ than Theorem IV, we shall nevertheless find Theorem IV more useful in our development. To illustrate (19), we shall take the operator log D, whose order is one. By direct calculation we have $A[\log u](t) = -1/t$. Hence, by (19) and (3), we get

$$(20)\quad \{\log D\}_{x_0}^{X}\phi(x) = \int_{x_0}^{X} \frac{\phi(X) - \phi(x)}{X - x}dx - [\gamma + \log(X - x_0)]\phi(X).$$

It will be seen that this second proof uses the hypothesis that $f(D)$ is of finite order ρ only to enable us to assume the existence of $A[u^{-\mu}f(u)](t)$, for $t>0$, $\mu=0, 1, \cdots, \rho$, and the validity of inequality (13). This suggests that we define $f(D)$ to be of *extended order* ρ if $A[u^{-\mu}f(u)](t)$ exists for $t>0$, $\mu=0, 1, \cdots, \rho$, and (13) is satisfied. The first proof also uses the fact that $D^{-\rho}f(D)$ is of order zero. But it can be proved that if $f(D)$ is of extended order ρ, $D^{-\rho}f(D)$ is of extended order zero. Hence all of our existence theorems are valid for operators of extended finite order.

6. **Existence theorems for operators of type zero.** The crucial theorem for operators of type zero is the following:

THEOREM VI. *If $f(D)$ is of type zero, while $\phi(x)$ is analytic for $x_1 \leqq x \leqq X$, with the radius of convergence at x_1 greater than $X - x_1$, then $\{f(D)\}_{x_1}^{X}\phi(x)$ exists, and is given by the convergent series*

$$(21)\quad \{f(D)\}_{x_1}^{X}\phi(x) = A[u^{-1}f(u)](X - x_1)\phi(x_1) + A[u^{-2}f(u)](X - x_1)\phi'(x_1) + \cdots.$$

As a consequence of this theorem, if $\phi(x)$ is continuous in (x_0, X), but analytic in some left neighborhood of X, then by choosing x_1 in this neighborhood, with $X - x_1$ less than half of the radius of convergence of $\phi(x)$ at X, both (9) and (21) become applicable, and so give

THEOREM VII. *If $f(D)$ is of type zero, while $\phi(x)$ is continuous in (x_0, X), and analytic in a left neighborhood of $x = X$, then $\{f(D)\}_{x_0}^{X}\phi(x)$ exists, and, if x_1 is chosen as indicated above, is given by*

$$(22)\quad \{f(D)\}_{x_0}^{X}\phi(x) = \sum_{\mu=0}^{\infty} A[u^{-\mu-1}f(u)](X - x_1)\phi^{(\mu)}(x_1) + \int_{x_0}^{x_1} A[f](X - x)\phi(x)dx.$$

It may be noted that with 1 for $f(D)$, formula (21) reduces to the Taylor expansion of $\phi(x)$ for $x = x_1$. More generally, (21) is the result of operating on this Taylor expansion with $\{f(D)\}_{x_1}^{X}$, term by term.

Our proof of Theorem VI is an extension of the first existence proof for operators of finite order. As in that proof, we use the finite difference reduction formula, with limits x_1, X, viz.

$$\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_1}^{X}\phi(x) = \left\{\left(\frac{\Delta}{\Delta x}\right)^{-m} f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_1}^{X} \frac{\Delta^m \phi(x)}{\Delta x^m}$$

$$+ \sum_{\mu=0}^{m-1} A\,[u^{-\mu-1} f(u)](q, \Delta x) \frac{\Delta^\mu \phi(x_1')}{\Delta x^\mu}.$$

On the other hand, we shall let m vary with Δx, in fact equal $\{\epsilon/\Delta x\}$, with fixed and positive ϵ. Our proof will require the following condition on ϵ:

$$0 < \epsilon < \frac{l - (X - x_1)}{2},$$

where l is between $X - x_1$ and the radius of convergence of $\phi(x)$ at x_1. Note that the left hand member of the reduction formula uses values of x only in the interval (x_1, X), whereas in the right hand member the values spread over the interval $(x_1 - m\Delta x, X)$. Since $m\Delta x < \epsilon < l$, all of these x's fall in the interval of convergence of the Taylor expansion of $\phi(x)$ at x_1; and so this Taylor expansion may be used to define $\phi(x)$ for $x < x_1$ for the purpose of our proof.

This proof consists essentially in establishing the following:

(a) For fixed N, $\displaystyle\lim_{\Delta x \to +0} \sum_{\mu=0}^{N-1} A\,[u^{-\mu-1} f(u)](q, \Delta x) \frac{\Delta^\mu \phi(x_1')}{\Delta x^\mu}$

$$= \sum_{\mu=0}^{N-1} A\,[u^{-\mu-1} f(u)](X - x_1)\phi^{(\mu)}(x_1).$$

(b) With $m = \{\epsilon/\Delta x\}$, $\displaystyle\lim_{\Delta x \to +0} \left\{\left(\frac{\Delta}{\Delta x}\right)^{-m} f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_1}^{X} \frac{\Delta^m \phi(x)}{\Delta x^m} = 0.$

(c) With $m = \{\epsilon/\Delta x\}$, $\displaystyle\limsup_{\Delta x \to +0} \sum_{\mu=N}^{m-1} A\,[u^{-\mu-1} f(u)](q, \Delta x) \frac{\Delta^\mu \phi(x_1')}{\Delta x^\mu}$

$$\leqq h(N); \quad \lim_{N \to \infty} h(N) = 0.$$

Since (b) allows us to neglect its term of the reduction formula, (a) and (c) together give us Theorem VI by means of the limit criterion.

(a) This is immediate.

(b) Formula (4) allows us to write

$$\left\{\left(\frac{\Delta}{\Delta x}\right)^{-m} f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_1}^{X} \frac{\Delta^m \phi(x)}{\Delta x^m} = \left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_1}^{X} \left\{\left(\frac{\Delta}{\Delta \xi}\right)^{-m}\right\}_{x_1}^{x} \frac{\Delta^m \phi(\xi)}{\Delta \xi^m}, \quad \Delta\xi = \Delta x.$$

By the law of the mean we have

$$\frac{\Delta^m\phi(\xi)}{\Delta\xi^m} = \phi^{(m)}(\xi - \theta m\Delta x);\ 0 < \theta < 1.^*$$

The Taylor expansion of $\phi(x)$ at x_1 enables us to define the function of a complex variable $\phi(z)$. Let M be the upper bound of the modulus of $\phi(z)$ on, and hence also within, the circle center x_1, radius l. The smallest distance from $\xi-\theta m\Delta x$ to this circle is $l-|\xi-\theta m\Delta x-x_1|$, which certainly exceeds $l-\epsilon-(\xi-x_1)$ for all ξ's in (x_1, X). By applying the standard inequality of complex variable theory to $|\phi^{(m)}(\xi-\theta m\Delta x)|$, we thus obtain

$$\left|\frac{\Delta^m\phi(\xi)}{\Delta\xi^m}\right| < \frac{Mm!}{[l - \epsilon - (\xi - x_1)]^m},$$

so that we can write

$$\left|\left\{\left(\frac{\Delta}{\Delta\xi}\right)^{-m}\right\}_{x_1}^{x} \frac{\Delta^m\phi(\xi)}{\Delta\xi^m}\right| < \sum_{s=0}^{\{(x-x_1)/\Delta x\}} A[u^{-m}](s, \Delta x)\frac{Mm!}{[l - \epsilon - (x - s\Delta x - x_1)]^m}\Delta x.$$

It is easily seen that

$$A[u^{-m}](s, \Delta x) = \frac{m(m + 1)\cdots(m + s - 1)}{s!}\Delta x^{m-1} < \frac{(\epsilon + s\Delta x)^{m-1}}{(m - 1)!}\cdot$$

Now the inequalities imposed on ϵ make $(\epsilon+s\Delta x)/[l-\epsilon-(x-s\Delta x-x_1)]$ an increasing function of $s\Delta x$. Since $s\Delta x<x-x_1$, we thereby obtain, on replacing $s\Delta x$ by $x-x_1$, the inequality

$$\frac{\epsilon + s\Delta x}{l - \epsilon - (x - s\Delta x - x_1)} < \frac{\epsilon + (x - x_1)}{l - \epsilon},$$

and so find in succession

$$\left|\left\{\left(\frac{\Delta}{\Delta\xi}\right)^{-m}\right\}_{x_1}^{x} \frac{\Delta^m\phi(\xi)}{\Delta\xi^m}\right| < \frac{M(x - x_1)}{\epsilon}\cdot m\left[\frac{\epsilon + (x - x_1)}{l - \epsilon}\right]^m,$$

$$\left|\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_1}^{X}\left\{\left(\frac{\Delta}{\Delta\xi}\right)^{-m}\right\}_{x_1}^{x} \frac{\Delta^m\phi(\xi)}{\Delta\xi^m}\right|$$
$$< \frac{M(X - x_1)}{\epsilon}\cdot m\left[\frac{\epsilon + (X - x_1)}{l - \epsilon}\right]^m \sum_{r=0}^{q} |A[f](r, \Delta x)|\,\Delta x.$$

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* This, as also the use of the law of the mean in §5, requires $\phi(x)$ to be a real function of the real variable x. If $\phi(x)$ is a complex function of a real variable, its real and imaginary parts separately will satisfy the assumption of existence and continuity of derivatives or of analyticity stated for $\phi(x)$ so that the demonstrations given are valid for these parts. Combining the two results thus obtained therefore yields the same result for the complex $\phi(x)$.

If δ is a real fixed quantity in the analytic sector of $f(z)$, we can choose for the C''' of §3 a circle, center $1/\Delta x$, radius $1/\Delta x-\delta$, and so, through the corresponding contour integral and condition (b) of §3, obtain

$$| A [f](r, \Delta x) | < \frac{Ke^{\kappa(2/\Delta x-\delta)}}{(1 - \delta\Delta x)^r}\cdot$$

$\sum_{r=0}^{q} |A [f](r, \Delta x)|\Delta x$ is then less than $e^{2\kappa/\Delta x}$ times the sum of a geometric progression which is easily seen to have a finite limit as $\Delta x\to+0$. The essential factor is thus $e^{2\kappa/\Delta x}$. Now the inequalities imposed on ϵ can be rewritten

$$0 < \frac{\epsilon + (X - x_1)}{l - \epsilon} < 1.$$

Hence, by choosing κ sufficiently small, we obtain

$$\lim_{\Delta x\to+0} m\left[\frac{\epsilon + (X - x_1)}{l - \epsilon}\right]^m e^{2\kappa/\Delta x} = 0, \quad m = \left\{\frac{\epsilon}{\Delta x}\right\},$$

thereby establishing (b).

(c) As in part (b), we obtain

$$\left|\frac{\Delta^\mu\phi(x_1')}{\Delta x^\mu}\right| < \frac{M\mu!}{(l - \epsilon)^\mu}\cdot$$

On the other hand

$$A [u^{-\mu-1}f(u)](q, \Delta x) = -\frac{1}{2\pi i}\int_W \frac{z^{-\mu-1}f(z)}{(1 - z\Delta x)^{q+1}}dz,$$

where W encloses $1/\Delta x$, but excludes the origin. The contour W will consist of W_1, that part of the C''' of §3 for which $|z|>R$, $R<1/\Delta x$, joined to W_2, the arc of the circle center the origin, radius R, cut off by C''', and excluding, with W_1, the origin. Such a contour will be valid when R exceeds the distance of the vertex of C from the origin. We have

$$\left|\frac{1}{2\pi}\int_{W_2}\frac{z^{-\mu-1}f(z)}{(1 - z\Delta x)^{q+1}}dz\right| < K\frac{R^{-\mu}e^{\kappa R}}{(1 - R\Delta x)^{q+1}}\cdot$$

Now $R^{-\mu}/(1-R\Delta x)^{q+1}$ has a minimum with respect to R for

$$R = \frac{\mu}{(q + 1)\Delta x + \mu\Delta x}\cdot$$

This value of R is less than $1/\Delta x$, and exceeds $\mu/[(X-x_1)+\epsilon]$, so that, for sufficiently large μ, it will give a valid contour. With it we have

$$\frac{R^{-\mu}e^{\kappa R}}{(1-R\Delta x)^{q+1}}=\frac{[(q+1)\Delta x+\mu\Delta x]^{\mu}}{\mu^{\mu}}e^{\kappa\mu/[(q+1)\Delta x+\mu\Delta x]}\left(1+\frac{\mu}{q+1}\right)^{q+1},$$

and, as $\mu<m$, we obtain

$$\left|\frac{1}{2\pi}\int_{W_2}\frac{z^{-\mu-1}f(z)}{(1-z\Delta x)^{q+1}}dz\right|<K[(X-x_1)+\epsilon]^{\mu}e^{\kappa\mu/(X-x_1)}\mu^{-\mu}e^{\mu}.$$

For W_1 we have $|z|>R$. Since W_1 is part of C''', we easily find that

$$\left|\frac{1}{2\pi}\int_{W_1}\frac{z^{-\mu-1}f(z)}{(1-z\Delta x)^{q+1}}dz\right|<\frac{[(X-x_1)+\epsilon]^{\mu+1}}{\mu^{\mu+1}}\cdot\frac{1}{2\pi}\int_{C'''}\frac{|f(z)|}{|1-z\Delta x|^{q+1}}|dz|.$$

Now our discussion of $A[f](r,\Delta x)$ in §3 proves that this integral has a finite limit as $\Delta x\to+0$. Symbolizing the product of $1/(2\pi)$ and this limit by $\|A[f](X-x_1)\|$, we are thus led to the following form for the $h(N)$ of the statement of this part of the proof, viz.,

$$\begin{aligned}h(N)=MK\sum_{\mu=N}^{\infty}&\left\{\left[\frac{(X-x_1)+\epsilon}{l-\epsilon}\right]^{\mu}e^{\kappa\mu/(X-x_1)}\frac{\mu!}{\mu^{\mu}e^{-\mu}}\right\}\\&+M(l-\epsilon)\|A[f](X-x_1)\|\sum_{\mu=N}^{\infty}\left\{\left[\frac{(X-x_1)+\epsilon}{l-\epsilon}\right]^{\mu+1}\frac{\mu!}{\mu^{\mu+1}}\right\}.\end{aligned}$$

Recalling that we have $0<[(X-x_1)+\epsilon]/(l-\epsilon)<1$, we see that the second series converges. By choosing κ sufficiently small to have

$$\frac{(X-x_1)+\epsilon}{l-\epsilon}e^{\kappa/(X-x_1)}<1,$$

the convergence of the first series is assured. As a consequence of their convergence, these series approach zero as limit as N increases indefinitely. The same is therefore true of $h(N)$. (c) has thus been proved, and with it Theorem VI.

We turn now to the formal properties mentioned in the introduction.

