

# ON NORMAL DIVISION ALGEBRAS OF TYPE $R$ IN THIRTY-SIX UNITS\*

BY

A. ADRIAN ALBERT

1. Introduction. A normal division algebra in  $n^2$  units over a non-modular field  $F$  is of type  $R$  if it contains a quantity  $i$  whose minimum equation with respect to  $F$ ,  $\phi(\omega) = 0$ , has degree  $n$  and  $n$  distinct roots which are polynomials in  $i$  with coefficients in  $F$ . Algebras of type  $R$  occupy a central position in the theory of division algebras as they are the only normal division algebras whose structure is known, and all division algebras of order less than twenty-five are expressible as algebras of type  $R$ .

The normal division algebras  $D$  whose structure is the simplest are those for the case where  $\phi(\omega) = 0$  has the cyclic group with respect to  $F$ . When  $n$  is six and  $\phi(\omega) = 0$  is cyclic,  $D$  is expressible as the direct product of a generalized quaternion division algebra and a cyclic division algebra of order nine, while conversely every such direct product is a cyclic division algebra of order thirty-six. The group of  $\phi(\omega) = 0$  is evidently regular and hence the only other type of equation to be considered for algebras of order thirty-six and type  $R$  is one which has the single non-cyclic, non-abelian regular group on six letters, a case giving a very complicated algebra.

It has never been demonstrated that there exist normal division algebras which are not cyclic algebras. The author showed, in a recent paper,<sup>†</sup> that the algebras which had been constructed by F. Cecioni<sup>‡</sup> and which were based on a non-cyclic quartic were cyclic algebras. We show here that *all normal division algebras of type  $R$  in thirty-six units are cyclic algebras.*

2. Algebras based on a non-cyclic sextic with regular group. Let  $D$  be an associative normal division algebra of order thirty-six and type  $R$ , and let  $i$  be the quantity of  $D$  which defines the type of  $D$ . If  $\phi(\omega) = 0$ , the minimum equation of  $i$ , is a cyclic sextic,  $D$  is called a cyclic algebra. There remains to be considered the case where the group of  $\phi(\omega) = 0$  is non-cyclic. The author has shown<sup>||</sup> that  $\phi(\omega)$  may be taken to have only even powers of the indeterminate  $\omega$  and that there exists a polynomial  $\theta(i)$  in  $F(i)$  such that

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\* Presented to the Society, October 25, 1930; received by the editors in August, 1930.

† These Transactions, vol. 32 (1930), pp. 171-195.

‡ Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 209-254.

|| See Theorem 12 of the author's paper, American Journal of Mathematics, vol. 52 (1930), pp. 283-292.

$$(1) \quad \phi(\omega) \equiv [\omega + \theta^2(i)][\omega - \theta^2(i)][\omega + \theta(i)][\omega - \theta(i)](\omega + i)(\omega - i),$$

while for the non-cyclic case

$$(2) \quad \theta^3(i) = i, \theta(-i) = -\theta^2(i), \theta^2(-i) = -\theta(i).$$

Evidently  $i^2$  satisfies a cubic equation irreducible in  $F$ , and  $F(i^2)$  is a cubic field over  $F$ . The set of all quantities in  $F(i)$  which are symmetric in  $i, \theta(i), \theta^2(i)$ , form a quadratic sub-field

$$(3) \quad K = F(v), \quad v^2 = \rho \text{ in } F,$$

of  $F(i)$ . A cubic field contains no quadratic sub-field so  $v$  is not in  $F(i^2)$ . Hence  $1, v$  are linearly independent with respect to  $F(i^2)$ , and every quantity in  $F(i)$  is expressible in the form

$$(4) \quad a = a(i) = a_1 + a_2v \quad (a_1 \text{ and } a_2 \text{ in } F(i^2)).$$

But then

$$i = p_1 + pv$$

with  $p$  and  $p_1$  in  $F(i^2)$ , so that

$$i^2 = (p_1^2 + p^2\rho) + 2p_1pv, \quad 0 = (p_1^2 + p^2\rho - i^2) + 2p_1pv.$$

