ON NORMAL DIVISION ALGEBRAS OF TYPE R IN THIRTY-SIX UNITS*

BY

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1. **Introduction.** A normal division algebra in n^2 units over a non-modular field F is of type R if it contains a quantity i whose minimum equation with respect to F, $\phi(\omega)=0$, has degree n and n distinct roots which are polynomials in i with coefficients in F. Algebras of type R occupy a central position in the theory of division algebras as they are the only normal division algebras whose structure is known, and all division algebras of order less than twenty-five are expressible as algebras of type R.

The normal division algebras D whose structure is the simplest are those for the case where $\phi(\omega)=0$ has the cyclic group with respect to F. When n is six and $\phi(\omega)=0$ is cyclic, D is expressible as the direct product of a generalized quaternion division algebra and a cyclic division algebra of order nine, while conversely every such direct product is a cyclic division algebra of order thirty-six. The group of $\phi(\omega)=0$ is evidently regular and hence the only other type of equation to be considered for algebras of order thirty-six and type R is one which has the single non-cyclic, non-abelian regular group on six letters, a case giving a very complicated algebra.

It has never been demonstrated that there exist normal division algebras which are not cyclic algebras. The author showed, in a recent paper,† that the algebras which had been constructed by F. Cecioni‡ and which were based on a non-cyclic quartic were cyclic algebras. We show here that *all normal division algebras of type R in thirty-six units are cyclic algebras.*

2. **Algebras based on a non-cyclic sextic with regular group.** Let D be an associative normal division algebra of order thirty-six and type R, and let i be the quantity of D which defines the type of D. If $\phi(\omega)=0$, the minimum equation of i, is a cyclic sextic, D is called a cyclic algebra. There remains to be considered the case where the group of $\phi(\omega)=0$ is non-cyclic. The author has shown|| that $\phi(\omega)$ may be taken to have only even powers of the indeterminate ω and that there exists a polynomial $\theta(i)$ in $F(i)$ such that

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† These Transactions, vol. 32 (1930), pp. 171–195.

‡ Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 209–254.

|| See Theorem 12 of the author's paper, American Journal of Mathematics, vol. 52 (1930), pp. 283–292.

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$$(1)\qquad \phi(\omega) \equiv [\omega + \theta^2(i)][\omega - \theta^2(i)][\omega + \theta(i)][\omega - \theta(i)](\omega + i)(\omega - i),$$

while for the non-cyclic case

$$(2)\qquad \theta^3(i) = i,\ \theta(-i) = -\theta^2(i),\ \theta^2(-i) = -\theta(i).$$

Evidently i^2 satisfies a cubic equation irreducible in F, and $F(i^2)$ is a cubic field over F. The set of all quantities in $F(i)$ which are symmetric in i, $\theta(i)$, $\theta^2(i)$, form a quadratic sub-field

$$(3)\qquad K = F(v),\ v^2 = \rho \text{ in } F,$$

of $F(i)$. A cubic field contains no quadratic sub-field so v is not in $F(i^2)$. Hence 1, v are linearly independent with respect to $F(i^2)$, and every quantity in $F(i)$ is expressible in the form

$$(4)\qquad a = a(i) = a_1 + a_2 v \qquad (a_1 \text{ and } a_2 \text{ in } F(i^2)).$$

But then

$$i = p_1 + pv$$

with p and p_1 in $F(i^2)$, so that

$$i^2 = (p_1^2 + p^2\rho) + 2p_1pv,\ 0 = (p_1^2 + p^2\rho - i^2) + 2p_1pv.$$

It follows that $2p_1p=0$. If p were zero then i would be in $F(i^2)$, a cubic field, contrary to the fact that $F(i)$ is a field of order six. Hence p_1 is zero and

$$(5)\qquad i = pv,\ p \text{ in } F(i^2).$$

It is known* that D has a basis

$$(6)\qquad i^s j^t,\ i^s j^t z \qquad (s = 0, 1, \cdots, 5;\ t = 0, 1, 2),$$

and a multiplication table

$$(7)\qquad \begin{aligned} &\phi(i) = 0, && j^t a = a[\theta^t(i)]j^t \qquad (t = 0, 1, \cdots), \\ &za(i) = a(-i)z, && zj = \alpha j^2 z,\ zj^2 = \alpha\alpha[\theta^2(i)]gjz, \\ &j^3 = g, && z^2 = \gamma, \end{aligned}$$

for every a in $F(i)$ where g, α, γ are in $F(i)$ Since $zz^2=z^2z$ we have $\gamma=\gamma(-i)$ is in $F(i^2)$. Similarly $jg=gj$ gives $g=g[\theta(i)]$ is in $F(v)$. Write

$$\gamma = \gamma_1 + \gamma_2 i^2 + \gamma_3 i^4 \qquad (\gamma_1, \gamma_2, \gamma_3 \text{ in } F),$$

and suppose that $\gamma_3 \neq 0$. We can then define scalars γ_5, γ_6 in F by $\gamma_1=\gamma_3\gamma_5$, $\gamma_2=2\gamma_3\gamma_6$ and