7. **Differentiation with respect to the upper limit.** We can obtain the derivative of $A[f](t)$ under a general hypothesis. If $A[f(u)](t)$ and $A[uf(u)](t)$ exist for $0<t_1\leqq t<t_2$,* then the terms of the relation

$$\{Df(D)\}_{-t}^{0}1=\{Df(D)\}_{-t_1}^{0}1+\int_{-t}^{-t_1}A[uf(u)](-\xi)d\xi$$

* The relation given below shows that if $A[uf(u)](t)$ exists in a neighborhood, and $A[f(u)](t)$ exists for one t in that neighborhood, it exists for every t therein.

obtained from Theorem II, exist, since by (12) this relation becomes

$$A[f(u)](t) = A[f(u)](t_1) + \int_{t_1}^{t} A[uf(u)](\eta)d\eta.$$

By keeping t_1 constant, and differentiating with respect to t, we get

THEOREM VIII. *If $A[f(u)](t)$ and $A[uf(u)](t)$ exist in a certain neighborhood of t, then $(d/dt)\, A[f(u)](t)$ exists, and is given by*

$$\frac{d}{dt}A[f(u)](t) = A[uf(u)](t). \tag{23}$$

By the use of (9), we immediately obtain the

COROLLARY. *If $A[u^{-1}f(u)](X-x_0)$ and $A[f(u)](X-x_0)$ exist in a certain neighborhood of X, then $(d/dX)\ \{f(D)\}_{x_0}^{X}1$ exists, and is given by*

$$\frac{d}{dX}\{f(D)\}_{x_0}^{X}1 = \{Df(D)\}_{x_0}^{X}1. \tag{24}$$

In extending this corollary to $\phi(x)$ as operand, it is desirable to replace the X left neighborhood of our existence theorems by a complete neighborhood, so that d/dX can stand for the derivative as ordinarily used. If, however, we retain the left neighborhood, then the following still remains valid, provided d/dX is understood to mean left derivative, an observation that is essential to most of our applications of these formulas.

For $f(D)$ either of finite order, or of type zero, we have, as a result of (23),

$$\frac{d}{dX}\int_{x_0}^{x_1} A[f(u)](X-x)\phi(x)dx = \int_{x_0}^{x_1} A[uf(u)](X-x)\phi(x)dx,$$

since the continuity of $A[uf(u)](X-x)$, for $x_0 \leqq x \leqq x_1$, and a neighborhood of X, permits differentiation under the integral sign. Hence, by (9), we shall have $(d/dX)\{f(D)\}_{x_0}^{X}\phi(x) = \{Df(D)\}_{x_0}^{X}\phi(x)$, provided we first prove this relation for lower limit x_1. We shall choose x_1, so that the special expansions hold.

First let $Df(D)$ be of order zero. $f(D)$ is then also of order zero, so that we can use (15), and write

$$\{f(D)\}_{x_1}^{X}\phi(x) = \lim_{h\to+0}\int_{x_1}^{X-h} A[f(u)](X-x)\phi(x)dx.$$

By Leibnitz's rule we have

$$\frac{d}{dX}\int_{x_1}^{X-h} A[f(u)](X-x)\phi(x)dx = A[f(u)](h)\phi(X-h) + \int_{x_1}^{X-h} A[uf(u)](X-x)\phi(x)dx.$$

Furthermore, by (14), we have

$$|A[uf(u)](X-x)| \leqq L(X-x)^{-1+\epsilon}, \quad |A[f(u)](h)| \leqq L'h^{\epsilon}.*$$

These inequalities show that the result of the last differentiation uniformly approaches $\int_{x_1}^{X} A[uf(u)](X-x)\phi(x)dx$ as limit in a neighborhood of X, as $h \to +0$. Hence by a well known criterion of differentiation,† we have, as desired,

$$\frac{d}{dX}\{f(D)\}_{x_1}^{X}\phi(x) = \int_{x_1}^{X} A[uf(u)](X-x)\phi(x)dx.$$

If now $Df(D)$ is of order ρ, we can use the first existence theorem for operators of finite order with $f(D)$ as of order ρ, and write

$$\{f(D)\}_{x_1}^{X}\phi(x) = \{D^{-\rho}f(D)\}_{x_1}^{X}\phi^{(\rho)}(x) + \sum_{\mu=0}^{\rho-1} A[u^{-\mu-1}f(u)](X-x_1)\phi^{(\mu)}(x_1).$$

Since $D\cdot D^{-\rho}f(D)$ is of order zero, we can use the special case just proved, along with (23), in differentiating both members of this equation, thereby obtaining on the right the expansion of $\{Df(D)\}_{x_1}^{X}\phi(x)$. Hence

THEOREM IX. *If $Df(D)$ is of order ρ, and if $\phi(x)$ is continuous in (x_0, X), and has a continuous ρth derivative in a neighborhood of X, then*

$$\frac{d}{dX}\{f(D)\}_{x_0}^{X}\phi(x) = \{Df(D)\}_{x_0}^{X}\phi(x). \tag{25}$$

COROLLARY. *If $D^n f(D)$ is of order ρ, and if $\phi(x)$ is continuous in (x_0, X), and has a continuous ρth derivative in a neighborhood of X, then*

$$\frac{d^n}{dX^n}\{f(D)\}_{x_0}^{X}\phi(x) = \{D^n f(D)\}_{x_0}^{X}\phi(x). \tag{26}$$

If $f(D)$ is replaced by $D^{-\rho}f(D)$, and n by ρ, the hypothesis of this corollary, restricted to a left neighborhood of X, reduces to the hypothesis of Theorem IV. Under this hypothesis, (26) therefore reduces to

* The first because $\rho=0$ for $uf(u)$; the second because the ρ of $f(u)$ may be taken to be -1. Though ρ, as defined in §5, is either zero or a positive integer, it can be assigned a negative value, as is convenient here, with (14) remaining valid, provided a positive ϵ can still be chosen.

† Goursat, *Cours d'Analyse*, vol. 1, 1910, p. 74.

$$\{f(D)\}_{x_0}^X \phi(x) = \frac{d^p}{dX^p}\int_{x_0}^X A[u^{-p}f(u)](X - x)\phi(x)dx. \tag{27}$$

Formula (27) is a direct extension of Riemann's form for D^n for $R(n) > 0$.

The extension of Theorem IX to operators of type zero offers no difficulty. We first observe that the result of differentiating series (21) term by term is the expansion of $\{Df(D)\}_{x_1}^X \phi(x)$. Hence it is only necessary to prove the resulting series uniformly convergent in a neighborhood of X to obtain the desired result. Since $Df(D)$ is also of type zero, no loss of generality ensues if we prove the series for $\{f(D)\}_{x_1}^X \phi(x)$ uniformly convergent. Turning to the discussion of (c) §6, we observe that the single series in which $h(N)$ can be written is a majorant for the corresponding part of the series for $\{f(D)\}_{x_1}^X \phi(x)$. Now the discussion of §3 shows that we can write

$$\| A[f](X - x_1) \| = \frac{1}{2\pi}\int_C e^{R(z)(X-x_1)} \, |f(z)| \, |dz| .$$

This integral is continuous in X, and hence bounded for the neighborhood of X in question. On the other hand, through the choice of κ made in (c) §6, we shall have, for a sufficiently small neighborhood of X,

$$0 < \frac{(X - x_1) + \epsilon}{l - \epsilon} e^{\kappa/(X-x_1)} < \lambda < 1.$$

As a result, the terms of $\{f(D)\}_{x_1}^X \phi(x)$ will be less in absolute value than those of a convergent series of positive constants, and so the series is uniformly convergent. We thus have

THEOREM X. *If $f(D)$ is of type zero, and $\phi(x)$ satisfies the hypothesis of Theorem* VII, *extended to a complete neighborhood of X, then* $(d/dX)\{f(D)\}_{x_0}^X \phi(x)$ *exists, and is given by* (25).

8. **The law of successive operations.** In the present section we shall consider certain conditions under which the relation

$$\{f_1(D)\}_{x_0}^X \{f_2(D)\}_{x_0}^x \phi(\xi) = \{f_1(D) f_2(D)\}_{x_0}^X \phi(x) \tag{28}$$

is valid. Inasmuch as $\{f_2(D)\}_{x_0}^x \phi(\xi)$, which serves as operand for $f_1(D)$, may be discontinuous at $x = x_0$, we shall introduce the following extension of our fundamental definition. If $\phi(x)$ is discontinuous for $x = x_0$, while $\{f(D)\}_{x_0+h}^X \phi(x)$ exists for sufficiently small h, then $\{f(D)\}_{x_0}^X \phi(x)$ is to be defined by

$$\{f(D)\}_{x_0}^X \phi(x) = \lim_{h \to +0} \{f(D)\}_{x_0+h}^X \phi(x) \tag{29}$$

provided the limit in question exists. Under the hypothesis of Theorem II, changed to allow for a discontinuity of $\phi(x)$ for $x=x_0$, the existence of this limit is equivalent to the convergence of the improper integral thus occurring in formula (9). This convergence is amply assured if $\phi(x)$ satisfies in a right neighborhood of x_0 the inequality

$$(30) \qquad | \phi(x) | \leqq M(x - x_0)^{-1+\eta}, \ \eta > 0.$$

It will be convenient to say that $\phi(x)$ then has $M(x-x_0)^{-1+\eta}$ as majorant in the neighborhood. With (30) to replace continuity of $\phi(x)$ at $x=x_0$, Theorem II, with formula (9), continues to hold. Our existence theorems therefore go over, as do also the results of the preceding section.*

Consider, however, the special case $f_2(D)=D^n$, n other than zero, or a positive integer. We have, by (12) and (6),

$$\{D^n\}_{x_0}^{x}1 = A[u^{n-1}](x - x_0) = (x - x_0)^{-n}/\Gamma(-n + 1).$$

This is discontinuous for $x=x_0$ when $R(n)>0$, so that, without the above extension of our definition, (28) could only hold for $R(n)<0$. With this extension, we can have $R(n)<1$, since $\{D^n\}_{x_0}^{x}1$ then satisfies (30). On the other hand, for $R(n)>1$, let $f_1(D)=D^{-1}$. We have

$$\{D^{-1}\}_{x_0+h}^{X}\{D^n\}_{x_0}^{x}1 = \int_{x_0+h}^{X} \frac{(x - x_0)^{-n}}{\Gamma(-n + 1)}dx,$$

which diverges as $h\to+0$. The left hand member of (28) therefore fails to exist in this case.

Volterra has encountered the same difficulty in the related theory of functions of composition; but his solution appears to the writer to be but a verbal evasion.† Among other possibilities, the difficulty might be removed by a study of the commonly neglected arbitrary series that Riemann adds to the definite integral in his formula for D^n. In the absence of a definitive solution, the writer leaves the breach open to view.‡ This possible failure

* The differentiation of an improper integral under the integral sign that is required here is easily justified by the criterion of differentiation referred to in §7.

† V. Volterra, *Functions of composition*, The Rice Institute Pamphlet, vol. 7 (1920), p. 202.

‡ This difficulty does not appear in the theory for infinite lower limit; but the validity of our definition for finite lower limit is thereby rendered questionable. It is to be noted, however, that this failure occurs in the first place for the commonly accepted Riemann-Grünwald form for D^n. Furthermore, in all other respects the theory for finite lower limit is satisfactory; and if not for its own value, it would still be required as a foundation for the theory for infinite lower limit. Finally, in the various tentative modifications of the definition considered by the writer, the present theory remains the indispensable basis for the extension.

of (28), because of the non-existence of the second of the two successive operations, forces us to restrict $f_2(D)$ to finite orders zero or one, the latter only made possible by the extension of definition just given. Because of this restriction of $f_2(D)$, we shall only consider $f_1(D)$'s of finite order.

Our first consideration is to find conditions on $\phi(x)$ which will insure the existence of the left hand member of (28). For this purpose $f_1(D)$ may be taken of order zero, the extension to arbitrary finite order being easily made. We shall then want $\{f_2(D)\}_{x_0}\phi(\xi)$ to exist and be continuous for $x_0<x\leqq X$, and to satisfy in a right neighborhood of x_0 an inequality of type (30).

First let $f_2(D)$ be of order zero. Then, to insure the existence of $\{f_2(D)\}_{x_0}^x\phi(\xi)$, we shall want $\phi(\xi)$ to be continuous for $x_0<\xi\leqq X$, and to have $M(\xi-x_0)^{-1+\eta}$, $\eta>0$, as majorant in a right neighborhood of x_0. The same conditions turn out to be sufficient to yield the remaining requirements for $\{f_2(D)\}_{x_0}^x\phi(\xi)$. In fact, note that, due to its continuity, $\phi(\xi)$ will have $M'(\xi-x_0)^{-1+\eta}$ as majorant over the whole interval (x_0, X). By applying (15), and using $Lt^{-1+\epsilon}$ as majorant for $A[f_2](t)$, in accordance with (14), we easily establish the continuity of $\{f_2(D)\}_{x_0}^x\phi(\xi)$. Furthermore, by the substitution of majorants, and reduction to the first Eulerian integral, (15) yields

$$\left|\ \{f_2(D)\}_{x_0}^x\phi(\xi)\ \right| \leqq LM'B(\epsilon, \eta)(x - x_0)^{-1+\epsilon+\eta},$$

which is of type (30).

When $f_2(D)$ is of order one, the use of the first existence theorem for operators of finite order requires $\phi(\xi)$ to possess a continuous first derivative for $x_0<\xi\leqq X$, since x must vary from x_0 to X. This time, $\phi'(\xi)$ is assumed to have $M(\xi-x_0)^{-2+\eta}$, $\eta>0$, as majorant. Since no loss of generality ensues if η is assumed less than one, $\phi(\xi)$ will then have $M'(\xi-x_0)^{-1+\eta}$ as majorant. By applying (17) with $x-x_1=(x-x_0)/2$, i.e. by writing

$$\{f_2(D)\}_{x_0}^x\phi(\xi) = \int_{x_1}^x A[u^{-1}f(u)](x-\xi)\phi'(\xi)d\xi + A[u^{-1}f_2(u)](x-x_1)\phi(x_1)$$
$$+ \int_{x_0}^{x_1} A[f_2(u)](x-\xi)\phi(\xi)d\xi,$$

and using the majorants $Lt^{-2+\epsilon}$ and $L't^{-1+\epsilon}$ for $A[f_2](t)$ and $A[u^{-1}f_2(u)](t)$ respectively, as given by (14), in conjunction with the above majorants for $\phi'(\xi)$ and $\phi(\xi)$, we establish the stated requirements for $\{f_2(D)\}_{x_0}^x\phi(\xi)$, under the added condition $\epsilon+\eta>1$.

The desired conditions are therefore the following:

(a) for $f_2(D)$ of order zero: $\phi(x)$ is continuous for $x_0<x\leqq X$, with $M(x-x_0)^{-1+\eta}, \eta>0$, as majorant;

(b) for $f_2(D)$ of order one: $\phi'(x)$ exists and is continuous for $x_0<x\leqq X$, with $M(x-x_0)^{-2+\eta}$ as majorant, where $\eta>0$, and $\epsilon+\eta>1$.

We proceed now to establish (28) for $f_1(D)$ of order zero under the above conditions on $\phi(x)$. The following preliminary result is fundamental.