It follows that  $2p_1p=0$ . If  $p$  were zero then  $i$  would be in  $F(i^2)$ , a cubic field, contrary to the fact that  $F(i)$  is a field of order six. Hence  $p_1$  is zero and

$$(5) \quad i = pv, \quad p \text{ in } F(i^2).$$

It is known\* that  $D$  has a basis

$$(6) \quad i^s j^t, i^s j^t z \quad (s = 0, 1, \dots, 5; t = 0, 1, 2),$$

and a multiplication table

$$(7) \quad \begin{aligned} \phi(i) &= 0, & j^t a &= a[\theta^t(i)]j^t & (t = 0, 1, \dots), \\ za(i) &= a(-i)z, & zj &= \alpha j^2 z, & zj^2 &= \alpha\alpha[\theta^2(i)]gjz, \\ j^3 &= g, & z^2 &= \gamma, \end{aligned}$$

for every  $a$  in  $F(i)$  where  $g, \alpha, \gamma$  are in  $F(i)$ . Since  $zz^2 = z^2z$  we have  $\gamma = \gamma(-i)$  is in  $F(i^2)$ . Similarly  $jg = gj$  gives  $g = g[\theta(i)]$  is in  $F(v)$ . Write

$$\gamma = \gamma_1 + \gamma_2 i^2 + \gamma_3 i^4 \quad (\gamma_1, \gamma_2, \gamma_3 \text{ in } F),$$

and suppose that  $\gamma_3 \neq 0$ . We can then define scalars  $\gamma_6, \gamma_8$  in  $F$  by  $\gamma_1 = \gamma_3 \gamma_6, \gamma_2 = 2\gamma_3 \gamma_8$  and

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\* See L. E. Dickson, *Algebren und ihre Zahlentheorie*, pp. 75-79, where  $q=2, z=j_3, j=j_1, \theta_q(i) \equiv -i, \theta_1(i) \equiv \theta(i)$ .

$$\gamma = \gamma_3(i^4 + 2\gamma_6i^2 + \gamma_6^2 + \gamma_5 - \gamma_6^2) = \gamma_3(i^2 + \gamma_6)^2 + \gamma_3(\gamma_5 - \gamma_6^2).$$

Consider the quantity

$$i_1 = (i^2 + \gamma_6)v.$$

Since  $v \neq 0$  we have  $v^2 = \rho \neq 0$  in a division algebra and

$$i_1^2 = (i^2 + \gamma_6)^2\rho = \rho i^4 + 2\rho i^2\gamma_6 + \rho\gamma_6^2$$

is in  $F(i^2)$  but not in  $F$ , since in particular the coefficient of  $i^4$  is not zero. But  $F(i^2)$  has no proper sub-field other than  $F$ , so that  $F(i_1^2) = F(i^2)$ . The quantities

$$(8) \quad (i^2 + \gamma_6)v, - (i^2 + \gamma_6)v, [\theta(i)^2 + \gamma_6]v, \\ - [\theta^2(i)^2 + \gamma_6]v, [\theta^2(i)^2 + \gamma_6]v, [\theta(i)^2 + \gamma_6]v$$

are transforms

$$i_1, \quad zi_1z^{-1}, \quad j i_1 j^{-1}, \quad zj i_1 (zj)^{-1}, \quad j^2 i_1 j^{-2}, \quad zj^2 i_1 (zj^2)^{-1}$$

of  $i_1$  and are roots of its minimum equation. If they were not distinct, two of

$$\pm (i^2 + \gamma_6), \pm [\theta(i)^2 + \gamma_6], \pm [\theta^2(i) + \gamma_6]$$

would be equal, which is impossible since those with plus signs are the distinct roots of the irreducible cubic minimum equation of  $i^2 + \gamma_6$ , while this cubic has not the negative of any one of its roots as a root since *it has not even powers only*. The minimum equation of  $i_1$  has thus six distinct roots in  $F(i)$  so that its degree is six,  $F(i_1)$  contained in  $F(i)$  has order six, and  $F(i_1) = F(i)$ . Evidently  $zi_1 = -i_1z$ , while  $j$  transforms  $i_1$  into a quantity in  $F(i_1)$ , that is a polynomial in  $i_1$ . We may thus replace  $i$  by  $i_1$  in the basis of  $D$  without loss of generality, and, since  $\gamma = (\gamma_3\rho^{-1})i_1^2 + \gamma_3(\gamma_5 - \gamma_6^2)$ , for *this new*  $i$  we have  $\gamma$  expressed as a linear combination with coefficients in  $F$  of 1 and  $i^2$ . When  $\gamma_3 = 0$  we also have immediately such an expression, so that we have proved