* See L. E. Dickson, *Algebren und ihre Zahlentheorie*, pp. 75–79, where $q=2$, $z\equiv j_q$, $j\equiv j_1$, $\theta_q(i)\equiv -i$, $\theta_1(i)\equiv\theta(i)$.

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$$\gamma = \gamma_3(i^4 + 2\gamma_6 i^2 + \gamma_6^2 + \gamma_5 - \gamma_6^2) = \gamma_3(i^2 + \gamma_6)^2 + \gamma_3(\gamma_5 - \gamma_6^2).$$

Consider the quantity

$$i_1 = (i^2 + \gamma_6)v.$$

Since $v \neq 0$ we have $v^2 = \rho \neq 0$ in a division algebra and

$$i_1^2 = (i^2 + \gamma_6)^2\rho = \rho i^4 + 2\rho i^2\gamma_6 + \rho\gamma_6^2$$

is in $F(i^2)$ but not in F, since in particular the coefficient of i^4 is not zero. But $F(i^2)$ has no proper sub-field other than F, so that $F(i_1^2) = F(i^2)$. The quantities

$$(8)\qquad \begin{aligned}&(i^2 + \gamma_6)v,\ -(i^2 + \gamma_6)v,\ [\theta(i)^2 + \gamma_6]v,\\ &\qquad -[\theta^2(i)^2 + \gamma_6]v,\ [\theta^2(i)^2 + \gamma_6]v,\ [\theta(i)^2 + \gamma_6]v\end{aligned}$$

are transforms

$$i_1, \quad z i_1 z^{-1}, \quad j i_1 j^{-1}, \quad zj i_1 (zj)^{-1}, \quad j^2 i_1 j^{-2}, \quad zj^2 i_1 (zj^2)^{-1}$$

of i_1 and are roots of its minimum equation. If they were not distinct, two of

$$\pm\,(i^2 + \gamma_6),\ \pm\,[\theta(i)^2 + \gamma_6],\ \pm\,[\theta^2(i) + \gamma_6]$$

would be equal, which is impossible since those with plus signs are the distinct roots of the irreducible cubic minimum equation of $i^2 + \gamma_6$, while this cubic has not the negative of any one of its roots as a root since *it has not even powers only*. The minimum equation of i_1 has thus six distinct roots in $F(i)$ so that its degree is six, $F(i_1)$ contained in $F(i)$ has order six, and $F(i_1) = F(i)$. Evidently $zi_1 = -i_1 z$, while j transforms i_1 into a quantity in $F(i_1)$, that is a polynomial in i_1. We may thus replace i by i_1 in the basis of D without loss of generality, and, since $\gamma = (\gamma_3\rho^{-1})i_1^2 + \gamma_3(\gamma_5 - \gamma_6^2)$, *for this new* i we have γ expressed as a linear combination with coefficients in F of 1 and i^2. When $\gamma_3 = 0$ we also have immediately such an expression, so that we have proved

LEMMA 1. *The quantity i may be so chosen that, without altering any other property of D,*

$$(9)\qquad \gamma = \gamma_1 + \gamma_2 i^2 \qquad (\gamma_1 \textit{ and } \gamma_2 \textit{ in } F).$$

We shall utilize the notations

$$(10)\qquad a' = a(-i),\ a_\theta = a[\theta(i)],\ a_{\theta\theta} = (a_\theta)_\theta,$$

so that from (2) we immediately have

$$(11)\qquad \begin{aligned}(a')' &= a, & (a_\theta)_{\theta\theta} &= (a_{\theta\theta})_\theta = a,\\ (a')_\theta &= (a_{\theta\theta})', & (a')_{\theta\theta} &= (a_\theta)',\end{aligned}$$

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for every a of $F(i)$. Also

(12) $$ja = a_\theta j, \; j^2 a = a_{\theta\theta} j^2, \; za = a' z,$$

from (7), while

(13) $$i' = -i, \; v' = -v, \; v_\theta = v, \; (i^2)' = i^2, \; g = g_\theta = g_{\theta\theta}, \; \gamma = \gamma'.$$