As in the derivation of (4) §1, we find

$$(31)\quad A[f_1(u)f_2(u)](p, \Delta x) = \sum_{r=0}^{p} A[f_1(u)](r, \Delta x)A[f_2(u)](p-r, \Delta x)\Delta x.$$

If we let $p=\{(X-x_0)/\Delta x\}$, and have $\Delta x\rightarrow +0$, then, by a slight modification of the method used in establishing (15), we obtain

$$(32)\quad A[f_1(u)f_2(u)](X-x_0) = \int_{x_0}^{X} A[f_1(u)](X-x)A[f_2(u)](x-x_0)dx,\text{*}$$

provided $f_1(D)$ and $f_2(D)$ are both of order zero. By (15) this becomes

$$(33)\qquad \{f_1(D)\}_{x_0}^{X} A[f_2(u)](x-x_0) = A[f_1(u)f_2(u)](X-x_0).$$

Let then $f_1(D)$ and $f_2(D)$ first both be of order zero, with $\phi(x)$ satisfying condition (a) given above. We can in this case apply (15), and write

$$\{f_1(D)\}_{x_0}^{X}\{f_2(D)\}_{x_0}^{x}\phi(\xi) = \int_{x_0}^{X} A[f_1(u)](X-x)\int_{x_0}^{x} A[f_2(u)](x-\xi)\phi(\xi)d\xi dx.$$

The improper double integral corresponding to this iterated integral exists, since the latter, and hence the former, is absolutely convergent.† Consequently the order of integration can be changed to give

$$\{f_1(D)\}_{x_0}^{X}\{f_2(D)\}_{x_0}^{x}\phi(\xi) = \int_{x_0}^{X}\left[\int_{\xi}^{X} A[f_1(u)](X-x)A[f_2(u)](x-\xi)dx\right]\phi(\xi)d\xi.$$

By (32) and (15) this reduces to the desired relation (28).

Now let $f_1(D)$ be of order zero, $f_2(D)$ of order one, with $\phi(x)$ satisfying condition (b). Let x_1 be chosen between x_0 and X. We have, by Theorem IV,

$$\{f_1(D)\}_{x_1}^{X}\{f_2(D)\}_{x_1}^{x}\phi(\xi) = \{f_1(D)\}_{x_1}^{X}\{D^{-1}f_2(D)\}_{x_1}^{x}\phi'(\xi) + \phi(x_1)\{f_1(D)\}_{x_1}^{X}A[u^{-1}f_2(u)](x-x_1).$$

* This formula is of special interest in connection with Volterra's functions of composition of the closed cycle group. (See V. Volterra, loc. cit., pp. 181–251.) If $A[f_1(u)](X-x)$ is written $g_1(X-x)$, and $A[f_2(u)](x-x_0)$, $g_2(x-x_0)$, then (32) shows $A[f_1(u)f_2(u)](X-x_0)$ to be Volterra's $\overset{*}{g}_1\overset{*}{g}_2$, so that this symbolic product of the g's corresponds to the actual product of the related f's.

† See de la Vallée Poussin, *Cours d'Analyse*, vol. 2, 1925, pp. 19–22.

Since $D^{-1}f_2(D)$ is of order zero, (28) can be applied to the first term, and (33) to the second term, of the right hand member to give

$$\{f_1(D)\}_{x_1}^{X}\{f_2(D)\}_{x_1}^{x}\phi(\xi) = \{D^{-1}f_1(D)f_2(D)\}_{x_1}^{X}\phi'(x)+A[u^{-1}f_1(u)f_2(u)](X-x_1)\phi(x_1)$$
$$= \{f_1(D)f_2(D)\}_{x_1}^{X}\phi(x).$$

(29) then shows that to demonstrate (28) in this case, we need but establish the relation

$$\lim_{x_1\to x_0} \{f_1(D)\}_{x_1}^{X}\{f_2(D)\}_{x_1}^{x}\phi(\xi) = \{f_1(D)\}_{x_0}^{X}\{f_2(D)\}_{x_0}^{x}\phi(\xi).$$

Now we have directly

$$\{f_1(D)\}_{x_0}^{X}\{f_2(D)\}_{x_0}^{x}\phi(\xi) - \{f_1(D)\}_{x_1}^{X}\{f_2(D)\}_{x_1}^{X}\phi(\xi)$$
$$= \int_{x_0}^{x_1} A[f_1(u)](X-x)[\{f_2(D)\}_{x_0}^{x}\phi(\xi)]dx$$
$$+ \{f_1(D)\}_{x_1}^{X}\left[\int_{x_0}^{x_1} A[f_2(u)](x-\xi)\phi(\xi)d\xi\right].$$

The first term of the second member of this equation is immediately seen to approach zero as limit as $x_1\to x_0$. As for the second term, return to the discussion of the conditions imposed on $\phi(x)$. We can assume without loss of generality the inequalities $0<\epsilon<1$, $0<\eta<1$. Since $\epsilon+\eta>1$ we can choose a positive λ such that $\lambda<\epsilon+\eta-1$. We therefore also have $\lambda<\epsilon$. Then in connection with the integral appearing in the second term, we have for the corresponding majorants

$$\int_{x_0}^{x_1}(x-\xi)^{-2+\epsilon}(\xi-x_0)^{-1+\eta}d\xi < (x-x_1)^{-1+\lambda}\int_{x_0}^{x_1}(x_1-\xi)^{-1+\epsilon-\lambda}(\xi-x_0)^{-1+\eta}d\xi$$
$$= (x-x_1)^{-1+\lambda}(x_1-x_0)^{\epsilon+\eta-1-\lambda}B(\epsilon-\lambda,\eta).$$

This second term will therefore be less in absolute value than

$$M''(X-x_1)^{\mu}(x_1-x_0)^{\epsilon+\eta-1-\lambda}.$$

Since $\epsilon+\eta-1-\lambda>0$, this term also approaches zero as limit as $x_1\to x_0$. The desired relation is thus established, and with it, (28).

The extension to $f_1(D)$ of arbitrary finite order ρ_1, with $f_2(D)$ of order zero or one, is now easily made. $D^{-\rho_1}f_1(D)$ will be of order zero, so that, under the above conditions on $\phi(x)$,

$$\{D^{-\rho_1}f_1(D)\}_{x_0}^{X}\{f_2(D)\}_{x_0}^{x}\phi(\xi) = \{D^{-\rho_1}f_1(D)f_2(D)\}_{x_0}^{X}\phi(x).$$

We have observed that the results of §7 remain valid under (29) and (30). Note also that the derivative appearing in Theorem IX and Corollary I is continuous in a left neighborhood of $x=X$. Suppose then that $D^{\rho_1}f_2(D)$ is of order ρ, and that $\phi(x)$ possesses a continuous ρth derivative in a left neighborhood of $x=X$. Then $\{f_2(D)\}_{x_0}^{x}\phi(\xi)$ will possess a continuous ρ_1th derivative in a left neighborhood of $x=X$. Now $D^{\rho_1}\cdot D^{-\rho_1}f_1(D)$ is of order ρ_1; $D^{\rho_1}\cdot D^{-\rho_1}f_1(D)f_2(D)$ is of order not greater than ρ. Hence we can operate on both sides of the above equation by d^{ρ_1}/dX^{ρ_1} in accordance with the corollary of Theorem IX, and by (26) obtain (28).

These results can be stated as

THEOREM XI. *If $f_1(D)$ and $f_2(D)$ are of orders ρ_1 and ρ_2 respectively, with ρ_2 equal to zero or one, while $D^{\rho_1}f_2(D)$ is of order ρ; and if $\phi(x)$ has a continuous ρth derivative in a left neighborhood of $x=X$, a continuous ρ_2th derivative for $x_0<x\leqq X$, while for this interval $|\phi^{(\rho_2)}(x)|\leqq M(x-x_0)^{-\rho_2-1+\eta}$, where η is positive and such that $D^{-\eta}f_2(D)$ is of order zero,* then* (28) *is valid.*

The restriction of $f_2(D)$ to order zero or one was necessary in order that the result should apply to $\phi(x)\equiv 1$, the simplest case. However, by requiring $\phi(x)$ and a sufficient number of its derivatives to vanish for $x=x_0$, the order of $f_2(D)$ can be arbitrarily large. In fact under the specific conditions stated below, formula (17) gives

$$\{f_2(D)\}_{x_0}^{x}\phi(\xi) = \{D^{-\rho_2+1}f_2(D)\}_{x_0}^{x}\phi^{(\rho_2-1)}(\xi).$$

As $D^{-\rho_2+1}f_2(D)$ will then be of order one, the theorem applies. Hence the

COROLLARY. *The relation* (28) *holds for $\rho_2>1$, provided the corresponding conditions of the theorem are replaced by the following: $\phi^{(\rho_2-2)}(x)$ exists, and is continuous, for $x_0\leqq x\leqq X$, with $\phi^{(\mu)}(x_0)=0$, for $\mu=0, 1, \cdots, \rho_2-2$; $\phi^{(\rho_2)}(x)$ exists, and is continuous, for $x_0<x\leqq X$, with $|\phi^{(\rho_2)}(x)|\leqq M(x-x_0)^{-2+\eta}$, where η is positive, and such that $D^{-\eta}f_2(D)$ is of order ρ_2-1.*

The reader can apply these results to more than two successive operations, and to the commutativity of the operators.

When either of the f's is a polynomial, no restriction of the order of $f_2(D)$ is needed. In fact, if $P(d/dx)$ is a polynomial in d/dx we obtain, by (26),

$$P\left(\frac{d}{dX}\right)\{f(D)\}_{x_0}^{X}\phi(x) = \{P(D)f(D)\}_{x_0}^{X}\phi(x), \tag{34}$$

under the condition that, if $P(D)f(D)$ is of order ρ, $\phi(x)$ satisfies the hypothesis of Theorem IX. On the other hand, through (17), we get

* This is another way of stating the condition $\epsilon+\eta>1$ used in the proof.

$$\{f(D)\}_{x_0}^{X} P\left(\frac{d}{dx}\right)\phi(x) = \{P(D)f(D)\}_{x_0}^{X}\phi(x) - \sum_{\mu=0}^{n-1} A\,[u^{n-\mu-1}f(u)\,](X - x_0)P_\mu(x_0), \tag{35}$$

where

$$P(D) = \sum_{\nu=0}^{n} a_\nu D^{n-\nu},\quad P_\mu(x_0) = \sum_{\nu=0}^{\mu} a_\nu \phi^{(\mu-\nu)}(x_0).$$

For (35), $\phi(x)$ is to possess a continuous nth derivative in (x_0, X), and a continuous ρth derivative in a left neighborhood of X, where ρ is the order of $P(D)f(D)$. Formulas (34) and (35) can easily be extended to the case where $f(D)$ is of type zero. They probably also admit of extension to the case where $P(D)$ is not a polynomial, but of type zero over the plane. (See §14.) Though the disagreement between (35) and (28) was to be expected as a result of the finite difference reduction formula, it may suggest a solution of the difficulty discussed above.

9. **Leibnitz's theorem generalized.** The operators $f^n(D)$, $n=1, 2, 3, \cdots$, are intimately associated with the generalization of Leibnitz's Theorem for repeated differentiation of a product of two functions. A simple relation between $A\,[f^{(n)}(u)\,](t)$ and $A\,[f(u)\,](t)$ follows directly from our definition. We have

$$A\,[f'(u)\,](r, \Delta x) = \frac{(-1)^r f^{(r+1)}(1/\Delta x)}{r!\Delta x^{r+1}} = -(r+1)\Delta x A\,[f(u)\,](r+1, \Delta x).$$

Hence, *for* $t>0$, $A\,[f'(u)\,](t)$ *and* $A\,[f(u)\,](t)$ *coexist, and satisfy the relation*

$$A\,[f'(u)\,](t) = -tA\,[f(u)\,](t).^* \tag{36}$$

By induction, we obtain

$$A\,[f^{(n)}(u)\,](t) = (-t)^n A\,[f(u)\,](t). \tag{37}$$

It is easily proved by the use of Cauchy's integral formula with circular contour for $f^{(n)}(z)$, and the corresponding conditions on $f(z)$, that, if $f(z)$ is of type zero over a certain sector, $f^{(n)}(z)$ is also of type zero over any sector interior to the sector of $f(z)$, and of the same angle. Likewise it can be shown that if $f(z)$ is of finite order ρ, $f^{(n)}(z)$ is of order $\rho-n$, or zero (according as $\rho>n$, or $\leqq n$), over any sector interior to that of $f(z)$, and of angle less than that of the sector of $f(z)$. However, the finite difference relation given above immediately shows that if $f(D)$ is of generalized order ρ, $f'(D)$ is of

* Comparison of this formula with (23) suggests a duality which is strikingly borne out, under special hypotheses, in a study of $A\,[f\,](t)$ for complex t, coupled with the extension to type one suggested in §17.

generalized order $\rho-1$ (or zero if $\rho=0$), with a corresponding extension to $f^{(n)}(D)$. A more complete statement is that $D^n f^{(n)}(D)$ is of the same order or generalized order as $f(D)$.

Turning now to the theorem desired, we shall first prove the special case

$$\{f(D)\}_{x_0}^{X} x\phi(x) = X\{f(D)\}_{x_0}^{X}\phi(x) + \{f'(D)\}_{x_0}^{X}\phi(x), \tag{38}$$

where $\phi(x)$ satisfies the conditions associated in our theorems with the existence of $\{f(D)\}_{x_0}^{X}\phi(x)$. We have, by definition,

$$\begin{aligned}\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X} x\phi(x) &= \sum_{r=0}^{\{(X-x_0)/\Delta x\}} \frac{(-1)^r f^{(r)}(1/\Delta x)}{r!\Delta x^r}(X - r\Delta x)\phi(X - r\Delta x) \\ &= X\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x) + \left\{f'\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X-\Delta x}\phi(x) \\ &= X\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x) + \left\{f'\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x) \\ &\quad - \Delta x\left\{\frac{\Delta}{\Delta x} f'\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X}\phi(x).\end{aligned}$$

Since $Df'(D)$ is of the same type, or finite order, as $f(D)$, $\{Df'(D)\}_{x_0}^{X}\phi(x)$ will exist, so that the last term vanishes with Δx. Hence (38).

We can rewrite (38) in the form

$$\{f(D)\}_{x_0}^{X}(x - X)\phi(x) = \{f'(D)\}_{x_0}^{X}\phi(x),$$

and obtain, by induction,

$$\{f(D)\}_{x_0}^{X}(x - X)^n\phi(x) = \{f^{(n)}(D)\}_{x_0}^{X}\phi(x). \tag{39}$$

If then $P(x)$ is a polynomial of degree m, we can expand it in powers of $(x-X)$, and obtain, by (39), the terminating Leibnitz expansion

$$\begin{aligned}\{f(D)\}_{x_0}^{X}P(x)\phi(x) = P(X)\{f(D)\}_{x_0}^{X}\phi(x) &+ \frac{P'(X)}{1!}\{f'(D)\}_{x_0}^{X}\phi(x) \\ &+ \cdots + \frac{P^{(m)}(X)}{m!}\{f^{(m)}(D)\}_{x_0}^{X}\phi(x).\end{aligned}$$

This method will now be extended to $\{f(D)\}_{x_0}^{X}\psi(x)\phi(x)$.