LEMMA 1. *The quantity  $i$  may be so chosen that, without altering any other property of  $D$ ,*

$$(9) \quad \gamma = \gamma_1 + \gamma_2 i^2 \quad (\gamma_1 \text{ and } \gamma_2 \text{ in } F).$$

We shall utilize the notations

$$(10) \quad a' = a(-i), \quad a_\theta = a[\theta(i)], \quad a_{\theta\theta} = (a_\theta)_\theta,$$

so that from (2) we immediately have

$$(11) \quad (a')' = a, \quad (a_\theta)_{\theta\theta} = (a_{\theta\theta})_\theta = a, \\ (a')_\theta = (a_{\theta\theta})', \quad (a')_{\theta\theta} = (a_\theta)'$$

for every  $a$  of  $F(i)$ . Also

$$(12) \quad ja = a_0j, \quad j^2a = a_{00}j^2, \quad za = a'z,$$

from (7), while

$$(13) \quad i' = -i, \quad v' = -v, \quad v_0 = v, \quad (i^2)' = i^2, \quad g = g_0 = g_{00}, \quad \gamma = \gamma'.$$

Consider the quantities

$$(14) \quad d = \lambda_1 + \lambda_4i, \quad e = \lambda_2 + \lambda_3i,$$

where

$$(15) \quad 2\lambda_1 = 1 + \gamma_1, \quad 2\lambda_2 = 1 - \gamma_1, \quad 2\lambda_3 = 1 + \gamma_2, \quad 2\lambda_4 = 1 - \gamma_2,$$

so that  $\lambda_1, \dots, \lambda_4$  are in  $F$ ,  $\lambda_1^2 - \lambda_2^2 = \gamma_1$ ,  $\lambda_3^2 - \lambda_4^2 = \gamma_2$ . Then

$$dd' - ee' = \lambda_1^2 - \lambda_4^2 i^2 - (\lambda_2^2 - \lambda_3^2 i^2) = \gamma_1 + \gamma_2 i^2 = \gamma.$$

But  $\gamma = \gamma'$  and if we put  $f = d\gamma^{-1}$ ,  $h = e\gamma^{-1}$ , we have

$$(ff' - hh')\gamma = (dd' - ee')\gamma^{-1} = 1,$$

and obtain

**LEMMA 2.** *There exist polynomials  $f$  and  $h$  in  $F(i)$  such that*

$$(16) \quad (ff' - hh')\gamma = 1.$$

Let now  $r$  and  $s$  be defined by

$$(17) \quad r = \alpha f, \quad s = \alpha h,$$

where  $\alpha$  is the quantity of (7) such that  $zj = \alpha j^2 z$  and  $\alpha$  is in  $F(i)$ . Then  $r$  and  $s$  are in  $F(i)$ , and

$$(18) \quad (rr' - ss')\gamma(\alpha^{-1})(\alpha^{-1})' = 1.$$

But if  $\delta = (jz)^2 = jzjz = j\alpha j^2 z z = \alpha_0 g \gamma$ , then  $\delta_0 = j(jz)^2 j^{-1} = (jjzj^{-1})^2 = (\alpha^{-1} \alpha j^2 z j^{-1})^2 = (\alpha^{-1} z j j^{-1})^2 = (\alpha^{-1} z)^2 = (\alpha^{-1})(\alpha^{-1})' \gamma$ , so that (18) gives