Consider the quantities

(14) $$d = \lambda_1 + \lambda_4 i, \; e = \lambda_2 + \lambda_3 i,$$

where

(15) $$2\lambda_1 = 1 + \gamma_1, \; 2\lambda_2 = 1 - \gamma_1, \; 2\lambda_3 = 1 + \gamma_2, \; 2\lambda_4 = 1 - \gamma_2,$$

so that $\lambda_1, \cdots, \lambda_4$ are in F, $\lambda_1^2 - \lambda_2^2 = \gamma_1$, $\lambda_3^2 - \lambda_4^2 = \gamma_2$. Then

$$dd' - ee' = \lambda_1^2 - \lambda_4^2 i^2 - (\lambda_2^2 - \lambda_3^2 i^2) = \gamma_1 + \gamma_2 i^2 = \gamma.$$

But $\gamma = \gamma'$ and if we put $f = d\gamma^{-1}$, $h = e\gamma^{-1}$, we have

$$(ff' - hh')\gamma = (dd' - ee')\gamma^{-1} = 1,$$

and obtain

Lemma 2. *There exist polynomials f and h in $F(i)$ such that*

(16) $$(ff' - hh')\gamma = 1.$$

Let now r and s be defined by

(17) $$r = \alpha f, \; s = \alpha h,$$

where α is the quantity of (7) such that $zj = \alpha j^2 z$ and α is in $F(i)$. Then r and s are in $F(i)$, and

(18) $$(rr' - ss')\gamma(\alpha^{-1})(\alpha^{-1})' = 1.$$

But if $\delta = (jz)^2 = jzjz = j\alpha j^2 zz = \alpha_\theta g\gamma$, then $\delta_\theta = j(jz)^2 j^{-1} = (jjzj^{-1})^2 = (\alpha^{-1}\alpha j^2 zj^{-1})^2 = (\alpha^{-1} zjj^{-1})^2 = (\alpha^{-1}z)^2 = (\alpha^{-1})(\alpha^{-1})'\gamma$, so that (18) gives

Lemma 3. *There exist quantities r and s in $F(i)$ such that, if*

(19) $$\delta = (jz)^2 = \alpha_\theta g\gamma,$$

then

(20) $$(rr' - ss')\delta_\theta = 1.$$

If a and b are defined by

(21) $$a = s\alpha^{-1}, \; b = r_{\theta\theta},$$

so that $b_\theta = r$, then

(22) $$Q \equiv [b_\theta(b_\theta)' - a\alpha a'\alpha'], \; Q\delta_\theta = 1.$$

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3. **The cyclic property.** We shall now proceed to prove that D is a cyclic algebra by the use of our fundamental existence theorem, Lemma 3. Consider the quantity

$$X = a + bj + cj^2, \tag{23}$$

where we take a and b to be the polynomials of Lemma 3 which satisfy (22) and where c will be chosen to be a polynomial in i with coefficients in F. For every c in $F(i)$, we have

$$zXz^{-1} \equiv X' = a' + c'\alpha\alpha_{\theta\theta}gj + b'\alpha j^2, \tag{24}$$

so that

$$XX' = (a + bj + cj^2)(a' + c'\alpha\alpha_{\theta\theta}gj + b'\alpha j^2) = A + Bj + Ej^2, \tag{25}$$

where A is a polynomial in i and

$$B = ac'\alpha\alpha_{\theta\theta}g + b(a')_\theta + c(b'\alpha)_{\theta\theta}g = Rc + Sc' + T, \tag{26}$$

$$E = ab'\alpha + b(c'\alpha\alpha_{\theta\theta}z)_\theta + c(a')_{\theta\theta} = Gc + H(c')_\theta + K. \tag{27}$$

The quantities B and E are polynomials in i and we have defined above

$$R \equiv (b'\alpha)_{\theta\theta}g = (b_\theta)'\alpha_{\theta\theta}g, \quad S \equiv a\alpha\alpha_{\theta\theta}g, \quad T \equiv b(a')_\theta = b(a_{\theta\theta})', \tag{28}$$

$$G = (a')_{\theta\theta} = (a_\theta)', \;\; H = b\alpha_\theta\alpha g, \;\; K = ab'\alpha, \tag{29}$$

all in $F(i)$. Now

$$(g')_\theta = (g_{\theta\theta})' = g', \;\; (g')_{\theta\theta} = [(g')_\theta]_\theta = (g')_\theta = g. \tag{30}$$