We shall assume $\psi(x)$ to be analytic in the closed interval (x_0, X) with the radius of convergence of its Taylor expansion at $x=X$ greater than $(X-x_0)$. Then, for this interval, we have

$$\psi(x) = \psi(X) + \frac{\psi'(X)}{1!}(x - X) + \frac{\psi''(X)}{2!}(x - X)^2 + \cdots.$$

When $f(D)$ is of finite order ρ, $f^{(\rho)}(D)$ is of order zero. Writing the remainder after ρ terms of the $\psi(x)$ expansion $(x-X)^\rho\chi(x)$ we have, by (39),

$$\{f(D)\}_{x_0}^{X}(x - X)^\rho\chi(x)\phi(x) = \{f^{(\rho)}(D)\}_{x_0}^{X}\chi(x)\phi(x).$$

Since $f^{(\rho)}(D)$ is of order zero, the integral formula (15) is applicable. Furthermore, as the series for $\chi(x)$ is uniformly convergent over (x_0, X), while the integral for $\{f^{(\rho)}(D)\}_{x_0}^{X}\phi(x)$ is absolutely convergent, we can integrate term by term. This is the same as operating with $f^{(\rho)}(D)$ term by term, or with $f(D)$ in the original series. We thus obtain, by (39), the generalized Leibnitz expansion

$$\{f(D)\}_{x_0}^{X}\psi(x)\phi(x) = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(X)}{n!}\{f^{(n)}(D)\}_{x_0}^{X}\phi(x). \tag{40}$$

For $f(D)$ of type zero, we will designate the remainder after m terms of the $\psi(x)$ series by $R_m(x)$, and prove directly that

$$\lim_{m\to\infty} \{f(D)\}_{x_0}^{X}R_m(x)\phi(x) = 0.$$

Applying Theorem II to this expression, we observe that $R_m(x)$ uniformly approaches zero as limit along (x_0, x_1) as $m\to\infty$, so that the integral likewise has zero for limit. On the other hand, for x_1 sufficiently near X, the expansion of Theorem VI is applicable to $\{f(D)\}_{x_1}^{X}R_m(x)\phi(x)$. The first N terms can be directly treated. On the other hand, the $h(N)$ of part (c) of §6, with $R_m(x)\phi(x)$ in place of $\phi(x)$, is an upper bound for the absolute value of the rest of this expansion. We thus see that

$$\left|\{f(D)\}_{x_0}^{X}R_m(x)\phi(x)\right| \leqq M_mL_m,$$

where M_m is the upper bound of the function of the complex variable $R_m(z)\phi(z)$ over the circle center x_1, radius l, while L_m depends only on $f(u)$, $(X-x_1)$, and the radius of convergence of the expansion of $R_m(z)\phi(z)$ about $z=x_1$. Now the radius of convergence of $R_m(z)$ is the same as that of $\psi(z)$, so that L_m is independent of m. On the other hand, $R_m(z)$ uniformly approaches zero over the circle of radius l, so that $M_m\to 0$, as $m\to\infty$. The limit in question is thus proved to be zero, and we can state

THEOREM XII. *If* $\{f(D)\}_{x_0}^{X}\phi(x)$ *exists under the hypotheses of any of our existence theorems, and if* $\psi(x)$ *is analytic along the closed interval* (x_0, X), *with the radius of convergence of its Taylor expansion at* $x=X$ *greater than* $(X-x_0)$, *then* $\{f(D)\}_{x_0}^{X}\psi(x)\phi(x)$ *is given by the generalized Leibnitz expansion* (40).

As a corollary, we have, by letting $\phi(x)$ identically equal 1,

$$(41)\qquad \{f(D)\}_{x_0}^{X}\psi(x) = \psi(X)\{f(D)\}_{x_0}^{X}1 + \frac{\psi'(X)}{1!}\{f'(D)\}_{x_0}^{X}1 + \cdots,$$

which is to be compared with series (21), which may be written

$$(42)\qquad \{f(D)\}_{x_0}^{X}\psi(x) = \psi(x_0)\{f(D)\}_{x_0}^{X}1 + \psi'(x_0)\{D^{-1}f(D)\}_{x_0}^{X}1 + \cdots.$$

In each case $\psi(x)$ is analytic along the closed interval (x_0, X); but in the first the radius of convergence at $x=X$ exceeds $X-x_0$, in the second that at $x=x_0$ exceeds $X-x_0$.

We reserve for §14 the application of (41) to entire operators.

10. $e^{ax}\phi(x)$ **as operand;** $f(D+a)$. From our definition we have

$$A[f(u+a)](r,\Delta x) = \frac{(-1)^r f^{(r)}\left(\dfrac{1}{\Delta x}+a\right)}{r!\Delta x^{r+1}}.$$

For real a, and sufficiently small Δx, the equation

$$\frac{1}{\Delta x} + a = \frac{1}{\Delta_1 x}$$

results in a positive $\Delta_1 x$. We can then write

$$A[f(u+a)](r,\Delta x) = (1 - a\Delta_1 x)^{r+1}A[f(u)](r,\Delta_1 x).$$

As $\Delta x \to +0$, and $r\Delta x \to t$, we have simultaneously $\Delta_1 x \to +0$, $r\Delta_1 x \to t$. Hence under the sole condition that a is real, we obtain

$$(43)\qquad A[f(u+a)](t) = e^{-at}A[f(u)](t).$$

When $f(z)$ is of type zero, and hence also when it is of finite order, this result is obtained immediately, for a both real and complex, by the use of the contour integral formula (8) of Theorem I.

The theorem of this section will be restricted to $f(D)$ of finite order. When $f(D)$ is of order zero, $f(D+a)$ also is. By (15), we have

$$\begin{aligned}\{f(D)\}_{x_0}^{X}e^{ax}\phi(x) &= \int_{x_0}^{X} A[f(u)](X-x)e^{ax}\phi(x)dx \\ &= e^{aX}\int_{x_0}^{X} e^{-a(X-x)}A[f(u)](X-x)\phi(x)dx,\end{aligned}$$

and so, by (43), and (15) again, we obtain

$$(44)\qquad \{f(D)\}_{x_0}^{X}e^{ax}\phi(x) = e^{aX}\{f(D+a)\}_{x_0}^{X}\phi(x).$$

When $f(D)$ is of order ρ, $D^{-\rho}f(D)$ is of order zero. We then have

$$\begin{aligned}\{f(D)\}_{x_0}^{X} e^{ax}\phi(x) &= d^\rho/dX^\rho\{D^{-\rho}f(D)\}_{x_0}^{X} e^{ax}\phi(x)\\ &= e^{aX}(d/dX + a)^\rho\{(D+a)^{-\rho}f(D+a)\}_{x_0}^{X}\phi(x)\\ &= e^{aX}\{f(D+a)\}_{x_0}^{X}\phi(x);\end{aligned}$$

the first by (26), the second by (44) for operators of order zero and polynomials in d/dX, the last by (34). We can therefore state

THEOREM XIII. *If $f(D)$ is of finite order, and $\phi(x)$ satisfies the hypothesis of the existence Theorem* IV, *then relation* (44) *is valid.*

By letting $\phi(x)\equiv 1$, (44) and (8) yield a contour integral for $\{f(D)\}_{x_0}^{X} e^{ax}$.

11. $x_0 = -\infty$; $\phi(x) = e^{ax}$. Let $f(z)$ be analytic to the right of the line $R(z) = c$. Then

$$\begin{aligned}\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{-\infty}^{X} e^{ax} &= f\left(\frac{1}{\Delta x}\right)e^{aX} - \frac{f'\left(\dfrac{1}{\Delta x}\right)}{1!\Delta x}e^{a(X-\Delta x)} + \cdots\\ &= e^{aX}f\left(\frac{1}{\Delta x} - \frac{e^{-a\Delta x}}{\Delta x}\right),\end{aligned}$$

provided $f(1/\Delta x - e^{-a\Delta x}/\Delta x)$ can be expanded in powers of $e^{-a\Delta x}/\Delta x$. Now

$$|e^{-a\Delta x}/\Delta x| = e^{-R(a)\Delta x}/\Delta x = 1/\Delta x - R(a) + \epsilon_{\Delta x}, \quad \lim_{\Delta x\to+0}\epsilon_{\Delta x} = 0.$$

Hence if $R(a) > c$, this absolute value will be less than $1/\Delta x - c$, for Δx sufficiently small. As the radius of convergence of the Taylor expansion of $f(z)$ about $1/\Delta x$ is at least $1/\Delta x - c$, the above is valid, so that

$$\{f(D)\}_{-\infty}^{X} e^{ax} = \lim_{\Delta x\to+0} e^{aX}f\left(\frac{1 - e^{-a\Delta x}}{\Delta x}\right) = e^{aX}f(a).$$

Conversely, if $f(z)$ is an analytic function of z, and $\{f(D)\}_{-\infty}^{X} e^{ax}$ exists for some a, $\{f(\Delta/\Delta x)\}_{-\infty}^{X} e^{ax}$ must converge for all positive Δx's less than some positive ϵ. Then $f(z)$ will be analytic within all circles center $1/\Delta x$, radius $|e^{-a\Delta x}/\Delta x|$ where $0 < \Delta x < \epsilon$. As these circular regions cover the half-plane $R(z) > R(a)$, $f(z)$ is analytic to the right of the line $R(z) = R(a)$. Hence

THEOREM XIV. *The necessary and sufficient condition that $\{f(D)\}_{-\infty}^{X} e^{ax}$ exist for some a, when $f(z)$ is an analytic function of z, is that $f(z)$ is analytic to the right of some line $R(z) = c$. In that case $\{f(D)\}_{-\infty}^{X} e^{ax}$ exists for $R(a) > c$, and is given by*

$$\{f(D)\}_{-\infty}^{X} e^{ax} = e^{aX}f(a). \tag{45}$$

If $c<0$, we can let $a=0$, and so obtain

$$\{f(D)\}_{-\infty}^{X} 1 = f(0). \tag{46}$$

The above results were obtained without assuming $f(z)$ of type zero, though the analyticity condition of Theorem XIV is a weaker form of the analyticity condition for an operator of type zero. If $f(z)$ is of type zero, we can apply Theorem XV of the next section to obtain

$$\lim_{x_0 \to -\infty} \{f(D)\}_{x_0}^{X} e^{ax} = \{f(D)\}_{-\infty}^{X} e^{ax}. \tag{47}$$

This can also be obtained directly by expressing the remainder after p terms of the Taylor expansion $\{f(\Delta/\Delta x)\}_{-\infty}^{X} e^{ax}$ as a contour integral, finding for its limit as $\Delta x \to +0$, $p = \{(X-x_0)/\Delta x\}$, an integral like that of formula (8), and observing that this approaches zero as limit as $x_0 \to -\infty$. If we further restrict $f(z)$ to order zero, we can use (15) in (47), and by the change of variable $X-x=t$, obtain through (45)

$$\int_0^\infty e^{-at} A[f](t)dt = f(a).* \tag{48}$$

That is, $A[f](t)$ satisfies the Laplace integral equation when $f(z)$ is of order zero. The derivation of (48) suggests that we may consider relation (45) a generalization of the Laplace integral equation.

12. $x_0 = -\infty$: **general case.** Our definition, for $x_0 = -\infty$, is

$$\{f(D)\}_{-\infty}^{X} \phi(x) = \lim_{\Delta x \to +0} \left\{ f\left(\frac{\Delta}{\Delta x}\right)\right\}_{-\infty}^{X} \phi(x).$$

Hence the relation

$$\{f(D)\}_{-\infty}^{X} \phi(x) = \lim_{x_0 \to -\infty} \{f(D)\}_{x_0}^{X} \phi(x) \tag{49}$$

is subject to proof. Since, in our definition, we consider the limit of an infinite series, some restriction on the behavior of $\phi(x)$ as $x \to -\infty$ will have to be introduced to insure the existence of the limits involved. The same condition will turn out to be adequate for (49).

We assume $f(D)$ to be of type zero. Then $f(z)$ will certainly be analytic

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* When $A[f](t)$ can be directly found from its definition, formula (48) leads to the evaluation of a corresponding definite integral. Thus, (6), the formula for $A[u^n](t)$, yields the well known infinite integral for the gamma function for $R(n)<0$, while (3), as transformed by (12), results in an infinite integral for the Eulerian constant γ. In a similar manner, formula (8) leads to the evaluation of corresponding contour integrals.

to the right of some line $R(z)=c$. *Let d designate the lower limit of these c's.* Then, *for Δx sufficiently small, and $r\Delta x$ greater than a positive h, there is a positive L for each b greater than d such that*

$$|A[f](r, \Delta x)| < Le^{b(r\Delta x)}. \tag{50}$$

To prove this fundamental inequality, we reconsider the discussion of the contour integral for $A[f](r, \Delta x)$ given in §3. For Δx sufficiently small, and $r\Delta x > h$, r will be sufficiently large to have, along C'',

$$\left|\int_{C''} \frac{f(z)dz}{(1 - z\Delta x)^{r+1}}\right| < L_1\lambda^r, \ 0 < \lambda < 1.$$

Since, for Δx sufficiently small, we shall have λ less than $e^{b\Delta x}$ even with b negative, a relation like (50) holds for this contribution to $A[f](r, \Delta x)$. Along $C'_{l,m}$, we can have, with l and m sufficiently large, $R(z) < b'$, where $b' < b$, and $b' < 0$. We can then write

$$\left|\int_{C'_{l,m}} \frac{f(z)dz}{(1 - z\Delta x)^{r+1}}\right| < \frac{1}{(1 - b'\Delta x)^{r+1-\{h/\Delta x\}}} \int_{C'_{l,m}} \frac{|f(z)||dz|}{|1 - z\Delta x|^{\{h/\Delta q\}}}.$$

The latter integral has a finite upper bound, for Δx sufficiently small, in accordance with §3. On the other hand, since $b' < b$, we have, again for sufficiently small Δx,

$$\frac{1}{1 - b'\Delta x} < e^{b\Delta x}.$$

Hence an inequality (50) exists for the $C'_{l,m}$ contribution. Finally, we may have to change $C_{l,m}$ so that it will consist of the segment of $R(z) = b''$, $d < b'' < b$, cut off by the former $C_{l,m}$, joined to the part of the former $C_{l,m}$ to the left of $R(z) = b''$. Since along this new $C_{l,m}$ we have

$$|1 - z\Delta x| \geqq 1 - b''\Delta x,$$

we find that

$$\left|\int_{C_{l,m}} \frac{f(z)dz}{(1 - z\Delta x)^{r+1}}\right| \leqq \frac{\bar{l}K'}{(1 - b''\Delta x)^{r+1}},$$

where $\bar{l}$ is the length of $C_{l,m}$, and K' the upper bound of $|f(z)|$ along it. Hence, as in the case of $C'_{l,m}$, a relation (50) holds. By combining these three results we get (50) itself.