**LEMMA 3.** *There exist quantities  $r$  and  $s$  in  $F(i)$  such that, if*

$$(19) \quad \delta = (jz)^2 = \alpha_0 g \gamma,$$

then

$$(20) \quad (rr' - ss')\delta_0 = 1.$$

If  $a$  and  $b$  are defined by

$$(21) \quad a = s\alpha^{-1}, \quad b = r_{00},$$

so that  $b_0 = r$ , then

$$(22) \quad Q \equiv [b_0(b_0)' - a\alpha a'\alpha'], \quad Q\delta_0 = 1.$$

3. **The cyclic property.** We shall now proceed to prove that  $D$  is a cyclic algebra by the use of our fundamental existence theorem, Lemma 3. Consider the quantity

$$(23) \quad X = a + bj + cj^2,$$

where we take  $a$  and  $b$  to be the polynomials of Lemma 3 which satisfy (22) and where  $c$  will be chosen to be a polynomial in  $i$  with coefficients in  $F$ . For every  $c$  in  $F(i)$ , we have

$$(24) \quad zXz^{-1} \equiv X' = a' + c'\alpha\alpha_{\theta\theta}gj + b'\alpha j^2,$$

so that

$$(25) \quad XX' = (a + bj + cj^2)(a' + c'\alpha\alpha_{\theta\theta}gj + b'\alpha j^2) = A + Bj + Ej^2,$$

where  $A$  is a polynomial in  $i$  and

$$(26) \quad B = ac'\alpha\alpha_{\theta\theta}g + b(a')_{\theta} + c(b'\alpha)_{\theta\theta}g = Rc + Sc' + T,$$

$$(27) \quad E = ab'\alpha + b(c'\alpha\alpha_{\theta\theta}z)_{\theta} + c(a')_{\theta\theta} = Gc + H(c')_{\theta} + K.$$

The quantities  $B$  and  $E$  are polynomials in  $i$  and we have defined above

$$(28) \quad R \equiv (b'\alpha)_{\theta\theta}g = (b_{\theta})'\alpha_{\theta\theta}g, \quad S \equiv a\alpha\alpha_{\theta\theta}g, \quad T \equiv b(a')_{\theta} = b(a_{\theta\theta})',$$

$$(29) \quad G \equiv (a')_{\theta\theta} = (a_{\theta})', \quad H = b\alpha_{\theta}\alpha g, \quad K = ab'\alpha,$$

all in  $F(i)$ . Now

$$(30) \quad (g')_{\theta} = (g_{\theta\theta})' = g', \quad (g')_{\theta\theta} = [(g')_{\theta}]_{\theta} = (g')_{\theta} = g.$$

Transforming  $B$  by  $z$  we have

$$(31) \quad B' = R'c' + S'c + T',$$

whence

$$(32) \quad R'B - SB' = R'Rc + R'Sc' + R'T - SR'c' - SS'c - ST' \\ = (RR' - SS')c - (ST' - R'T).$$

But

$$(33) \quad RR' - SS' = (b_{\theta})'\alpha_{\theta\theta}gb_{\theta}(\alpha_{\theta\theta})'g' - a\alpha\alpha_{\theta\theta}ga'(\alpha_{\theta\theta})'g' \\ = gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'[b_{\theta}(b_{\theta})' - a\alpha a'\alpha'] = gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'Q.$$

From (19)  $\delta_{\theta} = \alpha_{\theta\theta}g\gamma_{\theta}$ , so that, utilizing the relation  $Q\delta_{\theta} = 1$ , we have

$$(34) \quad RR' - SS' = gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'\delta_{\theta}^{-1} = g'(\alpha_{\theta\theta})'(\gamma_{\theta})^{-1} \neq 0,$$

since  $g$ ,  $\alpha$  and  $\gamma$  are all not zero in a division algebra. Hence  $RR' - SS'$  has an inverse  $(RR' - SS')^{-1}$  in  $F(i)$ , and if we define the quantity  $c$  by

$$(35) \quad c = (ST' - R'T)(RR' - SS')^{-1},$$

then

$$(36) \quad R'B - SB' = (ST' - R'T) - (ST' - R'T) = 0.$$

We shall henceforth consider the quantity  $X$  as completely defined in (23) with the  $a$  and  $b$  of Lemma 3 and the  $c$  of (35), so that (36) is satisfied. Transforming (36) by  $z$  we have

$$(37) \quad RB' - BS' = 0,$$

whence

$$(38) \quad R(R'B - SB') + S(-BS' + RB') = B(RR' - SS') = 0.$$

But  $RR' - SS'$  has an inverse in  $F(i)$ , whence  $B = 0$ .