Transforming B by z we have

$$B' = R'c' + S'c + T', \tag{31}$$

whence

$$\begin{aligned} R'B - SB' &= R'Rc + R'Sc' + R'T - SR'c' - SS'c - ST' \\ &= (RR' - SS')c - (ST' - R'T). \end{aligned} \tag{32}$$

But

$$\begin{aligned} RR' - SS' &= (b_\theta)'\alpha_{\theta\theta}gb_\theta(\alpha_{\theta\theta})'g' - a\alpha\alpha_{\theta\theta}ga'\alpha'(\alpha_{\theta\theta})'g' \\ &= gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'[b_\theta(b_\theta)' - a\alpha a'\alpha'] = gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'Q. \end{aligned} \tag{33}$$

From (19) $\delta_\theta = \alpha_{\theta\theta}g\gamma_\theta$, so that, utilizing the relation $Q\delta_\theta = 1$, we have

$$RR' - SS' = gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'\delta_\theta^{-1} = g'(\alpha_{\theta\theta})'(\gamma_\theta)^{-1} \neq 0, \tag{34}$$

since g, α and γ are all not zero in a division algebra. Hence $RR' - SS'$ has an inverse $(RR' - SS')^{-1}$ in $F(i)$, and if we define the quantity c by

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$$c = (ST' - R'T)(RR' - SS')^{-1}, \tag{35}$$

then

$$R'B - SB' = (ST' - R'T) - (ST' - R'T) = 0. \tag{36}$$

We shall henceforth consider the quantity X as completely defined in (23) *with the a and b of Lemma* 3 *and the c of* (35), *so that* (36) *is satisfied.* Transforming (36) by z we have

$$RB' - BS' = 0, \tag{37}$$

whence

$$R(R'B - SB') + S(- BS' + RB') = B(RR' - SS') = 0. \tag{38}$$

But $RR' - SS'$ has an inverse in $F(i)$, whence $B = 0$.

We consider now the polynomial E. We first compute

$$ST' - R'T = a\alpha\alpha_{\theta\theta}gb'a_{\theta\theta} - b_\theta(\alpha_{\theta\theta})'g'b(a_{\theta\theta})'. \tag{39}$$

Next

$$\begin{aligned} H(K')_\theta - (G')_\theta K &= b\alpha_\theta\alpha g(a')_\theta b_\theta(\alpha')_\theta - a_{\theta\theta}ab'\alpha \\ &= -\left[\alpha a a_{\theta\theta}b' - \alpha\alpha_\theta gbb_\theta(\alpha_{\theta\theta})'(a_{\theta\theta})'\right], \end{aligned} \tag{40}$$

so that

$$-\alpha_{\theta\theta}g[H(K')_\theta - (G')_\theta K] = a\alpha a_{\theta\theta}\alpha_{\theta\theta}b'g - bb_\theta(\alpha\alpha_\theta\alpha_{\theta\theta}g^2)(a_{\theta\theta})'(\alpha_{\theta\theta})'. \tag{41}$$

But $j^3 = g$, $g' = zgz^{-1} = zj^3z^{-1} = (zjz^{-1})^3 = (\alpha j^2zz^{-1})^3 = (\alpha j^2)^3 = \alpha j^2\alpha j^2\alpha j^2 = \alpha\alpha_{\theta\theta}\alpha_\theta g^2$, and we have the relations

$$g' = \alpha\alpha_\theta\alpha_{\theta\theta}g^2, \tag{42}$$

$$g = \alpha'(\alpha_{\theta\theta})'(\alpha_\theta)'(g')^2 = \alpha'(\alpha')_\theta(\alpha')_{\theta\theta}(g')^2. \tag{43}$$

Substituting (42) in (41) and comparing with (39) we write immediately

$$ST' - R'T = -\alpha_{\theta\theta}g[H(K')_\theta - (G')_\theta K]. \tag{44}$$

We also have, by the use of (11), $[(\alpha_\theta)']_\theta = [(\alpha')_{\theta\theta}]_\theta = \alpha'$, and

$$G(G')_\theta - H(H')_\theta = (a_\theta)'a_{\theta\theta} - b\alpha\alpha_\theta g(b')_\theta(\alpha')_\theta\alpha'g'. \tag{45}$$