If in (50) we let $\Delta x \to +0$, $r\Delta x \to t$, we obtain, for $t > h$,

$$|A[f](t)| \leqq Le^{bt}. \tag{51}$$

We can now prove

THEOREM XV. *If $f(D)$ is of type zero, with specified d, and if, for some positive M, and real c greater than d, $\phi(x)$ satisfies the inequality*

$$|\phi(x)| \leqq Me^{cx}$$

for every x less than some real x_1, relation (49) *will be valid.*

Choose b between c and d. By (50), and the condition on $\phi(x)$, we have

$$\left| \left\{ f\left(\frac{\Delta}{\Delta x}\right)\right\}_{-\infty}^{X} \phi(x) - \left\{ f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X} \phi(x) \right|$$
$$< LMe^{cX} \sum_{r=p+1}^{\infty} e^{-(c-b)r\Delta x}\Delta x, \qquad p = \left\{\frac{X - x_0}{\Delta x}\right\}$$

provided $x_0 < x_1$, and also $x_0 < X - h$. Since this geometric progression converges, $\{f(\Delta/\Delta x)\}_{-\infty}^{X}\phi(x)$ also converges. Furthermore, on summing this progression, we find its limit, as $\Delta x \to +0$, to be $[LM/(c-b)]e^{cX}e^{-(c-b)(X-x_0)}$. This expression approaches zero as limit as $x_0 \to -\infty$. We can therefore apply the limit criterion, with x_0 in place of ν, to the identity

$$\left\{ f\left(\frac{\Delta}{\Delta x}\right)\right\}_{-\infty}^{X} \phi(x) = \left\{ f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X} \phi(x)$$
$$+ \left[\left\{ f\left(\frac{\Delta}{\Delta x}\right)\right\}_{-\infty}^{X} \phi(x) - \left\{ f\left(\frac{\Delta}{\Delta x}\right)\right\}_{x_0}^{X} \phi(x) \right],$$

and obtain the theorem.

The use of the limit criterion assumes the existence of $\{f(D)\}_{x_0}^{X}\phi(x)$ for all x_0's less than X. Granting this, its corollary gives the existence of both members of (49). Hence

COROLLARY I. *If $\phi(x)$ is continuous for $x < X$, and satisfies the X-neighborhood conditions of any of our existence theorems for finite x_0, then, under the above hypothesis, both members of* (49) *exist, and are finite.*

It is also not difficult to establish

COROLLARY II. *If the X-neighborhood condition of Corollary* I *is not known to be fulfilled, but instead $\{f(D)\}^{X}{}_{\infty}\phi(x)$ is known to exist, then the existence of $\{f(D)\}_{x_0}^{X}\phi(x)$ for finite x_0 follows.*

The rest of this section will be devoted to the law of successive operations for $x_0 = -\infty$. For finite x_0, the order of $f_2(D)$ had to be zero or one, unless $\phi(x)$, and a sufficient number of its derivatives, vanished at $x = x_0$. In the present case this restriction on the order of $f_2(D)$ disappears. We shall, however, still restrict ourselves to operators of finite order.

First, in Theorem XV, let $f(D)$ be of order zero. Then, by (49) and (15), we obtain

$$\{f(D)\}_{-\infty}^{X}\phi(x) = \int_{-\infty}^{X} A[f](X-x)\phi(x)dx. \tag{52}$$

Break up this integral over limits $(-\infty, X-a)$, and $(X-a, X)$. By applying to the resulting integrals inequality (51), with b between c and d, and the hypothesis of Theorem XV, we obtain, for $X<x_1$, and $a>h$,

$$\left|\int_{-\infty}^{X} A[f](X-x)\phi(x)dx\right| < Me^{cX+|c|a}\int_{0}^{a} |A[f](t)|\,dt$$
$$+ [LM/(c-b)]e^{-a(c-b)}\cdot e^{cX} = M'e^{cX}.$$

That is, $|\{f(D)\}_{-\infty}^{X}\phi(x)|$ satisfies the same inequality that $|\phi(x)|$ satisfies except for the factor M. It is also easily seen that $\{f(D)\}_{-\infty}^{X}\phi(x)$ is continuous in X.

Let then $f_1(D)$ and $f_2(D)$ both be of order zero, with d's equal to d_1 and d_2 respectively. If, in the hypothesis of Theorem XV, we have $c>d_1$ as well as $c>d_2$, our last result shows that the existent $\{f_2(D)\}_{-\infty}^{X}\phi(x)$ can be used as operand for $\{f_1(D)\}_{-\infty}^{X}$, so that we can write

$$\{f_1(D)\}_{-\infty}^{X}\{f_2(D)\}_{-\infty}^{x}\phi(\xi) = \int_{-\infty}^{X} A[f_1](X-x)\left[\int_{-\infty}^{x} A[f_2](x-\xi)\phi(\xi)d\xi\right]dx.$$

Since this iterated integral is absolutely convergent, the corresponding improper double integral exists.* It can therefore be rewritten

$$\int_{-\infty}^{X}\left[\int_{x}^{X} A[f_1](X-\xi)A[f_2](\xi-x)d\xi\right]\phi(x)dx.$$

Now the d of $f_1(D)f_2(D)$ does not exceed both d_1 and d_2. We can therefore use (52), with $f_1(D)f_2(D)$ as operator, $\phi(x)$ as operand, and, by applying (32), obtain

$$\{f_1(D)\}_{-\infty}^{X}\{f_2(D)\}_{-\infty}^{x}\phi(\xi) = \{f_1(D)f_2(D)\}_{-\infty}^{X}\phi(x). \tag{53}$$

Before (53) can be extended to operators of arbitrary finite order, several preliminary results must be obtained. First, we have

$$\frac{d}{dX}\{f(D)\}_{-\infty}^{X}\phi(x) = \{Df(D)\}_{-\infty}^{X}\phi(x) \tag{54}$$

under the same X-neighborhood conditions as for finite x_0, along with the

* The reasoning is the same as in §8 for which a reference has been given.

usual inequality on $|\phi(x)|$.* In fact, our inequalities easily show that the integral resulting from differentiating $\int_{-\infty}^{x_2} A[f](X-x)\phi(x)dx$ under the integral sign is uniformly convergent in X. Secondly, we have to discard the reduction formula (17), since the d of $D^{-\rho}f(D)$ will be greater than that of $f(D)$ when the latter is negative and ρ is not zero. By §10 and formula (17) with x_1 and x_0 identical, we obtain in its place

$$\begin{aligned}\{f(D)\}_{x_0}^{X}\phi(x) &= \{(D-l)^{-\rho}f(D)\}_{x_0}^{X}\left(\frac{d}{dx}-l\right)^{\rho}\phi(x)\\ &\quad+\sum_{\mu=0}^{\rho-1}A[(u-l)^{-\mu-1}f(u)](X-x_0)\cdot\left[\left(\frac{d}{dx}-l\right)^{\mu}\phi(x)\right]_{x=x_0},\end{aligned}\tag{55}$$

where, for validity, $\phi(x)$ and its first ρ derivatives are continuous in (x_0, X).

To allow x_0 to approach $-\infty$ as limit in (55), we shall want $\phi(x)$, and its first ρ derivatives, to be continuous for $x \leqq X$. Also, for some c greater than d, and some positive M, we must have, for $x < x_1$,

$$|\phi^{(\mu)}(x)| \leqq Me^{cx}; \quad \mu = 0, 1, 2, \cdots, \rho.\dagger$$

As a result, we also have, for $x < x_1$,

$$\left|\left(\frac{d}{dx}-l\right)^{\mu}\phi(x)\right| \leqq M'e^{cx}; \quad \mu = 0, 1, 2, \cdots, \rho.$$

Now choose l less than c. The d of $(D-l)^{-\mu-1}f(D)$ will then also be less than c. Choose b between this d and c. Then, by (51),

$$|A[(u-l)^{-\mu-1}f(u)](X-x_0)| \leqq L_\mu e^{b(X-x_0)}.$$

As $b < c$, these two inequalities give us

$$\lim_{x_0\to-\infty} A[(u-l)^{-\mu-1}f(u)](X-x_0)\left[\left(\frac{d}{dx}-l\right)^{\mu}\phi(x)\right]_{x=x_0} = 0.$$

We thus obtain

$$\{f(D)\}_{-\infty}^{X}\phi(x) = \{(D-l)^{-\rho}f(D)\}_{-\infty}^{X}\left(\frac{d}{dx}-l\right)\phi(x).\tag{56}$$

Though (56) reduces all operators of finite order to operators of order zero, for which we have already established the law of successive operations, its conditions are stronger than those yielded by the following more extended

* See the first footnote of §8.

† If $\phi(x)$ behaves "regularly" as $x\to-\infty$, the inequalities for $\mu>0$ will follow from the one for $\mu=0$.

treatment. The operators $f_1(D)$ and $f_2(D)$ will now be of arbitrary finite orders ρ_1 and ρ_2 respectively with corresponding d_1 and d_2. We shall require the inequalities

$$|\phi^{(\mu)}(x)| \leqq M_1 e^{c_1 x}, \quad \mu = 0, 1, 2, \cdots, \rho_2, \quad \text{for} \quad x < x_1, \quad \text{with} \quad c_1 > d_1.$$

When $d_2 \geqq c_1$, we shall also require, to insure the existence of $\{f_2(D)\}_{-\infty}^{x}\phi(\xi)$,

$$|\phi(x)| \leqq M_2 e^{c_2 x}, \text{ for } x < x_1, \text{ with } c_2 > d_2.$$

In the latter case $c_2 > c_1$, so that, for some x_2, we shall have for $x < x_2$, $M_2 e^{c_2 x} < M_1 e^{c_1 x}$. In either case, we can prove that, if l is less than c_1,

$$\left|\frac{d^\mu}{dx^\mu}\{(D-l)^{-\rho_2} f_2(D)\}_{-\infty}^{x}\phi(\xi)\right| \leqq M_3 e^{c_1 x}, \text{ for } x < x_3,\ \mu = 0, 1, 2, \cdots, \rho_2.$$

In fact, through (54) extended, we can write

$$\frac{d^\mu}{dx^\mu}\{(D-l)^{-\rho_2} f_2(D)\}_{-\infty}^{x}\phi(\xi) = \{D^\mu (D-l)^{-\rho_2} f_2(D)\}_{x-a}^{x}\phi(\xi)$$
$$+ \int_{-\infty}^{x-a} A[u^\mu (u-l)^{-\rho_2} f_2(u)](x-\xi)\phi(\xi)d\xi.$$

By reducing the first term of the right hand member of this equation by formula (17) with $\rho = \mu$, $X = x$, $x_1 = x_0 = x - a$, we can easily apply our inequalities, by methods already made familiar, to obtain the desired result. We can therefore use (56), with ρ_2 in place of ρ, $(D-l)^{-\rho_1} f_1(D)$ in place of $f(D)$, and $\{(D-l)^{-\rho_2} f_2(D)\}_{-\infty}^{x}\phi(\xi)$ in place of $\phi(x)$ to obtain

$$\{(D-l)^{-\rho_1} f_1(D)\}_{-\infty}^{X}\{(D-l)^{-\rho_2} f_2(D)\}_{-\infty}^{x}\phi(\xi)$$
$$= \{(D-l)^{-(\rho_1+\rho_2)} f_1(D)\}_{-\infty}^{X}\left(\frac{d}{dx} - l\right)^{\rho_2}\{(D-l)^{-\rho_2} f_2(D)\}_{-\infty}^{x}\phi(\xi).$$

Now reduce the first member of this equation by our proved result for operators of order zero, the second member by operating with $(d/dx - l)^{\rho_2}$ formally, as is justified by (54). We thus obtain

$$\{(D-l)^{-(\rho_1+\rho_2)} f_1(D) f_2(D)\}_{-\infty}^{X}\phi(x) = \{(D-l)^{-(\rho_1+\rho_2)} f_1(D)\}_{-\infty}^{X}\{f_2(D)_{-\infty}^{x}\}\phi(\xi).$$

Operating on both sides of this equation by $(d/dx - l)^{\rho_1+\rho_2}$ yields our law of successive operations for operators of arbitrary finite orders.

In this discussion we have merely concerned ourselves with the convergence difficulties introduced by letting $x_0 = -\infty$. When we also supply the discussion of the existence of derivatives tacitly assumed above, a discussion that does not introduce anything essentially new, we obtain

THEOREM XVI. *Let $f_1(D)$ and $f_2(D)$ be of finite orders ρ_1 and ρ_2 respectively, with corresponding d_1 and d_2, and let $D^{\rho_1}f_2(D)$ be of order ρ. Let $\phi(x)$ and its first ρ_2 derivatives be continuous for $x \leqq X$, and also let the first ρ derivatives of $\phi(x)$ be continuous in a left neighborhood of $x = X$. Finally, let us have the inequalities*

$$|\phi^{(\mu)}(x)| \leqq M_1 e^{c_1 x}, \text{ for } x < X, \ \mu = 0, 1, 2, \cdots, \rho_2, \text{ with } c_1 > d_1,$$

and, if c_1 does not exceed d_2, let us also have

$$|\phi(x)| \leqq M_2 e^{c_2 x}, \text{ for } x < X, \text{ with } c_2 > d_2.$$

Then formula (53) *is valid, and both of its members exist.*

13. $x_0 = -\infty$: special case. When $d = 0$, the existence theorem of the preceding section still requires an exponential inequality for $|\phi(x)|$ as $x \to -\infty$. This can be replaced by an algebraic inequality if certain assumptions are made about the behavior of $f(z)$ in the neighborhood of those singularities that lie on the axis of imaginaries. The operator D^m is typical.

Let then $f(z)$ be of type zero, and analytic to the right of the axis of imaginaries. We shall first assume the origin to be the only singularity with real part zero. The assumptions required are the following:

(a) *$f(z)$ is analytic within a circular sector of radius λ, and angle greater than π, which has its vertex at the origin, and is bisected by the positive half of the real axis,*

(b) *in this sector, for some real m and positive K, $|f(z)|$ satisfies the inequality*

$$|f(z)| \leqq K|z|^m.$$

Let S designate the interior of the infinite analytic sector of the type zero condition, S' the rest of the z-plane; s the interior of the circular sector of radius λ, s' the rest of the interior of the circle. It will be convenient to have λ small enough for this circle to lie wholly within S'. Since the origin is the only singularity with real value zero, there will be a line $R(z) = -\kappa_1$, $\kappa_1 > 0$, to the right of which $f(z)$ is analytic, except for z's in s'. Now in the argument leading to inequality (50) of the preceding section, replace d by $-\kappa_1$, and choose the b'' of that argument between 0 and $-\kappa_1$, and also sufficiently near zero to have the line $R(z) = b''$ cut the sides of s'. If then we remove from C''', as modified in that argument, the portion of this line that is in s', we shall have along the resulting open contour C^{iv}

$$\left| \int_{C^{\text{iv}}} \frac{f(z)dz}{(1 - z\Delta x)^{r+1}} \right| < L_1 e^{-\kappa(r\Delta x)}$$

where $-\kappa$ is between 0 and b'', and hence is negative.* For the purposes of this section we further modify C''' as follows. Let C_δ^{v} consist of two segments in s, starting from a point $z=\delta$ on the real axis, running parallel to the sides of s, and terminated by $R(z)=b''$. Such a contour is possible for δ sufficiently small. Join this to C^{iv}, where we remove from C^{iv} the portions of $R(z)=b''$ that lie between the ends of C_δ^{v}. Then C_δ^{v} with the shortened C^{iv} forms an admissible C'''. Now if C^{iv} is thus shortened in the above inequality, the result is simply strengthened. As for C_δ^{v}, replace δ by $1/[2(r+1)\Delta x]$, and transform z by $\zeta=(r+1)\Delta xz$. Using condition (b), we obtain

$$\left|\int_{C_\delta^{\mathrm{v}}}\frac{f(z)dz}{(1-z\Delta x)^{r+1}}\right| \leqq \frac{K}{[(r+1)\Delta x]^{m+1}}\int_{C_{1/2}^{\mathrm{v}}}\frac{|\zeta|^m\,|d\zeta|}{\left|1-\dfrac{\zeta}{r+1}\right|^{r+1}}\cdot$$

If the sides of $C_{1/2}^{\mathrm{v}}$ be extended to infinity, we obtain the same integral that occurred in §11, except that m replaces $\rho-\epsilon$. This integral was there shown to converge, and be bounded with respect to r for $r\geqq\rho$. The same is thus true of the $C_{1/2}^{\mathrm{v}}$ integral for $r>m$. Combining these results, we are enabled to state that, *for Δx sufficiently small, and $r\Delta x$ greater than a fixed positive h, we have with a positive L,*

$$|A[f](r,\Delta x)| < \frac{L}{(r\Delta x)^{m+1}}\cdot \tag{57}$$

Hence, also, for t greater than h, we have

$$|A[f](t)| \leqq \frac{L}{t^{m+1}}\cdot\dagger \tag{58}$$

This result can be immediately extended to the case where there are a finite number of singularities of the above type on the axis of imaginaries. If these singularities are at points $z=z_i$, then on each we impose conditions like (a) and (b), where in (a) the vertex of the sector is now at z_i, while the inequality in (b) is replaced by

$$|f(z)| \leqq K_i|z-z_i|^{m_i}.$$

The line $R(z)=-\kappa_1$ can be chosen once for all, and C''' modified near each of the singularities. As a result, (57) and (58) will follow, provided *m is*

* Inequality (b) of this section, in conjunction with the choice of b'' made here, allows us to conclude that $|f(z)|$ has a finite upper bound along that part of $R(z)=b''$ used in C^{iv}, as is required in this proof.