We consider now the polynomial  $E$ . We first compute

$$(39) \quad ST' - R'T = a\alpha\alpha_{\theta\theta}g'b'a_{\theta\theta} - b_{\theta}(\alpha_{\theta\theta})'g'b(a_{\theta\theta})'.$$

Next

$$(40) \quad \begin{aligned} H(K')_{\theta} - (G')_{\theta}K &= b\alpha_{\theta}g(\alpha')_{\theta}b_{\theta}(\alpha')_{\theta} - a_{\theta\theta}ab'\alpha \\ &= - [\alpha a a_{\theta\theta} b' - \alpha \alpha_{\theta} g b b_{\theta} (\alpha_{\theta\theta})' (a_{\theta\theta})'], \end{aligned}$$

so that

$$(41) \quad -\alpha_{\theta\theta}g[H(K')_{\theta} - (G')_{\theta}K] = \alpha\alpha a_{\theta\theta}\alpha_{\theta\theta}b'g - bb_{\theta}(\alpha\alpha_{\theta}\alpha_{\theta\theta}g^2)(a_{\theta\theta})'(\alpha_{\theta\theta})'.$$

But  $j^3 = g, g' = zgz^{-1} = zj^3z^{-1} = (zjz^{-1})^3 = (\alpha j^2 z z^{-1})^3 = (\alpha j^2)^3 = \alpha j^2 \alpha j^2 \alpha j^2 = \alpha \alpha_{\theta\theta} \alpha_{\theta} g^2$ , and we have the relations

$$(42) \quad g' = \alpha\alpha_{\theta}\alpha_{\theta\theta}g^2,$$

$$(43) \quad g = \alpha'(\alpha_{\theta\theta})'(\alpha_{\theta})'(g')^2 = \alpha'(\alpha')_{\theta}(\alpha')_{\theta\theta}(g')^2.$$

Substituting (42) in (41) and comparing with (39) we write immediately

$$(44) \quad ST' - R'T = -\alpha_{\theta\theta}g[H(K')_{\theta} - (G')_{\theta}K].$$

We also have, by the use of (11),  $[(\alpha_{\theta})']_{\theta} = [(\alpha')_{\theta\theta}]_{\theta} = \alpha'$ , and

$$(45) \quad G(G')_{\theta} - H(H')_{\theta} = (a_{\theta})'a_{\theta\theta} - b\alpha\alpha_{\theta}g(b')_{\theta}(\alpha')_{\theta}\alpha'g'.$$

But then

$$(46) \quad \begin{aligned} -g\alpha_{\theta\theta}(\alpha')_{\theta\theta}[G(G')_{\theta} - H(H')_{\theta}] \\ = (\alpha\alpha_{\theta}\alpha_{\theta\theta}g^2)[\alpha'(\alpha')_{\theta}(\alpha')_{\theta\theta}g']b(b')_{\theta} - g a_{\theta\theta}(a')_{\theta\theta}\alpha_{\theta\theta}(\alpha')_{\theta\theta}, \end{aligned}$$

which by (42) and (43) has the value

$$(47) \quad g[b(b')_{\theta} - a_{\theta\theta}\alpha_{\theta\theta}(a')_{\theta\theta}(\alpha')_{\theta\theta}].$$

But if

$$(48) \quad Q = b_\theta(b_\theta)' - a\alpha a' \alpha' = b_\theta(b')_{\theta\theta} - a\alpha a' \alpha',$$

then, from (22),  $\delta_\theta Q = 1, Q \neq 0,$

$$(49) \quad Q_{\theta\theta} = b(b')_\theta - a_{\theta\theta}\alpha_{\theta\theta}(a')_{\theta\theta}(\alpha')_{\theta\theta}.$$

Hence

$$(50) \quad -\alpha_{\theta\theta}(\alpha')_{\theta\theta}g[G(G')_\theta - H(H')_\theta] = gQ_{\theta\theta} \neq 0,$$

so that

$$(51) \quad \frac{H(K')_\theta - (G')_\theta K}{G(G')_\theta - H(H')_\theta} = \frac{-\alpha_{\theta\theta}g[H(K')_\theta - (G')_\theta K]}{-\alpha_{\theta\theta}g[G(G')_\theta - H(H')_\theta]} \frac{(\alpha')_{\theta\theta}}{(\alpha')_{\theta\theta}} = \frac{(\alpha')_{\theta\theta}[ST' - R'T]}{gQ_{\theta\theta}}.$$