But then

$$\begin{aligned} &-g\alpha_{\theta\theta}(\alpha')_{\theta\theta}[G(G')_\theta - H(H')_\theta] \\ &\qquad = (\alpha\alpha_\theta\alpha_{\theta\theta}g^2)[\alpha'(\alpha')_\theta(\alpha')_{\theta\theta}g']b(b')_\theta - ga_{\theta\theta}(a')_{\theta\theta}\alpha_{\theta\theta}(\alpha')_{\theta\theta}, \end{aligned} \tag{46}$$

which by (42) and (43) has the value

$$g[b(b')_\theta - a_{\theta\theta}\alpha_{\theta\theta}(a')_{\theta\theta}(\alpha')_{\theta\theta}]. \tag{47}$$

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But if

$$Q = b_\theta(b_\theta)' - a\alpha a'\alpha' = b_\theta(b')_{\theta\theta} - a\alpha a'\alpha', \tag{48}$$

then from (22), $\delta_\theta Q=1$, $Q\neq 0$,

$$Q_{\theta\theta} = b(b')_\theta - a_{\theta\theta}\alpha_{\theta\theta}(a')_{\theta\theta}(\alpha')_{\theta\theta}. \tag{49}$$

Hence

$$-\alpha_{\theta\theta}(\alpha')_{\theta\theta}g[G(G')_\theta - H(H')_\theta] = gQ_{\theta\theta} \neq 0, \tag{50}$$

so that

$$\frac{H(K')_\theta - (G')_\theta K}{G(G')_\theta - H(H')_\theta} = \frac{-\alpha_{\theta\theta}g[H(K')_\theta - (G')_\theta K]}{-\alpha_{\theta\theta}g[G(G')_\theta - H(H')_\theta]}\cdot\frac{(\alpha')_{\theta\theta}}{(\alpha')_{\theta\theta}} = \frac{(\alpha')_{\theta\theta}[ST' - R'T]}{gQ_{\theta\theta}}. \tag{51}$$

From (22) we have $\delta_\theta Q=1$, so that

$$j^2(\delta_\theta Q)j^{-2} = 1 = \delta Q_{\theta\theta} = \delta_\theta Q, \tag{52}$$

whence

$$\frac{\delta_\theta Q}{\gamma_\theta} = \frac{\delta Q_{\theta\theta}}{\gamma_\theta}. \tag{53}$$

Now $\delta=\gamma\alpha_\theta g$ from (19), while $\delta=(jz)^2$ is commutative with jz and equals its transform by jz, the quantity $(\delta')_\theta$. Hence

$$g = g_\theta,\quad \alpha_\theta g = \frac{\delta}{\gamma},\quad \alpha_{\theta\theta}g = \frac{\delta_\theta}{\gamma_\theta},\quad \alpha g = \frac{\delta_{\theta\theta}}{\gamma_{\theta\theta}}, \tag{54}$$

and

$$\alpha'g' = \frac{(\delta_{\theta\theta})'}{(\gamma_{\theta\theta})'} = \frac{(\delta')_\theta}{(\gamma')_\theta} = \frac{\delta}{\gamma_\theta}, \tag{55}$$

since $\gamma=\gamma'$. Equation (53) becomes

$$\alpha_{\theta\theta}gQ = \alpha'g'Q_{\theta\theta}. \tag{56}$$

From (43) $g=\alpha'(\alpha')_\theta(\alpha')_{\theta\theta}(g')^2$, so that

$$\alpha'g'Q_{\theta\theta} = (\alpha'g')[(\alpha')_\theta(\alpha')_{\theta\theta}g']\alpha_{\theta\theta}Q,$$

and

$$Q_{\theta\theta} = (\alpha')_{\theta\theta}(\alpha_{\theta\theta})'\alpha_{\theta\theta}g'Q. \tag{57}$$

It follows now that

$$gQ_{\theta\theta}/(\alpha')_{\theta\theta} = gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'Q = RR' - SS', \tag{58}$$

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by (33). Using (35) and (51) we have

$$(59)\quad c=(ST'-R'T)(RR'-SS')^{-1}=[H(K')_\theta-(G')_\theta K][G(G')_\theta-H(H')_\theta]^{-1}.$$