† This inequality can be used as the basis of a simple derivation of certain of the Heaviside asymptotic expansions, and extensions thereof.

the least of the m_i's. For ease of reference we shall speak of a function and corresponding operator of this kind as being of *degree m.* We can now state

THEOREM XVII. *If an operator $f(D)$ of type zero, with $d=0$, is of degree m, and if for $x<x_1$, $x_1<0$, we have*

$$|\phi(x)| \leqq M(-x)^{m-\eta}$$

with positive η and M, the equivalence relation (49) *will hold.*

COROLLARY. *With this hypothesis replacing that of Theorem* XV, *Corollary* I *and Corollary* II *of the latter theorem continue to hold here.*

In the proof of this theorem the geometric progression of the preceding section is replaced by a series that can be written

$$LM\sum_{r=p+1}^{\infty}\left(1-\frac{X}{r\Delta x}\right)^{m+1}\frac{1}{(r\Delta x - X)^{1+\eta}}\Delta x,$$

provided $x_0<x_1$, and, also, $x_0<X-h$. Since $[1-X/(r\Delta x)]^{m+1}$ is bounded as $\Delta x\rightarrow +0$, and $x_0\rightarrow -\infty$, while we have

$$\sum_{r=p+1}^{\infty}\frac{1}{(r\Delta x - X)^{1+\eta}}\Delta x < \int_{X-x_0-\Delta x}^{\infty}(t-X)^{-1-\eta}dt = \frac{(-x_0-\Delta x)^{-\eta}}{\eta},$$

recourse can again be had to the limit criterion to give the theorem.

The case $\phi(x)\equiv 1$ is of special interest. When $d<0$, the discussion of §11 applies, as well as the resulting formula (46). When $d>0$, the same discussion shows $\{f(D)\}_{-\infty}^{X}1$ to be non-existent. When $d=0$, (46) still holds, provided $f(z)$ is analytic at the origin; but the equivalence relation (49) seems to require further assumptions. The following theorem, which we state without proof, contains several simpler cases, and can be considered the extension of (46) for $d=0$.

THEOREM XVIII. *If $f(z)$ is of type zero, with $d=0$, and if, except for the origin, it is of degree of greater than -1, while for the analytic sector vertexed at the origin it approaches a finite limit K as $z\rightarrow 0$ in the sector, then*

$$\{f(D)\}_{-\infty}^{X}1 = K = \lim_{x_0\rightarrow -\infty}\{f(D)\}_{x_0}^{X}1. \tag{59}$$

We turn now to the law of successive operations. As before, $f_1(z)$ and $f_2(z)$ will be of finite order. The case of most interest is the one in which both d_1 and d_2 are zero, the treatment of the cases where but one d is zero then being obvious. On the whole, the development of the preceding section can be followed.

First let $f(D)$ be of order zero, while otherwise it, and $\phi(x)$, satisfy the hypothesis of Theorem XVII. We can again use (46), and break up the infinite integral over limits $(-\infty, X-a)$, $(X-a, X)$. Choose a greater than h, and let X be less than x_1. We then easily obtain

$$\left|\int_{X-a}^{X} A[f](X-x)\phi(x)dx\right| \leqq M(-X+\theta a)^{m-\eta}\int_0^a |A[f](t)|\,dt < M''(-X)^{m-\eta},$$

$$\left|\int_{-\infty}^{X-a} A[f](X-x)\phi(x)dx\right| < LM\int_{-\infty}^{X-a} \frac{(-x)^{m-\eta}}{(X-x)^{m+1}}dx$$

$$= LM(-X)^{-\eta}\int_0^{-X/a}(1+t)^{m-\eta}t^{\eta-1}dt,$$

where, in the latter, we have set $t=-X/(X-x)$. It is convenient to assume $X<-a$, so that $-X/a>1$. We can then break up this integral over limits $(0,1)$, and $(1,-X/a)$. The first part converges, and is independent of X. For the second, we can write, with some fixed k,

$$\int_1^{-X/a}(1+t)^{m-\eta}t^{\eta-1}dt < k\int_1^{-X/a} t^{m-1}dt.$$

The case $m=0$ can be avoided. For $m\neq 0$ we thus obtain the inequalities

$$m>0:\quad |\{f(D)\}_{-\infty}^{X}\phi(x)| < M'(-X)^{m-\eta}, \tag{60}$$

$$m<0:\quad |\{f(D)\}_{-\infty}^{X}\phi(x)| < M'(-X)^{-\eta}. \tag{61}$$

Now let $f_1(D)$ and $f_2(D)$ both be of order zero, with d_1 and d_2 zero, and associated m_1 and m_2. No loss of generality ensues if we assume the d of $f_1(D)f_2(D)$ to be zero.* The corresponding m_3 is evidently not less than m, the least of the three quantities m_1, m_2, m_1+m_2. We can then show that if $\phi(x)$ satisfies the hypothesis of Theorem XVII for m so defined, the law of successive operations will be satisfied. It will be sufficient to show that $\{f_2(D)\}_{-\infty}^{x}\phi(\xi)$, $\{f_1(D)\}_{-\infty}^{X}\{f_2(D)\}_{-\infty}^{x}\phi(\xi)$ and $\{f_1(D)f_2(D)\}_{-\infty}^{X}\phi(x)$ exist in accordance with Theorem XVII, since the argument of the preceding section will then be valid here. The inequality of Theorem XVII is the primary consideration. Since the three operators involved, $f_2(D)$, $f_1(D)$, $f_1(D)f_2(D)$, are of degrees m_2, m_1, and m_3 respectively, we have to show that the corresponding operands $\phi(\xi)$, $\{f_2(D)\}_{-\infty}^{x}\phi(\xi)$ and $\phi(x)$ satisfy inequalities with exponents $m_2-\eta_2$, $m_1-\eta_1$ and $m_3-\eta_3$ respectively, where the η's are all positive. Note that our hypothesis makes $\phi(x)$ satisfy an inequality with

* This d may be less than zero; but in that case the actual inequalities are even stronger than on the assumption that it is zero.

exponent $m-\eta$ where η is positive. This suffices for the first and last cases; for by identifying this inequality with the desired inequalities, we are setting $m_2-\eta_2=m-\eta$, $m_3-\eta_3=m-\eta$. Since m does not exceed m_2 or m_3, we thus have $\eta_2\geqq\eta$, $\eta_3\geqq\eta$, that is, η_2 and η_3 are positive, as was desired. For η_1, we must use the inequalities (60) and (61). The case $m_2=0$ can be avoided since m_2 can always be decreased in the inequalities on $|f_2(z)|$, at least for $|z-z_i|\leqq 1$, and by making this decrease less than η there will at worst result a corresponding decrease in m and η in the original inequality on $\phi(x)$, with the new η still positive. For $m_2>0$, we see from (60) that $\{f_2(D)\}_{-\infty}^{x}\phi(\xi)$ satisfies an inequality with exponent equal to that of the inequality for $\phi(\xi)$. That is, we may set $m_1-\eta_1=m-\eta$, and as before, obtain $\eta_1\geqq\eta$. For $m_2<0$, (61) shows that the exponent for $\{f_2(D)\}_{-\infty}^{x}\phi(\xi)$ is but $-\eta_2$, if $m_2-\eta_2$ is the exponent for $\phi(\xi)$. That is, we may set $m_1-\eta_1=-\eta_2$, where we have identically $m_2-\eta_2=m-\eta$. By combining the two equations, we obtain $\eta_1=m_1+m_2-m+\eta\geqq\eta$. A positive η_1 therefore results. The sufficiency of the $m-\eta$ exponent for $\phi(x)$ has thus been demonstrated.

It is unnecessary to go into the details of the extension of these results to operators of arbitrary finite orders, as the steps of the procedure of the previous section are easily verified here. We can thus state

THEOREM XIX. *Let $f_1(D)$ and $f_2(D)$ be of finite orders ρ_1 and ρ_2 respectively with d_1 and d_2 zero, and associated m_1 and m_2, and let $D^{\rho_1}f_2(D)$ be of order ρ. Let $\phi(x)$ and its first ρ_2 derivatives be continuous for $x\leqq X$, and also let the first ρ derivatives of $\phi(x)$ be continuous in a left neighborhood of $x=X$. Finally, for all x's such that $x<x_1<0$, let*

$$|\phi(x)|\leqq M(-X)^{m-\eta},\quad |\phi^{(\mu)}(x)|\leqq M_1(-X)^{m_1-\eta_1},\quad \mu=1,2,\cdots,\rho_2,$$

where η and η_1 are positive, and m is the least of the three quantities m_1, m_2, m_1+m_2. Then formula (53) *holds, and both of its members exist.*

14. **Entire operators;** $A[f](t)\equiv 0$. If $f(z)$ is an entire function of the complex variable z, and satisfies condition (b) of §3 over the whole z-plane, then $f(D)$ will be said to be of *type zero over the plane*. Since entire transcendental functions of genus zero are known to satisfy this condition, their corresponding operators are of type zero over the plane. Polynomials in D are also included.

We first prove that, for such operators, $A[f](t)$ *vanishes identically*. In formula (8) of §3 choose C so that its vertex is at the origin. Let $\overline{C}_N$ be the portion of the line $R(z)=-N$, $N>0$, between the half-lines of C. Since $f(z)$ is entire, the integral in (8), limited to that part of C which is to the right of this line, will equal the same integral over $\overline{C}_N$. Hence we have

$$A[f](t) = \frac{1}{2\pi i} \lim_{N\to\infty} \int_{\bar{C}_N} e^{tz} f(z) dz.$$

By choosing κ of condition (b) less than $t \cos \alpha$, this limit is seen to be zero.

An immediate consequence of this result is that, for continuous $\phi(x)$, $\{f(D)\}_{x_0}^{X}\phi(x)$ *is independent of the finite lower limit* x_0, since by (9),

$$\{f(D)\}_{x_0}^{X}\phi(x) = \{f(D)\}_{x_1}^{X}\phi(x).$$

The same is evidently true for $x_0 = -\infty$, if $\phi(x)$ satisfies the hypothesis of Theorem XV. We can now easily show that when $f(D)$ is of type zero over the plane, $\{f(D)\}_{x_0}^{X}\phi(x)$ *reduces to the formal result obtained by expanding* $f(D)$ *in powers of* D. The function $\phi(x)$ is of course assumed to be analytic in a left neighborhood of $x = X$. We can then choose x_1 sufficiently near X to apply the Leibnitz expansion (41) to $\{f(D)\}_{x_1}^{X}\phi(x)$. We thus have

$$\{f(D)\}_{x_0}^{X}\phi(x) = \phi(X)\{f(D)\}_{x_1}^{X}1 + \frac{\phi'(X)}{1!}\{f'(D)\}_{x_1}^{X}1 + \cdots .$$

Now $f^{(n)}(D)$ is of type zero over the plane for every n. Hence

$$\{f^{(n)}(D)\}_{x_1}^{X}1 = \{f^{(n)}(D)\}_{-\infty}^{X}1 = f^{(n)}(0),$$

the last by (46). We therefore have the standard expansion

$$\{f(D)\}_{x_0}^{X}\phi(x) = f(0)\phi(X) + \frac{f'(0)}{1!}\phi'(X) + \frac{f''(0)}{2!}\phi''(X) + \cdots . \tag{62}$$

Suppose now, conversely, that $\{f(D)\}_{x_0}\phi(x)$ is independent of x_0 in a certain x_0 interval. Differentiation of (9) with respect to x_0 gives

$$A[f](X - x)\phi(x_0) = 0.$$

If then $\phi(x_0)$ does not vanish in this interval, $A[f](t)$ must vanish over a corresponding interval. Now formula (8) shows $A[f](t)$ to be an analytic function of t for every real and positive t. Hence if $A[f](t)$ vanishes over an interval, it vanishes identically. The question thus raised is answered by

THEOREM XX. *If $f(D)$ is of finite order, $A[f](t)$ vanishes identically when, and only when, $f(D)$ is a polynomial; if $f(D)$ is of type zero, $A[f](t)$ vanishes identicallly when, and only when, $f(D)$ is of type zero over the plane.*

The direct part of this theorem has already been demonstrated. As for the converses, let $f(D)$ first be of finite order ρ. Then $f^{(\rho)}(D)$ will be of order zero, so that, for $R(a) > d$, we shall have, by (48),

$$f^{(p)}(a) = \int_0^\infty e^{-at} A\,[f^{(p)}](t)dt.$$

Since under our hypothesis $A\,[f](t) \equiv 0$, (37) gives us $A\,[f^{(p)}](t) \equiv 0$, so that $f^{(p)}(a) = 0$ for all a's in at least a half-plane. $f(a)$ is therefore a polynomial in that half-plane, and, being analytic, is a polynomial throughout.

Now let $f(D)$ be of type zero, with $A\,[f](t) \equiv 0$. Then, for $R(a) > d$,

$$\{f(D)\}_{-\infty}^{X} e^{ax} = \{f(D)\}_{x_1}^{X} e^{ax}.$$

We can apply (45) to the first member of this equation; and, as e^{ax} is an entire function of x, (41) will yield a convergent series for the second. We thus get

$$f(a) = \{f(D)\}_{x_1}^{X} 1 + \frac{\{f'(D)\}_{x_1}^{X} 1}{1!} a + \frac{\{f''(D)\}_{x_1}^{X} 1}{2!} a^2 + \cdots .$$

Since this power series in a converges in a half-plane of a, its radius of convergence must be infinite, and so $f(a)$ is entire. To prove that it is of type zero over the plane, note that, from the series, we have

$$\{f^{(n)}(D)\}_{x_1}^{X} 1 = f^{(n)}(0).$$

Now let X be any positive number, however small, and choose x_1 between 0 and X. Formula (41) will then yield the convergent series

$$\{f(D)\}_{x_1}^{X} x^{-1} = X^{-1} f(0) - X^{-2} f'(0) + X^{-3} f''(0) - \cdots .$$

Since this converges for every positive X we must have

$$\lim_{n \to \infty} [f^{(n)}(0)]^{1/n} = 0.$$

This condition on the coefficients of the expansion of an entire $f(z)$ in powers of z is equivalent to condition (b) of §3 holding over the z-plane. Hence $f(z)$ is of type zero over the plane.