From (22) we have  $\delta_\theta Q = 1,$  so that

$$(52) \quad j^2(\delta_\theta Q)j^{-2} = 1 = \delta Q_{\theta\theta} = \delta_\theta Q,$$

whence

$$(53) \quad \frac{\delta_\theta Q}{\gamma_\theta} = \frac{\delta Q_{\theta\theta}}{\gamma_\theta}.$$

Now  $\delta = \gamma\alpha_\theta g$  from (19), while  $\delta = (jz)^2$  is commutative with  $jz$  and equals its transform by  $jz,$  the quantity  $(\delta')_\theta.$  Hence

$$(54) \quad g = g_\theta, \alpha_\theta g = \frac{\delta}{\gamma}, \alpha_{\theta\theta}g = \frac{\delta_\theta}{\gamma_\theta}, \alpha g = \frac{\delta_{\theta\theta}}{\gamma_{\theta\theta}},$$

and

$$(55) \quad \alpha' g' = \frac{(\delta_{\theta\theta})'}{(\gamma_{\theta\theta})'} = \frac{(\delta')_\theta}{(\gamma')_\theta} = \frac{\delta}{\gamma},$$

since  $\gamma = \gamma'.$  Equation (53) becomes

$$(56) \quad \alpha_{\theta\theta}gQ = \alpha' g' Q_{\theta\theta}.$$

From (43)  $g = \alpha'(\alpha')_\theta(\alpha')_{\theta\theta}(g')^2,$  so that

$$\alpha' g' Q_{\theta\theta} = (\alpha' g') [(\alpha')_\theta(\alpha')_{\theta\theta}g'] \alpha_{\theta\theta}Q,$$

and

$$(57) \quad Q_{\theta\theta} = (\alpha')_{\theta\theta}(\alpha_{\theta\theta})' \alpha_{\theta\theta}g' Q.$$

It follows now that

$$(58) \quad gQ_{\theta\theta}/(\alpha')_{\theta\theta} = gg' \alpha_{\theta\theta}(\alpha_{\theta\theta})' Q = RR' - SS',$$

by (33). Using (35) and (51) we have

$$(59) \quad c = (ST' - R'T)(RR' - SS')^{-1} = [H(K')_{\theta} - (G')_{\theta}K][G(G')_{\theta} - H(H')_{\theta}]^{-1}.$$

We have now demonstrated that

$$(60) \quad [G(G')_{\theta} - H(H')_{\theta}]c - [H(K')_{\theta} - (G')_{\theta}K] = 0,$$

a relation very similar to (36). In fact, since  $E$  is given by (27), and  $[(c')_{\theta}]' = c_{\theta\theta}$ ,  $(c_{\theta\theta})_{\theta} = c$ ,

$$(61) \quad (E')_{\theta} = (G')_{\theta}(c')_{\theta} + (H')_{\theta}c + (K')_{\theta},$$

so that, by (60),

$$(62) \quad \begin{aligned} (G')_{\theta}E - H(E')_{\theta} &= (G')_{\theta}Gc + (G')_{\theta}H(c')_{\theta} + (G')_{\theta}K \\ &\quad - H(G')_{\theta}(c')_{\theta} - H(H')_{\theta}c - H(K')_{\theta} \\ &= [G(G')_{\theta} - H(H')_{\theta}]c - [H(K')_{\theta} - (G')_{\theta}K] = 0. \end{aligned}$$

Transforming (62) by  $jz$  we have

$$(63) \quad G(E')_{\theta} - (H')_{\theta}E = 0,$$

and

$$(64) \quad 0 = G[(G')_{\theta}E - H(E')_{\theta}] + H[G(E')_{\theta} - (H')_{\theta}E] = [G(G')_{\theta} - H(H')_{\theta}]E = 0.$$

It follows from (50) that  $E=0$  and that (25) becomes

$$(65) \quad XX' = A \text{ in } F(i).$$

But then

$$(66) \quad (Xz)^2 = XzXz = XX'\gamma = A\gamma = t \text{ in } F(i),$$

since  $\gamma$  is in  $F(i)$ , and  $X'$  was defined so that  $zX = X'z$  in (24).