We have now demonstrated that

$$(60)\qquad [G(G')_\theta-H(H')_\theta]c-[H(K')_\theta-(G')_\theta K]=0,$$

a relation very similar to (36). In fact, since E is given by (27), and $[(c')_\theta]'=c_{\theta\theta}$, $(c_{\theta\theta})_\theta=c$,

$$(61)\qquad (E')_\theta=(G')_\theta(c')_\theta+(H')_\theta c+(K')_\theta,$$

so that, by (60),

$$(62)\qquad \begin{aligned}(G')_\theta E-H(E')_\theta &= (G')_\theta Gc+(G')_\theta H(c')_\theta+(G')_\theta K\\ &\qquad -H(G')_\theta(c')_\theta-H(H')_\theta c-H(K')_\theta\\ &=[G(G')_\theta-H(H')_\theta]c-[H(K')_\theta-(G')_\theta K]=0.\end{aligned}$$

Transforming (62) by jz we have

$$(63)\qquad G(E')_\theta-(H')_\theta E=0,$$

and

$$(64)\ 0=G[(G')_\theta E-H(E')_\theta]+H[G(E')_\theta-(H')_\theta E]=[G(G')_\theta-H(H')_\theta]E=0.$$

It follows from (50) that $E=0$ and that (25) becomes

$$(65)\qquad XX'=A \text{ in } F(i).$$

But then

$$(66)\qquad (Xz)^2=XzXz=XX'\gamma=A\gamma=t \text{ in } F(i),$$

since γ is in $F(i)$, and X' was defined so that $zX=X'z$ in (24).

Let first b and a be both not zero, so that since Xz is commutative with its square,

$$tXz=(ta+tbj+tcj^2)z=Xzt=(a+bj+cj^2)t'z.$$

Hence

$$ta+tbj+tcj^2=t'a+(t')_\theta bj+(t')_{\theta\theta}cj^2,$$

and since (6) are a basis of D, $ta=t'a$, $tb=(t')_\theta b$. Since a is not zero, $t=t'$ is in $F(i^2)$. Since also b is not zero, $t=(t')_\theta=t_\theta$ is in F. It follows that when $ab\neq 0$ we have shown that there exists a quantity X in the algebra

$$\Sigma=(i^sj^r)\qquad (s=0,1,\cdots,5;\ r=0,1,2),$$

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such that $X \neq 0$ and

$$(Xz)^2 = \lambda \text{ in } F. \tag{67}$$

Suppose next that a were zero so that from its origin (21) we have $s=0$ and $h=0$ in (17). Then (16) becomes

$$ff'\gamma = (fz)^2 = 1,$$

while then obviously f cannot be zero and $f = \alpha^{-1}r = \alpha^{-1}b_\theta$ is in $F(i)$ and in Σ. Again we have (67) for $X \neq 0$ in Σ. Finally the only remaining case is b zero. Then $r_{\theta\theta} = 0$ so that $r=0$ in (17), and hence the quantity f is zero. Equation (16) now becomes $hh'\gamma = (hz)^2 = -1$, and since then $h = \alpha^{-1}s = a$ in $F(i)$ cannot be zero when $(hz)^2 = -1$, we have again proved the existence of $X \neq 0$ in Σ and satisfying (67). Hence in all cases we have

Lemma 4. *There exists a quantity X in Σ such that $X \neq 0$, and if $y = Xz$ then*

$$y^2 = \lambda \ \text{in } F. \tag{68}$$

The quantities $1, v, y, vy$ are linearly independent with respect to F, for otherwise (6) could not be a basis of D when $X \neq 0$. A relation of the form

$$\xi_1 + \xi_2 v + (\xi_3 + \xi_4 v)Xz = 0,$$

with $\xi_1, \xi_2, \xi_3, \xi_4$ not all zero and in F, would then evidently express z as a quantity of Σ. Also $v^2 = \rho$, $y^2 = \lambda$, $yv = Xzv = -Xvz = -vXz = -vy$, since v, a polynomial in i commutative with j, is commutative with X. But the linear set

$$\Gamma = (1, v; y, vy)$$

is evidently a generalized quaternion algebra over F, and is a normal division cyclic algebra over F. Hence D, containing Γ, is the direct product* of Γ and another algebra Ω of order nine over F. Since D is a normal algebra, so is necessarily Ω,† so that Ω is a cyclic algebra‡ of order nine. Hence D, the direct product of algebra Γ and algebra Ω, is a cyclic algebra.†

Theorem. *Every normal division algebra of type R in thirty-six units is a cyclic algebra.*

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* A theorem of Wedderburn; cf. *Algebras and their Arithmetics*, p. 237.

† For the first and second of the above references respectively see Theorems 7 and 16 of the author's paper, *On direct products, cyclic division algebras, and pure Riemann matrices*, which appears in the present number of these Transactions.

‡ A theorem of Wedderburn, these Transactions, vol. 22 (1921), pp. 129–135.

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