The scope of Theorem XX is clarified by the following two observations. First, $f(z)$ *may be entire, and of type zero in a sector of angle greater than* π, *and yet not of type zero over the plane*; secondly, $f(z)$ *may be a transcendental function of type zero over the plane without being of genus zero.* For the first case consider the function

$$f(z) = \int_0^\infty e^{-zt} e^{-t^2} dt.$$

Since the integral converges uniformly over every bounded region of the

z-plane, $f(z)$ is an entire analytic function of z. Now it can easily be seen that, if t is considered as a complex variable, the positive half of the real t axis, used in the above integral for contour, can be replaced by any half-line from the origin which makes an angle θ between $-\pi/4$ and $\pi/4$ with that positive t axis. For any one θ, we find

$$|f(z)| < \frac{A_\theta}{R(e^{i\theta}z) + B_\theta},$$

for $R(e^{i\theta}z) > -B_\theta$. By combining the inequalities for $\theta=\theta_1$ and $\theta=-\theta_1$ we thus easily see that $f(z)$ is in fact of order zero over any sector of angle less than $3\pi/2$ bisected by the positive half of the real z axis. It is therefore also of type zero. That it is not of type zero over the plane is seen by considering negative real values of z, for which we have, with arbitrarily large N,

$$f(z) > \int_0^N e^{-zt}e^{-t^2}dt > \frac{e^{-N^2}}{(-z)}\,[e^{N(-z)} - 1].$$

For the second observation consider

$$f(z) = \prod_{n=2}^{\infty}\left[1 - \frac{z^2}{(n\log n)^2}\right],$$

whose zeros are $\pm n\log n$, $n=2, 3, \cdots$. As $\sum 1/(n\log n)$ does not converge, $f(z)$ is not of genus zero. On the other hand

$$|f(z)| < \prod_{n=2}^{\infty}\left[1 + \frac{|z|^2}{(n\log n)^2}\right] < P_N(|z|)\,\frac{\sinh(\pi|z|/\log N)}{\pi|z|/\log N},$$

where $P_N(|z|)$ is the polynomial in $|z|$ formed from the first $N-2$ factors. By expressing the hyperbolic sine in terms of exponentials, we see that condition (b) of §3 is satisfied over the plane, that is, $f(z)$ is of type zero over the plane.*

15. **The Laplace integral equation.** Except for a few results that followed directly from our definition of generalized differentiation, the preceding sections studied that definition for operators which we called of type zero. In the present and following sections, we shall make an independent study of operators given as Laplace integrals. We have seen in §11 that $A[f](t)$ satisfies the Laplace integral equation (48) when $f(D)$ is of order zero. We turn now to an extension of the converse of this result.

It has been shown in the literature that if $\int_a^\infty \psi(t)e^{-zt}dt$, where $\psi(t)$ is

* For the standard derivation of the same inequalities in the case of functions of genus zero see Borel, *Fonctions Entières*, chapter 3.

continuous for $t \geqq a$, converges for $\zeta=\zeta_0$, then it converges for all ζ's with $R(\zeta)>R(\zeta_0)$, and represents an analytic function of ζ in that half-plane.* This proof is readily extended to the case where $\psi(t)$ is continuous only for $t>a$, by breaking up the interval of integration. With this in mind we state

THEOREM XXI. *Let $\psi(t)$ be a continuous function of t for $t>0$, and let $\int_0^\infty \psi(t)e^{-\zeta t}dt$, considered improper at both its limits, converge for some value of ζ. Then, if*

$$\int_0^\infty \psi(t)e^{-\zeta t}dt = f(\zeta)$$

in the resulting half-plane of convergence, we will have, for $t>0$,

$$\psi(t) = A[f](t) = \lim_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} \frac{(-1)^r f^{(r)}(1/\Delta x)}{r!\Delta x^{r+1}}. \tag{63}$$

Before we turn to the proof, note that when $f(D)$ is of order zero, $A[f](t)$ may be discontinuous for $t=0$. Hence the assumption of continuity for $\psi(t)$ only for positive t. Such an assumption of mere continuity is in accord with the literature. Since we have observed that for $f(D)$ not only of order zero, but of type zero, $A[f](t)$ is analytic for positive t, we see that the present development must be a quite independent study of our fundamental definition.

The proof of the convergence theorem for Laplace integrals can easily be extended to show that successive derivatives of the integral can be found by differentiating under the integral sign. We therefore find, for the $f(\zeta)$ of our theorem,

$$A[f](r, \Delta x) = \frac{(-1)^r f^{(r)}(1/\Delta x)}{r!\Delta x^{r+1}} = \int_0^\infty \frac{e^{-t/\Delta x}t^r}{r!\Delta x^{r+1}}\psi(t)dt,$$

provided $1/\Delta x$ is in the half-plane of convergence. Let $t=r\Delta x\cdot\tau$. With the help of Stirling's formula for $r!$ we obtain

$$A[f](r, \Delta x) = \frac{[r/(2\pi)]^{1/2}}{1+\epsilon_r}\int_0^\infty (e^{1-\tau}\tau)^r\psi(r\Delta x\cdot\tau)d\tau,$$

where $\epsilon_r \to 0$ as $r\to\infty$. The function $e^{1-\tau}\tau$ attains a maximum value of one for $\tau=1$. Hence $(e^{1-\tau}\tau)^r$ stays equal to one for $\tau=1$ as $r\to\infty$, but otherwise becomes inappreciable, as r increases, except for an ever narrowing neighborhood of $\tau=1$. This suggests that we break up the integral over limits $(0, 1-\lambda)$, $(1-\lambda, 1+\lambda)$, $(1+\lambda, \infty)$, where λ is between zero and one, and

* For references, see Pincherle, loc. cit., p. 40.

show that the first and third parts can be neglected in finding the limit of $A[f](r, \Delta x)$.

With this in mind, let b be a real number in the half-plane of convergence of the Laplace integral. That means, with our change of variable, that

$$\int_0^\infty e^{-b\cdot r\Delta x\cdot\tau}\psi(r\Delta x\cdot\tau)d\tau$$

converges. Now rewrite the integrand $(e^{1-\tau}\tau)^r\psi(r\Delta x\cdot\tau)$ so that it reads $(e^{1-\tau+b\Delta x\cdot\tau}\tau)^r e^{-b\cdot r\Delta x\cdot\tau}\psi(r\Delta x\cdot\tau)$. For fixed λ, and Δx sufficiently small, $(e^{1-\tau+b\Delta x\cdot\tau}\tau)^r$ will monotonically increase from $\tau=0$ to $\tau=1-\lambda$, and monotonically decrease from $\tau=1+\lambda$ to $\tau=\infty$. We can therefore apply the second law of the mean for integrals,* and obtain, with $0\leqq\theta\leqq 1-\lambda$,

$$\begin{aligned}\int_0^{1-\lambda}&(e^{1-\tau+b\Delta x\cdot\tau}\tau)^r e^{-b\cdot r\Delta x\cdot\tau}\psi(r\Delta x\cdot\tau)d\tau\\ &= \left[(e^{1-\tau+b\Delta x\cdot\tau}\tau)^r\right]_{\tau\to+0}\int_0^\theta e^{-b\cdot r\Delta x\cdot\tau}\psi(r\Delta x\cdot\tau)d\tau\\ &\qquad + \left[(e^{1-\tau+b\Delta x\cdot\tau}\tau)^r\right]_{\tau=1-\lambda}\int_\theta^{1-\lambda} e^{-b\cdot r\Delta x\cdot\tau}\psi(r\Delta x\cdot\tau)d\tau.\end{aligned}$$

The first term of the right hand member is always zero. As for the second term, the integral, expressed in the original variable t, is

$$\frac{1}{r\Delta x}\int_{r\Delta x\cdot\theta}^{r\Delta x(1-\lambda)} e^{-bt}\psi(t)dt,$$

and, due to the convergence of $\int_0^\infty e^{-bt}\psi(t)dt$, stays bounded irrespective of its limits of integration, as $r\Delta x$ approaches a finite limit. On the other hand, as $\Delta x\to+0$, $e^{1-\tau+b\Delta x\cdot\tau}\tau$, for $\tau=1-\lambda$, becomes and remains less than a fixed positive quantity which is itself less than one. Since at the same time r must increase indefinitely, we thus obtain

$$\lim_{\substack{\Delta x\to+0\\ r\Delta x\to t}}\frac{[r/(2\pi)]^{1/2}}{1+\epsilon_r}\left[(e^{1-\tau+b\Delta x\cdot\tau}\tau)^r\right]_{\tau=1-\lambda} = 0.$$

Hence the $(0, 1-\lambda)$ contribution to $A[f](r, \Delta x)$ can be neglected in studying its limit. An entirely similar proof shows the same to be true of the $(1+\lambda, \infty)$ contribution. We thus have, provided either limit exists,

* In the present instance we need the second law of the mean for an improper integral that may be only conditionally convergent. If, however, we first apply this law to the corresponding integral with positive lower limit ϵ (see de la Vallée Poussin, *Cours d'Analyse*, vol. 2, 1925, pp. 1–3) and if we then let θ be a limit value of the corresponding θ_ϵ's as $\epsilon\to+0$, we easily obtain the desired result.

$$\lim_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} A[f](r, \Delta x) = \lim_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} \frac{[r/(2\pi)]^{1/2}}{1 + \epsilon_r} \int_{1-\lambda}^{1+\lambda} (e^{1-\tau}\tau)^r \psi(r\Delta x \cdot \tau) d\tau.$$

Since $(e^{1-\tau}\tau)^r$ is positive, and $\psi(r\Delta x \cdot \tau)$ is continuous in τ, we have

$$\frac{[r/(2\pi)]^{1/2}}{1 + \epsilon_r} \int_{1-\lambda}^{1+\lambda} (e^{1-\tau}\tau)^r \psi(r\Delta x \cdot \tau) d\tau$$
$$= \psi[r\Delta x(1 + \theta\lambda)] \cdot \frac{[r/(2\pi)]^{1/2}}{1 + \epsilon_r} \int_{1-\lambda}^{1+\lambda} (e^{1-\tau}\tau)^r d\tau,$$

with $-1 < \theta < 1$. Now consider the special case $\psi(t) \equiv 1$. From the Laplace integral we find

$$f(\zeta) = \int_0^\infty e^{-\zeta t} dt = 1/\zeta.$$

For this f direct calculation gives

$$A[f](r, \Delta x) = \frac{(-1)^r f^{(r)}(1/\Delta x)}{r!\Delta x^{r+1}} \equiv 1,$$

so that, from the above expression for lim $A[f](r, \Delta x)$, we obtain

$$\lim_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} \frac{[r/(2\pi)]^{1/2}}{1 + \epsilon_r} \int_{1-\lambda}^{1+\lambda} (e^{1-\tau}\tau)^r d\tau \equiv 1.$$

Combining these results, we thus see that

$$\limsup_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} A[f](r, \Delta x) \leqq M_\lambda, \quad \liminf_{\substack{\Delta x \to +0 \\ r\Delta x \to t}} A[f](r, \Delta x) \geqq m_\lambda,$$

where M_λ and m_λ are the upper and lower bounds of $\psi[t(1+\theta\lambda)]$ as θ varies from -1 to 1. Now let $\lambda \to +0$. The continuity of ψ makes M_λ and m_λ both approach $\psi(t)$ as limit. Hence the upper and lower limits of $A[f](r, \Delta x)$ both equal $\psi(t)$, i.e., $A[f](t)$ exists, and equals $\psi(t)$.

16. **Operator expressed as a Laplace integral.** With the help of the preceding section we shall now prove

THEOREM XXII. *If $f(\zeta)$ is given by the convergent Laplace integral*

$$f(\zeta) = \int_0^\infty \psi(t) e^{-\zeta t} dt,$$

where $\psi(t)$ is continuous for positive t, and the integral is considered improper at both its limits, and if $\phi(x)$ is continuous in the closed interval (x_0, X), then

$$\{f(D)\}_{x_0}^{X}\phi(x) = \int_{x_0}^{X}\psi(X-x)\phi(x)dx, \tag{64}$$

under either of the following two auxiliary conditions:

(a) *the improper integral* $\int_0^h \psi(t)dt$ *is absolutely convergent for positive* h;

(b) $\phi(x)$ *has a finite total variation in a left neighborhood of* $x=X$.

It may be worth noting that (a) is the necessary and sufficient condition on a $\psi(t)$ continuous for positive t that $\int_{x_0}^X \psi(X-x)\phi(x)dx$ converge for every continuous $\phi(x)$; while (b) is the necessary and sufficient condition on a continuous $\phi(x)$ that this integral converge for every $\psi(t)$ which is continuous for positive t, and for which $\int_0^h \psi(t)dt$ converges.

Since the preceding section proves that $A[f](t)$ exists, and equals $\psi(t)$, it is sufficient for the proof of our theorem, as in the treatment of operators of order zero given in §5, to show that

$$\lim_{h\to+0}\ \limsup_{\Delta x\to+0}\left|\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{X-h}^{X}\phi(x)\right| = 0.$$

We have by definition, and from the preceding section,

$$\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{X-h}^{X}\phi(x) = \sum_{r=0}^{q}A[f](r,\Delta x)\phi(X-r\Delta x)\Delta x,\quad q=\{h/\Delta x\},$$

$$A[f](r,\Delta x) = \int_0^\infty \frac{e^{-t/\Delta x}t^r}{r!\Delta x^{r+1}}\psi(t)dt.$$

We shall now break up this integral over limits $(0, k)$, and (k, ∞), where k is greater than h but independent of r and Δx, and consider the two sums into which the above sum is thus broken up.

The treatment of the sum arising from the integrals with limits (k, ∞) is independent of the special conditions (a), (b). Let

$$\alpha_r(t) = \frac{e^{-t/\Delta x}t^r}{r!\Delta x^{r+1}}\cdot$$

We observe that $\alpha_r(t)$ is a monotonically decreasing function of t in the interval (k, ∞), due to the inequalities $k>h>r\Delta x$. The same will then be true of $\alpha_r(t)e^{bt}$ for sufficiently small Δx. As in the last section, the second law of the mean for integrals is applicable, and gives

$$\left|\int_k^\infty \alpha_r(t)\psi(t)dt\right| = \alpha_r(k)e^{bk}\left|\int_k^\mu e^{-bt}\psi(t)dt\right| \leqq N\alpha_r(k)e^{bk}.$$

The sum in question is thus not greater in absolute value than

$$MN\sum_{r=0}^{q}\alpha_r(k)e^{bk}\Delta x,$$

where M is the upper bound of $|\phi(x)|$ in (x_0, X). The above inequalities $k>h>r\Delta x$ also show that $\alpha_r(k)$ increases with r, as r varies from 0 to q. Since $(q+1)\Delta x<h+\Delta x$, we find this sum to be less in absolute value than

$$(h+\Delta x)MN\alpha_q(k)e^{bk}.$$

As $\Delta x\to+0$, q increases indefinitely, and so $\alpha_q(k)\to 0$. This sum can therefore be neglected in our discussion.