Let first  $b$  and  $a$  be both not zero, so that since  $Xz$  is commutative with its square,

$$tXz = (ta + tbj + tcj^2)z = Xzt = (a + bj + cj^2)t'z.$$

Hence

$$ta + tbj + tcj^2 = t'a + (t')_{\theta}bj + (t')_{\theta\theta}cj^2,$$

and since (6) are a basis of  $D$ ,  $ta = t'a$ ,  $tb = (t')_{\theta}b$ . Since  $a$  is not zero,  $t = t'$  is in  $F(i^2)$ . Since also  $b$  is not zero,  $t = (t')_{\theta} = t_{\theta}$  is in  $F$ . It follows that when  $ab \neq 0$  we have shown that there exists a quantity  $X$  in the algebra

$$\Sigma = (i^s j^r) \quad (s = 0, 1, \dots, 5; r = 0, 1, 2),$$



such that  $X \neq 0$  and

$$(67) \quad (Xz)^2 = \lambda \text{ in } F.$$

Suppose next that  $a$  were zero so that from its origin (21) we have  $s=0$  and  $h=0$  in (17). Then (16) becomes

$$ff'\gamma = (fz)^2 = 1,$$

while then obviously  $f$  cannot be zero and  $f = \alpha^{-1}r = \alpha^{-1}b_\theta$  is in  $F(i)$  and in  $\Sigma$ . Again we have (67) for  $X \neq 0$  in  $\Sigma$ . Finally the only remaining case is  $b$  zero. Then  $r_{\theta\theta} = 0$  so that  $r=0$  in (17), and hence the quantity  $f$  is zero. Equation (16) now becomes  $hh'\gamma = (hz)^2 = -1$ , and since then  $h = \alpha^{-1}s = a$  in  $F(i)$  cannot be zero when  $(hz)^2 = -1$ , we have again proved the existence of  $X \neq 0$  in  $\Sigma$  and satisfying (67). Hence in all cases we have

LEMMA 4. *There exists a quantity  $X$  in  $\Sigma$  such that  $X \neq 0$ , and if  $y = Xz$  then*

$$(68) \quad y^2 = \lambda \text{ in } F.$$

The quantities  $1, v, y, vy$  are linearly independent with respect to  $F$ , for otherwise (6) could not be a basis of  $D$  when  $X \neq 0$ . A relation of the form

$$\xi_1 + \xi_2v + (\xi_3 + \xi_4v)Xz = 0,$$

with  $\xi_1, \xi_2, \xi_3, \xi_4$  not all zero and in  $F$ , would then evidently express  $z$  as a quantity of  $\Sigma$ . Also  $v^2 = \rho, y^2 = \lambda, yv = Xzv = -Xvz = -vXz = -vy$ , since  $v$ , a polynomial in  $i$  commutative with  $j$ , is commutative with  $X$ . But the linear set

$$\Gamma = (1, v, y, vy)$$

is evidently a generalized quaternion algebra over  $F$ , and is a normal division cyclic algebra over  $F$ . Hence  $D$ , containing  $\Gamma$ , is the direct product\* of  $\Gamma$  and another algebra  $\Omega$  of order nine over  $F$ . Since  $D$  is a normal algebra, so is necessarily  $\Omega$ ,† so that  $\Omega$  is a cyclic algebra‡ of order nine. Hence  $D$ , the direct product of algebra  $\Gamma$  and algebra  $\Omega$ , is a cyclic algebra.†

THEOREM. *Every normal division algebra of type R in thirty-six units is a cyclic algebra.*

\* A theorem of Wedderburn; cf. *Algebras and their Arithmetics*, p. 237.

† For the first and second of the above references respectively see Theorems 7 and 16 of the author's paper, *On direct products, cyclic division algebras, and pure Riemann matrices*, which appears in the present number of these Transactions.

‡ A theorem of Wedderburn, these Transactions, vol. 22 (1921), pp. 129-135.