For condition (a) it will be sufficient to observe that the sum for the $(0, k)$ integrals will not exceed in absolute value

$$M\int_0^k\left(\sum_{r=0}^{q}\frac{e^{-t/\Delta x}t^r}{r!\Delta x^r}\right)|\psi(t)|\,dt.$$

But we have directly

$$\sum_{r=0}^{q}\frac{e^{-t/\Delta x}t^r}{r!\Delta x^r}<e^{-t/\Delta x}\sum_{r=0}^{\infty}\frac{t^r}{r!\Delta x^r}\equiv 1.$$

Hence this sum is less in absolute value than $M\int_0^k|\psi(t)|dt$, and hence also

$$\limsup_{\Delta x\to+0}\left|\left\{f\left(\frac{\Delta}{\Delta x}\right)\right\}_{X-h}^{X}\phi(x)\right|\leqq M\int_0^k|\psi(t)|\,dt.$$

Now let $h\to+0$, and at the same time let $k\to+0$, while keeping $k>h$. Due to the convergence of the last integral under condition (a), it will approach zero as limit with k, thus proving our result.

For condition (b), we shall write this sum in the form

$$\sum_{r=0}^{q}\phi(X-r\Delta x)\beta_r\Delta x,\quad \beta_r=\int_0^k\alpha_r(t)\psi(t)dt,$$

with $\alpha_r(t)$ as already defined, and use the identity

$$\sum_{r=0}^{q}\phi(X-r\Delta x)\beta_r\Delta x=\phi(X)\sum_{r=0}^{q}\beta_r\Delta x+[\phi(X-\Delta x)-\phi(X)]\sum_{r=1}^{q}\beta_r\Delta x$$
$$+\cdots+[\phi(X-q\Delta x)-\phi(X-[q-1]\Delta x)]\beta_q\Delta x.$$

We can write

$$\sum_{r=s}^{q}\beta_r\Delta x=\int_0^k\gamma_s(t)\psi(t)dt,\quad \gamma_s(t)=\sum_{r=s}^{q}\alpha_r(t)\Delta x.$$

Integration by parts gives the relation

$$\int_0^k \gamma_s(t)\psi(t)dt = \gamma_s(k)\int_0^k \psi(t)dt - \int_0^k\left[\int_0^t \psi(\tau)d\tau\right]\gamma_s'(t)dt.$$

Let N_k be the upper bound of $|\int_0^t\psi(\tau)d\tau|$ in the interval $0\leqq t\leqq k$. Note that N_k is finite, since the convergence of the Laplace integral entails the convergence of this integral. We then have

$$\left|\int_0^k \gamma_s(t)\psi(t)dt\right| \leqq N_k\left[\gamma_s(k) + \int_0^k |\gamma_s'(t)|\,dt\right].$$

Now we easily verify that

$$\gamma_s(k) < 1,\ \gamma_s'(t) = \alpha_{s-1}(t) - \alpha_q(t).$$

Furthermore, we have, for $\alpha_{s-1}(t)$, and similarly for $\alpha_q(t)$,

$$\int_0^k \alpha_{s-1}(t)dt < \int_0^\infty \alpha_{s-1}(t)dt \equiv 1,$$

so that we find

$$\left|\sum_{r=s}^{q}\beta_r\Delta x\right| = \left|\int_0^k \gamma_s(t)\psi(t)dt\right| < 3N_k.$$

Returning to our identity we thus obtain

$$\left|\sum_{r=0}^{q}\beta_r\phi(X - r\Delta x)\Delta x\right| < 3N_k[\,|\phi(X)| + V_{X-h}^{X}\phi(x)]$$

where $V_{X-h}^{X}\phi(x)$ is the total variation of $\phi(x)$ in $(X-h, X)$. Under hypothesis (b), this is finite for sufficiently small h, and so does not then increase as $h\to 0$. On the other hand, N_k approaches zero as limit, as k, along with h, approaches zero as limit. Our theorem is thus proved under (b).

To illustrate (64), consider the operator $B(D, n)$. Through the change of variable $x=e^{-t}$, the usual integral for the beta function becomes

$$B(\zeta, n) = \int_0^\infty (1 - e^{-t})^{n-1}e^{-\zeta t}dt,$$

where $R(n)>0$ (and $R(\zeta)>0$). Since condition (a) is satisfied, we thus have, for $\phi(x)$ continuous in (x_0, X),

$$\{B(D, n)\}_{x_0}^{X}\phi(x) = \int_{x_0}^{X}[1 - e^{-(X-x)}]^{n-1}\phi(x)dx.* \tag{65}$$

* (65) in conjunction with (44) leads to the solution of certain linear differential equations with exponential polynomial coefficients by means of definite integrals. Note, of course, that $B(D, n)$ is of order zero for $R(n)>0$.

17. **Carson's form; e^{-aD} as operator.** Carson has given a form for generalized differentiation which, with our notation, reads, *if*

$$\frac{f(\zeta)}{\zeta} = \int_0^\infty \chi(t)e^{-\zeta t}dt,$$

then

$$\{f(D)\}_{x_0}^X \phi(x) = \frac{d}{dX}\int_{x_0}^X \chi(X - x)\phi(x)dx.* \tag{66}$$

We shall show that this form results from our definition in either of the following two cases:

(a) $\chi(t)$ *is continuous for* $t \geqq 0$, *with* $\lim_{t\to\infty} e^{-\zeta t}\chi(t) = 0$ *for sufficiently large* ζ; $\chi'(t)$ *exists for* $t > 0$, *and, with* $\phi(x)$, *satisfies the hypothesis of Theorem* XXII.

(b) $\chi(t)$ *is continuous for* $t > 0$, *with* $\int_0^k |\chi(t)|dt$ *convergent, and* $\phi'(x)$ *exists, and is continuous, for* $x_0 \leqq x \leqq X$.

It may be noted that in case (a) Carson's form reduces to that of §16, while in case (b) it is related to that of §16 much as our operators of order one are related to those of order zero.

In case (a), let first $\chi(0) = 0$. We have directly

$$\frac{d}{dX}\int_{x_0}^X \chi(X - x)\phi(x)dx = \int_{x_0}^X \chi'(X - x)\phi(x)dx.$$

On the other hand, integration by parts of the integral for $f(\zeta)/\zeta$ gives

$$f(\zeta) = \int_0^\infty \chi'(t)e^{-\zeta t}dt,$$

so that, by Theorem XXII, we have also

$$\{f(D)\}_{x_0}^X \phi(x) = \int_{x_0}^X \chi'(X - x)\phi(x)dx.$$

(66) thus follows. If $\chi(0) = c$, we have

$$\int_0^\infty [\chi(t) - c]e^{-\zeta t}dt = \frac{f(\zeta) - c}{\zeta}.$$

The case just proved can therefore be applied to $f(D) - c$, so that

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* J. R. Carson, *The Heaviside operational calculus*, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 43–68. Numerous papers on this subject by Carson are to be found in the Bell System Technical Journal.

$$\{f(D) - c\}_{x_0}^{X}\phi(x) = \frac{d}{dX}\int_{x_0}^{X}[\chi(X - x) - c]\phi(x)dx,$$

which easily reduces to (66)

In case (b), replace $X-x$ by a new variable. We thus get

$$\frac{d}{dX}\int_{x_0}^{X}\chi(X - x)\phi(x)dx = \int_{x_0}^{X}\chi(X - x)\phi'(x)dx + \chi(X - x_0)\phi(x_0)$$

where we returned to the old variable x in the end. Now by §§15 and 16

$$A[u^{-1}f(u)](t) = \chi(t),\ \{D^{-1}f(D)\}_{x_0}^{X}\phi'(x) = \int_{x_0}^{X}\chi(X - x)\phi'(x)dx.$$

If we turn then to the first existence proof for operators of finite order, with $\rho = 1$, we see that it will hold here provided

$$\sum_{r=0}^{p} | A[u^{-1}f(u)](r, \Delta x) | \Delta x$$

is bounded as $\Delta x \to +0$. But the discussion of §16, with $u^{-1}f(u)$ in place of $f(u)$, shows that for $k > X - x_0$

$$\limsup_{\Delta x \to +0} \sum_{r=0}^{p} | A[u^{-1}f(u)](r, \Delta x) | \Delta x \leqq \int_{0}^{k} | \chi(t) | dt,$$

which is here assumed finite. Hence formula (17), with $\rho = 1$, can be used to give

$$\{f(D)\}_{x_0}^{X}\phi(x) = \int_{x_0}^{X}\chi(X - x)\phi'(x)dx + \chi(X - x_0)\phi(x_0),$$

which is just what we found for the other member of (66).

More generally, if $\chi(t)$ and $\phi(x)$ satisfy the hypothesis of Theorem XXII, (66) is equivalent to the relation

$$\{f(D)\}_{x_0}^{X}\phi(x) = \frac{d}{dX}\{D^{-1}f(D)\}_{x_0}^{X}\phi(x),$$

which offers no difficulty when $f(D)$ is of finite order, or of type zero (see §7). However, the complete discussion of this relation when $f(\zeta)/\zeta$ is given by a Laplace integral offers considerable difficulty, and can only be made the subject of a separate investigation. We shall merely append an examination of the special operator e^{-aD}, $a > 0$, which, in addition to its relation to the Laplace integral treatment, throws considerable light on our previous work with operators of type zero.

This operator comes under Carson's treatment by assigning to $\chi(t)$ the value 0, for $0 \leqq t \leqq a$, 1, for $t > a$. Since $\chi(t)$ is not even continuous for $t > 0$, agreement of the results with our definition cannot be sought for in the above two cases, but must be obtained directly. From our definition,

$$\left\{e^{-a(\Delta/\Delta x)}\right\}_{x_0}^{X}\phi(x) = \sum_{r=0}^{p} \frac{e^{-a/\Delta x}a^r}{r!\Delta x^r}\phi(X - r\Delta x), \quad p = \left\{\frac{X - x_0}{\Delta x}\right\}.$$

The coefficient of $\phi(X - r\Delta x)$ attains its largest value for $r\Delta x < a \leqq (r+1)\Delta x$. As in the treatment of the Laplace integral equation, it can be shown that only values of $r\Delta x$ in a neighborhood of this value affect the limit as $\Delta x \to +0$. We thus similarly obtain, for continuous $\phi(x)$,

$$(67)\quad X - x_0 < a\colon \left\{e^{-aD}\right\}_{x_0}^{X}\phi(x) = 0;\ X - x_0 > a\colon \left\{e^{-aD}\right\}_{x_0}^{X}\phi(x) = \phi(X - a),^*$$

which agree exactly with Carson's results.† We may note in passing, that (31) gives the relation

$$A[e^{-au}f(u)](r, \Delta x) = \sum_{s=0}^{r} \frac{e^{-a/\Delta x}a^s}{s!\Delta x^s} A[f(u)](r - s, \Delta x),$$

from which, in a similar manner, we find

$$(68)\quad t < a\colon A[e^{-au}f(u)](t) = 0;\ t > a\colon A[e^{-au}f(u)](t) = A[f(u)](t - a),$$

provided $f(D)$ is of type zero. If we replace $f(u)$ by $u^{-1}f(u)$, we also obtain, by (12),

$$(69)\quad X - x_0 < a\colon \left\{e^{-aD}f(D)\right\}_{x_0}^{X}1 = 0;\ X - x_0 > a\colon \left\{e^{-aD}f(D)\right\}_{x_0}^{X}1 = \left\{f(D)\right\}_{x_0}^{X-a}1.$$

Consider now some non-formal aspects of this and related operators. We found in §3 that, when $f(D)$ is of type zero, $A[f](t)$ exists for every positive t. By contrast, the formula

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* For $X - x_0 = a$, terms on but one side of the maximum are included. Due to approximate symmetry of the coefficients with respect to this maximum, the result for $X - x_0 = a$ is seen to be $\phi(X - a)/2$.

† The reader may be interested in the following formal "derivation" of the formula of §16 from this formula. Writing symbolically

$$f(D) = \int_0^\infty \psi(t)e^{-tD}dt,$$

and noting that $\left\{e^{-tD}\right\}_{x_0}^{X}\phi(x)$ vanishes for $t > X - x_0$, we would be led to

$$\{f(D)\}_{x_0}^{X}\phi(x) = \int_0^\infty \psi(t)\left[\left\{e^{-tD}\right\}_{x_0}^{X}\phi(x)\right]dt = \int_0^{X-x_0} \psi(t)\phi(X - t)dt.$$

By replacing $X - t$ by x, we obtain the formula in question. A rigorous proof along these lines may be possible.

$$e^{-}A\left[{}^{au}\right](r,\Delta x) = \frac{e^{-a/\Delta x}a^{r}}{r!\Delta x^{r}}$$

shows that $A\left[e^{-au}\right](t)$, while zero for every other positive t, fails to exist for $t=a$ through becoming infinite. Stronger still is the contrast furnished by e^{aD}, $a>0$, for which $A\left[e^{au}\right](t)$ fails to exist for every positive t not exceeding a certain positive α, and is zero for $t>\alpha$. Of course e^{-aD} and e^{aD} are not of type zero. They are however closely related to operators of type zero, since the corresponding analytic functions satisfy the analyticity condition (a), given in §3, for operators of type zero, and also the inequality of (b), not, however, for each positive κ, but only for $\kappa>\kappa_0$, $\kappa_0>0$. Calling operators satisfying (a), and this qualified (b), operators of type one, we easily find from §3 that, for them, $A[f](t)$ exists for t greater than some positive α. Also, by a modification of the treatment of part (b) of §6, we obtain the existence and expansion in series (21) of $\{f(D)\}_{x_1}^{X}\phi(x)$, provided $X-x_1$ is greater than this α and the radius of convergence of $\phi(x)$ at x_1 is greater than $X-x_1$ by more than a certain fixed positive β. These results were not included in the paper since the method used gave values to α and β larger than those demanded by the operators themselves. They serve, however, to clarify the non-existence of $A\left[e^{-au}\right](t)$ and $A\left[e^{-au}\right](t)$ for certain t's,* and the difference in form of $\{e^{-aD}\}_{x_0}^{X}\phi(x)$ for $X-x_0<a$, and $X-x_0>a$.

We may, in fact, think of our definition of generalized differentiation as not beginning to work, in the latter case, until $X-x_0>a$. As a increases, this period of adjustment, as we may call it, increases. It is then interesting to observe that in the case of the operator e^{-D^2} this period of adjustment is never completed, since, for finite x_0,

$$\{e^{-D^2}\}_{x_0}^{X}\phi(x) \equiv 0. \tag{70}$$

Here, then, we must take $x_0=-\infty$, for which, at least, (45) holds. e^{-D^2} can be considered of type higher than one. This failure of our definition of generalized differentiation, with its partial failure in the case of operators of type one, throws into greater relief its peculiar applicability to those operators we have called of type zero.

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* That $A\left[e^{-au}\right](t)$ and $A\left[e^{au}\right](t)$ are identically zero for $t>\alpha$ corresponds to e^{-aD} and e^{aD} being operators of type one over the plane.

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